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EXISTENCE THEORY FOR SINGLE AND MULTIPLE SOLUTIONS TO SINGULAR POSITONE DISCRETE DIRICHLET BOUNDARY VALUE PROBLEMS TO THE ONE-DIMENSION p-LAPLACIAN

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Abstract. In this paper we establish the existence of single and multiple solutions to the positone discrete Dirichlet boundary value problem

$$\begin{cases} \Delta\big[\phi(\Delta u(t-1))\big] + q(t)f(t,u(t)) = 0\,, & t \in \{1,2,\dots,T\} \\ u(0) = u(T+1) = 0\,, \end{cases}$$

where $\phi(s) = |s|^{p-2}s$, $p > 1$ and our nonlinear term $f(t,u)$ may be singular at $u = 0$.

1. Introduction

In this paper we establish the existence of single and multiple solutions to the positone discrete Dirichlet boundary value problem

$$\begin{cases} \Delta\big[\phi(\Delta u(t-1))\big] + q(t)f(t,u(t)) = 0\,, & t \in N = \{1,2,\dots,T\} \\ u(0) = u(T+1) = 0\,, \end{cases} \tag{1.1}$$

where $\phi(s) = |s|^{p-2}s$, $p > 1$ and $T \in \{1,2,\dots\}$, $N^+ = \{0,1,\dots,T+1\}$ and $u : N^+ \to [0,\infty)$. Throughout this paper we will assume $f : N \times (0,\infty) \to (0,\infty)$ is continuous. As a result our nonlinearity $f(t,u)$ may be singular at $u = 0$.

Remark 1.1. Recall a map $f : N \times (0,\infty) \to (0,\infty)$ is continuous if it is continuous as a map of the topological space $N \times (0,\infty)$ into the topological space $(0,\infty)$. Throughout this paper the topology on N will be the discrete topology.

We will let $C(N^+, \mathbf{R})$ denote the class of maps u continuous on N^+ (discrete topology), with norm $\|u\| = \max_{t\in N^+} |u(t)|$. By a solution to (1.1) we mean a $u \in C(N^+, [0,\infty))$ such that u satisfies (1.1) for $t \in N$ and u satisfies the boundary (Dirichlet) conditions.

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It is of interest to note here that the existence of single and multiple solutions to singular positone boundary value problems in the continuous case have been studied in great detail in the literature [5, 6, 7, 8, 12] ($p = 2$). However, for the discrete case almost all papers in the literature [1, 3, 10, 11] ($p = 2$) are devoted to the existence of one solution for singular positone problems, and only recently in [13] has the existence of one solution for singular discrete problems to the one-dimension p-Laplacian been discussed.

This paper discusses the existence of single and multiple solutions for singular positone discrete problems. Existence principles for nonsingular discrete Dirichlet problem to the one-dimension p-Laplacian are presented in Section 2. Some general existence theorems will be presented in Section 3 and there we will show, for example, that the discrete boundary value problem

$$\begin{cases} \Delta(\phi(\Delta u(t-1))) + \sigma([u(t)]^{-\alpha} + [u(t)]^{\beta} + 1) = 0\,, \ t \in N\,, \\ u(0) = 0, \ u(T+1) = 0\,, \quad \alpha > 0\,, \quad \beta > 1\,, \quad \sigma > 0 \quad \text{small,} \end{cases}$$

has two nonnegative solutions. Existence in this paper will be established using a Leray-Schauder alternative [14] and a general cone fixed point theorem in [5, 9].

In this paper we only consider discrete Dirichlet boundary data. It is worth remarking here that we could consider Sturm Liouville boundary data also; however since the arguments are essentially the same (in fact easier if not Dirichlet data) we will leave the details to the reader.

2. Existence principles

Consider the discrete Dirichlet boundary value problem

$$\text{(2.1)} \qquad \begin{cases} \Delta\big[\phi(\Delta u(t-1))\big] + f(t, u(t)) = 0\,, \quad \text{for} \quad t \in N = \{1, 2, \dots, T\}, \\ u(0) = A\,, \quad u(T+1) = B\,, \end{cases}$$

where A and B are given real numbers, $\phi(s) = |s|^{p-2}s$, $p > 1$. Suppose the following two conditions are satisfied:

(A1) $f(t,u) : N \times \mathbf{R} \to \mathbf{R}$ is continuous;

(A2) for each $r > 0$ there exists $h_r \in C(N,\ [0,\infty))$ such that $|u| \le r$ implies $|f(t,u)| \le h_r(t)$ for $t \in N$.

Suppose that $D \subset E := C(N^+, \mathbf{R})$ is a bounded set, and there exists a constant $r > 0$ such that $\|u\| \le r$ for $u \in \bar{D}$. Thus $|F\big(t, u(t)\big)| \le h_r(t)$ for $u \in \bar{D}$.

For each fixed $u \in D$, we consider the discrete boundary value problem

$$\text{(2.2)} \qquad \begin{cases} \Delta\big[\phi(\Delta w(t-1))\big] + f(t, u(t)) = 0\,, \qquad t \in N\,, \\ w(0) = A\,, \quad w(T+1) = B\,. \end{cases}$$

Then (2.2) is equivalent to

$$\text{(2.3)} \qquad w(t) = (\Phi u)(t) = \begin{cases} A\,, \ t = 0\,, \\ B + \sum_{s=t}^{T} \phi^{-1}\Big(\tau + \sum_{r=1}^{s} f(r, u(r))\Big)\,, \quad t \in N\,, \\ B\,, \ t = T+1\,, \end{cases}$$

where $\tau = -\phi(\Delta w(0))$ is a solution of the equation

$$Z(\tau) := \phi^{-1}(\tau) + \sum_{s=1}^{T} \phi^{-1}\Big(\tau + \sum_{r=1}^{s} f(r, u(r))\Big) = A - B\,. \tag{2.4}$$

Lemma 2.1. *For each fixed $u \in D$, Eq.* (2.4) *has a unique solution $\tau \in \mathbf{R}$, and*

$$|\tau| \leq C_r\,,$$

where C_r is a positive constant independent of $u \in D$.

Proof. Let $u \in D$ be fixed. Then we have, by the definition of $Z(\tau)$,

$$(T+1)\phi^{-1}\big(\tau - \sum_{t=1}^{T} h_r(t)\big) \leq Z(\tau) \leq (T+1)\phi^{-1}\Big(\tau + \sum_{t=1}^{T} h_r(t)\Big)\,, \tag{2.5}$$

$\forall \tau \in \mathbf{R}$, where h_r is defined by $(A2)$. Because ϕ^{-1} is a continuous, strictly increasing function on $\mathbf{R}$ with $\phi^{-1}(\mathbf{R}) = \mathbf{R}$, so is Z (for each fixed $u \in D$). Thus, there exists an unique $\tau \in \mathbf{R}$ satisfying Eq (2.4). By (2.4) and (2.5), we have

$$\tau \leq \phi\Big(\frac{A-B}{T+1}\Big) + \sum_{t=1}^{T} h_r(t),\ \ \tau \geq \phi\Big(\frac{A-B}{T+1}\Big) - \sum_{t=1}^{T} h_r(t)\,,$$

i.e.,

$$|\tau| \leq \phi\Big(\frac{|A-B|}{T+1}\Big) + \sum_{t=1}^{T} h_r(t) =: C_r\,.$$

The Lemma is thus proved. □

From Lemma 2.1, we conclude that $\Phi : D \to E$ is well defined. Concerning the mapping Φ, the following Lemma holds.

Lemma 2.2. *$\Phi : \bar{D} \to E$ is bounded and continuous.*

Proof. Let $u \in \bar{D}$ be fixed and $\tau \in \mathbf{R}$ is the unique solution of (2.4) corresponding to u. Then by (2.3), (2.4) and (2.5), we have

$$\|\Phi u\| \leq M_r\,, \tag{2.6}$$

where M_r is a positive constant independent of $u \in \bar{D}$. This shows that $\Phi(\bar{D})$ is a bounded subset of E.

Now assume that $u_0, u_n \in \bar{D}$ and $u_n \to u_0$ in $\bar{D}$. Then we have

$$(\Phi u_n)(t) = \begin{cases} A\,,\ t = 0\,, \\ B + \sum_{s=t}^{T} \phi^{-1}\Big(\tau_n + \sum_{r=1}^{s} f(r, u_n(r))\Big)\,, & t \in N\,, \\ B\,,\ t = T+1\,, \end{cases} \tag*{$(2.3)_n$}$$

where τ_n, $n = 0, 1, 2, \ldots$, satisfies the condition

$$\phi^{-1}(\tau_n) + \sum_{s=1}^{T} \phi^{-1}\Big(\tau_n + \sum_{r=1}^{s} f(r, u_n(r))\Big) = A - B\,. \tag*{$(2.4)_n$}$$

From Lemma 2.1, we know that $|\tau_n| \le C_r$, $n = 0, 1, 2, \dots$, where C_r is independent of u_n. Suppose that $\tau^* \in [-C_r, C_r]$ is an accumulation point of $\{\tau_n\}$. Then there is a subsequence of $\{\tau_n\}$, $\{\tau_{n(j)}\}$ which converge to τ^*. It follows from $(2.4)_{n(j)}$ that

$$\phi^{-1}(\tau^*) + \sum_{s=1}^{T} \phi^{-1}\Big(\tau^* + \sum_{r=1}^{s} f(r, u_0(r))\Big) = A - B\,.$$

This shows that $\tau^* = \tau_0$, by Lemma 2.1. Thus $\{\tau_n\}$ has a unique accumulation, and hence $\tau_n \to \tau_0$. Thus, from $(2.3)_n$ and $(2.4)_n$, we have

$$\lim_{n\to\infty} (\Phi u_n)(t) = (\Phi u_0)(t)\,, \qquad t \in N^+\,.$$

This shows that Φ is continuous (and bounded) from $\bar{D}$ to E. The proof of the Lemma is complete. □

Since D is an arbitrary bounded subset in E, we have

Lemma 2.3. *$\Phi : E \to E$ is completely continuous.*

We obtain the following general existence principles for (2.1) by using Schauder fixed point theorem and a nonlinear alternative of Leray-Schauder type.

Theorem 2.1. *Suppose (A1) and (A2) hold. In addition suppose there is a constant $M > |A| + |B|$, independent of λ with*

$$\|u\| = \max_{t\in N^+} |u(t)| \ne M \tag{2.7}$$

for any solution $u \in C(N^+, \mathbf{R})$ to

$$(2.8)_\lambda \qquad \begin{cases} \Delta\big(\phi(\Delta u(t-1))\big) + \lambda^{p-1} f\big(t, u(t)\big) = 0\,, & t \in N\,, \\ u(0) = \lambda A\,, \quad u(T+1) = \lambda B\,, \end{cases}$$

for each $\lambda \in (0, 1)$. Then (2.1) has a solution u with $\|u\| \le M$.

Proof. $(2.8)_\lambda$ is equivalent to the fixed point problem

$$(2.9)_\lambda \qquad u(t) = \lambda(\Phi u)(t)\,, \quad t \in N^+\,,$$

where Φ is as in (2.3). Set

$$U = \{u \in C(N^+, \mathbf{R})\,, \quad \|u\| < M\}\,.$$

Since $\Phi : C(N^+, \mathbf{R}) \to C(N^+, \mathbf{R})$ is continuous and completely continuous, the nonlinear alterative [14] guarantees that Φ has a fixed point i.e., $(2.9)_1$ has a solution in $\bar{U}$. □

Theorem 2.2. *Suppose (A1) and (A2) hold. In addition suppose there is a constant $M > |A| + |B|$, independent of λ with*

$$\|u\| = \max_{t\in N^+} |u(t)| \ne M$$

for any solution $u \in C(N^+, \mathbf{R})$ *to*

$$(2.10)_\lambda \quad \begin{cases} \Delta\big(\phi(\Delta u(t-1) - (1-\lambda)(\frac{B-A}{T+1}))\big) + \lambda^{p-1} f(t, u(t)) = 0\,, & t \in N\,, \\ u(0) = A\,, \quad u(T+1) = B\,, \end{cases}$$

for each $\lambda \in (0,1)$*. Then* (2.1) *has a solution* u *with* $\|u\| \le M$*.*

Proof. $(2.10)_\lambda$ is equivalent to the fixed point problem

$$(2.11)_\lambda \qquad u = (1-\lambda)Q + \lambda \Phi u \quad \text{where} \quad Q = A + \frac{B-A}{T+1} t\,.$$

Set

$$U = \{u \in C(N^+, \mathbf{R})\,, \quad \|u\| < M\}\,.$$

Since $\Phi : C(N^+, \mathbf{R}) \to C(N^+, \mathbf{R})$ is continuous and completely continuous, the nonlinear alterative [14] guarantees that Φ has a fixed point i.e., $(2.11)_1$ has a solution in $\bar{U}$. □

Theorem 2.3. *Suppose that* $(A1)$ *holds, and there exists* $h \in C\big(N, [0,\infty)\big)$ *with* $|F(t,u)| \le h(t)$ *for* $t \in N$*. Then* (2.1) *has a solution* u*.*

Proof. Solving (2.1) is equivalent to the fixed point problem $u = \Phi u$. Since $\Phi : C(N^+, \mathbf{R}) \to C(N^+, \mathbf{R})$ is continuous and compact, the result follows from Schauder's fixed point theorem. □

3. Singular discrete boundary value problems

In this section we examine the singular Dirichlet boundary value problem

$$(3.1) \qquad \begin{cases} \Delta(\phi(\Delta u(t-1))) + q(t) f(t, u(t)) = 0\,, & t \in N\,, \\ u(0) = 0\,, \quad u(T+1) = 0\,, \end{cases}$$

where $\phi(s) = |s|^{p-2}s$, $p > 1$, and nonlinearity f may be singular at $u = 0$. We begin by showing that (3.1) has a solution. To do so we first establish, via Theorem 2.2, the existence of a solution, for each sufficiently large n, to the "modified" problem

$$(3.1)^n \qquad \begin{cases} \Delta\big(\phi(\Delta u(t-1))\big) + q(t) f\big(t, u(t)\big) = 0\,, & t \in N\,, \\ u(0) = \frac{1}{n}\,, \quad u(T+1) = \dfrac{1}{n}\,. \end{cases}$$

To show that (3.1) has a solution we let $n \to \infty$; the key idea in this step is Arzela-Ascoli theorem.

Before we prove our main results we first state one well known result [4].

Lemma 3.1 ([4])**.** *Let* $y \in C(N^+, \mathbf{R})$ *satisfy* $y(t) \ge 0$ *for* $t \in N^+$*. If* $u \in C(N^+, \mathbf{R})$ *satisfies*

$$\begin{cases} \Delta^2 u(t-1) + y(t) = 0\,, & t \in N\,, \\ u(0) = u(T+1) = 0\,, \end{cases}$$

then

$$u(t) \ge \mu(t)\|u\| \quad \text{for} \quad t \in N^+;$$

here

$$\mu(t) = \min\{\frac{T+1-t}{T+1}, \frac{t}{T}\}.$$

Theorem 3.1. *Suppose the following conditions are satisfied:*

(H_1) $q : N \to (0,\infty)$ *is continuous;*

(H_2) $f : N \times (0,\infty) \to (0,\infty)$ *is continuous;*

(H_3) $f(t,u) \le g(u) + h(u)$ *on* $N \times (0,\infty)$ *with* $g > 0$ *continuous and nonincreasing on* $(0,\infty)$*,* $h \ge 0$ *continuous on* $[0,\infty)$*, and* $\frac{h}{g}$ *nondecreasing on* $(0,\infty)$*;*

(H_4) *for each constant* $H > 0$ *there exists a function* ψ_H *continuous on* N^+ *and positive on* N *such that* $f(t,u) \ge \psi_H(t)$ *on* $N \times (0,H]$*;*

(H_5) *there exists a constant* $r > 0$ *such that*

$$\frac{1}{\phi^{-1}\left(1 + \frac{h(r)}{g(r)}\right)} \int_0^r \frac{dy}{\phi^{-1}(g(y))} > b_0, \tag{3.2}$$

where

$$b_0 = \max_{t\in N}\left(\sum_{s=1}^{t}\phi^{-1}(\sum_{r=s}^{t}q(r))\,, \quad \sum_{s=t}^{T}\phi^{-1}\Big(\sum_{r=t}^{s}q(r)\Big)\right).$$

Then (3.1) *has a solution* $u \in C\big(N^+,\ [0,\infty)\big)$ *with* $u > 0$ *on* N *and* $\|u\| < r$*.*

Proof. Choose $\epsilon > 0$, $\epsilon < r$ with

$$\frac{1}{\phi^{-1}\left(1 + \frac{h(r)}{g(r)}\right)} \int_\epsilon^r \frac{dy}{\phi^{-1}(g(y))} > b_0\,. \tag{3.3}$$

Let $n_0 \in \{1,2,\dots\}$ be chosen so that $\frac{1}{n_0} < \epsilon$ and let $Z^+ = \{n_0, n_0+1, \dots\}$. To show $(3.1)^n$, $n \in Z^+$, has a solution we examine

$$\begin{cases} \Delta\big(\phi(\Delta u(t-1))\big) + q(t)F\big(t,u(t)\big) = 0\,, & t \in N\,,\\ u(0) = \frac{1}{n}\,,\quad u(T+1) = \frac{1}{n}\,, & n \in Z^+\,, \end{cases} \tag*{$(3.4)^n$}$$

where

$$F(t,u) = \begin{cases} f(t,u)\,, & u \ge \frac{1}{n}\,,\\ f(t,\frac{1}{n})\,, & u \le \frac{1}{n}\,. \end{cases}$$

To show that $(3.4)^n$ has a solution for $n \in Z^+$, we will apply Theorem 2.2. Consider the family of problems

$$\begin{cases} -\Delta\big(\phi(\Delta u(t-1))\big) = \lambda^{p-1}q(t)F(t,u(t))\,, & t \in N\,,\\ u(0) = \frac{1}{n}\,,\quad u(T+1) = \frac{1}{n}\,, & n \in Z^+\,, \end{cases} \tag*{$(3.5)^n_\lambda$}$$

where $\lambda \in (0,1)$. Let u be a solution of $(3.5)^n_\lambda$. Since $\Delta[\phi(\Delta u(t-1))] \le 0$ on N implies $\Delta^2 u(t-1) \le 0$ on N, then $u(t) \ge \frac{1}{n}$ on N^+ and there exists $t_0 \in N$ with $\Delta u(t) \ge 0$ on $[0,t_0) = \{0,1,\dots,t_0-1\}$ and $\Delta u(t) \le 0$ on $[t_0, T+1) = \{t_0, t_0+1, \dots, T\}$, and $u(t_0) = \|u\|$.

Also notice that

$$F(t,u(t)) = f(t,u(t)) \le g(u(t)) + h(u(t))\,, \qquad t \in N\,,$$

so for $z \in N$, we have

$$-\Delta\big(\phi(\Delta u(z-1))\big) \le g\big(u(z)\big)\left(1 + \frac{h(u(t_0))}{g(u(t_0))}\right)q(z)\,. \tag{3.6}$$

We sum the equation (3.6) from $s+1 (0 \le s < t_0)$ to t_0 to obtain

$$\phi\big[\Delta u(t_0)\big] \ge \phi[\Delta u(s)] - \left(1 + \frac{h(u(t_0))}{g(u(t_0))}\right)\sum_{z=s+1}^{t_0} g(u(z))q(z)\,.$$

Since $\Delta u(t_0) \le 0$, and $u(z) \ge u(s+1)$ when $s+1 \le z \le t_0$, then we have

$$\begin{aligned}\phi\big[\Delta u(s)\big] &\le \phi\big[\Delta u(t_0)\big] + \left(1 + \frac{h(u(t_0))}{g(u(t_0))}\right)\sum_{z=s+1}^{t_0} g(u(z))q(z)\\ &\le g(u(s+1))\left(1 + \frac{h(u(t_0))}{g(u(t_0))}\right)\sum_{z=s+1}^{t_0} q(z)\,, \quad s < t_0\,,\end{aligned}$$

i.e.,

$$\frac{\Delta u(s)}{\phi^{-1}(g(u(s+1)))} \le \phi^{-1}\left(1 + \frac{h(u(t_0))}{g(u(t_0))}\right)\phi^{-1}\Big(\sum_{z=s+1}^{t_0} q(z)\Big)\,, \quad s < t_0\,. \tag{3.7}$$

Since $g(u(s+1)) \le g(u) \le g(u(s))$ for $u(s) \le u \le u(s+1)$ when $s < t_0$, then we have

$$\int_{u(s)}^{u(s+1)} \frac{du}{\phi^{-1}(g(u))} \le \frac{\Delta u(s)}{\phi^{-1}(g(u(s+1)))}\,, \quad s < t_0\,. \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$\int_{u(s)}^{u(s+1)} \frac{du}{\phi^{-1}(g(u))} \le \phi^{-1}\left(1 + \frac{h(u(t_0))}{g(u(t_0))}\right)\phi^{-1}\Big(\sum_{z=s+1}^{t_0} q(z)\Big)\,, \quad s < t_0\,,$$

and then we sum the above from 0 to $t_0 - 1$ to obtain

$$\begin{aligned}\int_{\frac{1}{n}}^{u(t_0)} \frac{du}{\phi^{-1}(g(u))} &\le \phi^{-1}\left(1 + \frac{h(u(t_0))}{g(u(t_0))}\right)\sum_{s=0}^{t_0-1}\phi^{-1}\Big(\sum_{z=s+1}^{t_0} q(z)\Big)\\ &= \phi^{-1}\left(1 + \frac{h(u(t_0))}{g(u(t_0))}\right)\sum_{s=1}^{t_0}\phi^{-1}\Big(\sum_{z=s}^{t_0} q(z)\Big)\,.\end{aligned} \tag{3.9}$$

Similarly, we sum the equation (3.6) from t_0 to $s (t_0 \le s < T+1)$ to obtain

$$\phi\big[\Delta u(s)\big] \ge \phi[\Delta u(t_0-1)] - \left(1 + \frac{h(u(t_0))}{g(u(t_0))}\right)\sum_{z=t_0}^{s} g(u(z))q(z)\,, \quad s \ge t_0\,.$$

Since $\Delta u(t_0-1)\ge 0$, then we have

$$-\phi\big[\Delta u(s)\big] \le -\phi[\Delta u(t_0-1)] + \left(1+\frac{h(u(t_0))}{g(u(t_0))}\right)\sum_{z=t_0}^{s} g(u(z))q(z)$$
$$\le g(u(s))\left(1+\frac{h(u(t_0))}{g(u(t_0))}\right)\sum_{z=t_0}^{s} q(z)\,, \quad s\ge t_0\,,$$

i.e.,

$$\frac{-\Delta u(s)}{\phi^{-1}(g(u(s)))} \le \phi^{-1}\left(1+\frac{h(u(t_0))}{g(u(t_0))}\right)\phi^{-1}\Big(\sum_{z=t_0}^{s} q(z)\Big)\,, \quad s\ge t_0\,.$$

So we have

$$\int_{u(s+1)}^{u(s)}\frac{du}{\phi^{-1}(g(u))} \le \frac{-\Delta u(s)}{\phi^{-1}(g(u(s)))} \le \phi^{-1}\left(1+\frac{h(u(t_0))}{g(u(t_0))}\right)\phi^{-1}\Big(\sum_{z=t_0}^{s} q(z)\Big)\,, \quad s\ge t_0\,,$$

and then we sum the above from t_0 to T to obtain

$$\int_{\frac{1}{n}}^{u(t_0)}\frac{du}{\phi^{-1}(g(u))} \le \phi^{-1}\left(1+\frac{h(u(t_0))}{g(u(t_0))}\right)\sum_{s=t_0}^{T}\phi^{-1}\Big(\sum_{z=t_0}^{s} q(z)\Big)\,. \tag{3.10}$$

Now (3.9) and (3.10) imply

$$\int_{\varepsilon}^{u(t_0)}\frac{du}{\phi^{-1}(g(u))} \le \int_{\frac{1}{n}}^{u(t_0)}\frac{du}{\phi^{-1}(g(u))} \le b_0\phi^{-1}\left(1+\frac{h(u(t_0))}{g(u(t_0))}\right).$$

This together with (3.3) implies $\|u\|=u(t_0)\ne r$. Then Theorem 2.2 implies that $(3.4)^n$ has a solution u_n with $||u_n||\le r$. In fact (as above)

$$\frac{1}{n}\le u_n(t)<r\,, \quad \text{for} \quad t\in N^+\,.$$

Thus $u_n(t)$ is a solution of $(3.1)^n$ also.

Next we obtain a sharper lower bound on u_n, namely we will show that there exists a constant $k>0$, independent of n, with

$$u_n(t)\ge k\mu(t)\,, \quad \text{for} \quad t\in N^+\,, \tag{3.11}$$

where μ is as in Lemma 3.1.

To see this notice (H_4) guarantees the existence of a function $\psi_r(t)$ continuous on N^+ and positive on N with $f(t,u)\ge\psi_r(t)$ for $(t,u)\in N\times(0,r]$. Let $y_r(t)\in C(N^+,\ \mathbf{R})$ be a unique solution to the problem

$$\begin{cases} \Delta\big(\phi(\Delta y_r(t-1))\big)+q(t)\psi_r(t)=0\,, & t\in N\,,\\ y_r(0)=0\,, \quad y_r(T+1)=0\,. \end{cases} \tag{3.12}$$

Since $\Delta\big(\phi(\Delta y_r(t-1))\big)\le 0$ on N, with $y_r(0)=y_r(T+1)=0$, then $\Delta^2 y_r(t-1)\le 0$ on N, and so Lemma 3.1 implies,

$$y_r(t)\ge\mu(t)\|y_r\|\,, \qquad t\in N^+\,. \tag{3.13}$$

Since $f(t,u)\ge\psi_r(t)$ for $(t,u)\in N\times(0,r]$, we claim that

$$u_n(t)\ge y_r(t)\,, \qquad t\in N^+\,. \tag{3.14}$$

Suppose (3.14) is false i.e. assume $u_n(t) < y_r(t)$ for some $t \in N^+$. Since $u_n(0) > y_r(0) = 0$, $u_n(T+1) > y_r(T+1) = 0$, the function $V(t) = y_r(t) - u_n(t)$ would have a positive maximum at a point $t_0 \in N$. Hence $\Delta V(t_0 - 1) \geq 0$, i.e., $\Delta y_r(t_0 - 1) \geq \Delta u_n(t_0 - 1)$. Notice that

$$\Delta\big(\phi(\Delta y_r(t-1))\big) - \Delta\big(\phi(\Delta u_n(t-1))\big) = -q(t)\psi_r(t) + q(t)f(t, u_n(t)) \geq 0\,, \quad \forall\, t \in N\,.$$

Sum both sides of the above inequality from t_0 to $t \in [t_0, T+1) = \{t_0, \ldots, T\}$ to get

$$\phi\big(\Delta y_r(t)\big) - \phi\big(\Delta y_r(t_0 - 1)\big) \geq \phi\big(\Delta u_n(t)\big) - \phi\big(\Delta u_n(t_0 - 1)\big)\,,$$

for all $t \in [t_0, T+1)$, and so

$$\phi\big(\Delta y_r(t)\big) - \phi\big(\Delta u_n(t)\big) \geq \phi\big(\Delta y_r(t_0 - 1)\big) - \phi\big(\Delta u_n(t_0 - 1)\big) \geq 0\,,$$

for all $t \in [t_0, T+1)$. That is

$$\Delta V(t) = \Delta y_r(t) - \Delta u_n(t) \geq 0\,,$$

for all $t \in [t_0, T+1)$, and so $V(t_0) \leq V(T+1) < 0$, a contradiction.

Now (3.14) together with (3.13) implies (3.11) holds for $k = \|y_r\|$.

The Arzela-Ascoli theorem guarantees the existence of a subsequence $Z^0 \subset Z^+$ and a function $u \in C(N^+, \mathbf{R})$ with $u_n \to u$ in $C(N^+, \mathbf{R})$ as $n \to \infty$ through Z^0. Also $u(0) = u(T+1) = 0$, $\|u\| \leq r$ for $t \in N^+$. In particular $u(t) \geq k\mu(t) \geq \frac{k}{T+1}$ on N. Fix $t \in N$, and we obtain

$$\begin{aligned}
\Delta\big[\phi(\Delta u_n(t-1))\big] &= \phi\big(\Delta u_n(t)\big) - \phi\big(\Delta u_n(t-1)\big) \\
&= \phi\big(u_n(t+1) - u_n(t)\big) - \phi\big(u_n(t) - u_n(t-1)\big) \\
&\to \Delta\big(\phi(\Delta u(t-1))\big)\,, \quad t \in N,\ n \in Z^0,\ n \to \infty\,,
\end{aligned}$$

and

$$f\big(t, u_n(t)\big) \to f\big(t, u(t)\big)\,, \qquad t \in N,\ n \in Z^0,\ n \to \infty\,.$$

Thus $\Delta(\phi(\Delta u(t-1))) + q(t)f(t, u(t)) = 0$ for $t \in N$, $u(0) = u(T+1) = 0$. Finally it is easy to see that $\|u\| < r$ (note if $\|u\| = r$, then following essentially the same argument from (3.6)–(3.10) will yield a contradiction).

This complete the proof of Theorem 3.1. $\square$

Example 3.1. Consider the singular boundary value problem

$$\left\{\begin{array}{ll} \Delta\big(\phi(\Delta u(t-1))\big) + \sigma([u(t)]^{-\alpha} + [u(t)]^{\beta}) = 0\,, & t \in N \\ u(0) = 0\,, \quad u(T+1) = 0\,, & \end{array}\right. \tag{3.15}$$

with $\alpha > 0$, $\beta \geq 0$, $\sigma > 0$ is such that

$$\sigma < \left[\frac{p-1}{b_1(\alpha + p - 1)}\right]^{p-1} \sup_{c \in (0,\infty)} \frac{c^{\alpha+p-1}}{1 + c^{\alpha+\beta}}\,; \tag{3.16}$$

here

$$b_1 = \max_{t \in N}\left(\sum_{s=1}^{t}(t - s + 1)^{\frac{1}{p-1}},\ \sum_{s=t}^{T}(s - t + 1)^{\frac{1}{p-1}}\right) = \sum_{t=1}^{T} t^{\frac{1}{p-1}}\,. \tag{3.17}$$

Then (3.15) has a solutions u with $u(t) > 0$ for $t \in N$.

To see this we will apply Theorem 3.1 with

$$q(s) = \sigma\,, \quad g(u) = u^{-\alpha}\,, \quad h(u) = u^{\beta}\,.$$

Clearly (H_1)–(H_4) hold. Also notice

$$\sum_{s=1}^{t}\phi^{-1}(\sum_{r=s}^{t}\sigma) = \sigma^{\frac{1}{p-1}}\sum_{s=1}^{t}(t-s+1)^{\frac{1}{p-1}}\,,$$

$$\sum_{s=t}^{T}\phi^{-1}(\sum_{r=t}^{s}\sigma) = \sigma^{\frac{1}{p-1}}\sum_{s=t}^{T}(s-t+1)^{\frac{1}{p-1}}\,,$$

and so

$$b_0 = \max_{t\in N}\left(\sigma^{\frac{1}{p-1}}\sum_{s=1}^{t}(t-s+1)^{\frac{1}{p-1}},\ \sigma^{\frac{1}{p-1}}\sum_{s=t}^{T}(s-t+1)^{\frac{1}{p-1}}\right) = \sigma^{\frac{1}{p-1}}b_1\,.$$

Consequently (H_5) holds since (3.16) implies there exists $r > 0$ such that

$$\sigma < \left[\frac{p-1}{b_1(\alpha+p-1)}\right]^{p-1}\frac{r^{\alpha+p-1}}{1+r^{\alpha+\beta}}\,,$$

and so

$$\frac{1}{\phi^{-1}(1+\frac{h(r)}{g(r)})}\int_0^r\frac{dy}{\phi^{-1}(g(y))} = \frac{p-1}{p-1+\alpha}\phi^{-1}\left(\frac{r^{\alpha+p-1}}{1+r^{\alpha+\beta}}\right) > b_0\,.$$

Thus all the conditions of Theorem 3.1 are satisfied so existence is guaranteed.

Remark 3.1. If $\beta < p-1$ then (3.16) is automatically satisfied.

Next we establish the existence of two positive solutions to (3.1). First we state the fixed point result we will use to establish multiplicity.

Lemma 3.2 ([5]). *Let $E = (E, \|\cdot\|)$ be a Banach space and let $K \subset E$ be a cone in E, and let $\|\cdot\|$ be increasing with respect to K. Also, r, R are constants with $0 < r < R$. Suppose $\Phi : \bar{\Omega}_R \cap K \to K($ here $\Omega_R = \{x \in E, \|x\| < R\})$ is a continuous, compact map and assume the conditions*

$$x \neq \lambda\Phi(x)\,, \quad \textit{for} \quad \lambda \in [0,1) \quad \textit{and} \quad x \in \partial\Omega_r \cap K \tag{3.18}$$

and

$$\|\Phi x\| > \|x\|\,, \quad \textit{for} \quad x \in \partial\Omega_R \cap K \tag{3.19}$$

hold. Then Φ has a fixed point in $K \cap \{x \in E : r \leq \|x\| \leq R\}$.

Remark 3.2. In Lemma 3.2 if (3.18) and (3.19) are replaced by

$$x \neq \lambda\Phi(x)\,, \quad \text{for} \quad \lambda \in [0,1) \quad \text{and} \quad x \in \partial\Omega_R \cap K \tag{3.18*}$$

and

$$\|\Phi x\| > \|x\|\,, \quad \text{for} \quad x \in \partial\Omega_r \cap K\,. \tag{3.19*}$$

Then Φ has a fixed point in $K \cap \{x \in E : r \leq \|x\| \leq R\}$.

Let K be the cone in $E = C(N^+, \mathbf{R})$ given by

$$K := \{u \in C(N^+, \mathbf{R}) :\ u(t) \geq \mu(t)\|u\|\,, \quad t \in N^+\}\,.$$

Theorem 3.2. *Assume that* (H$_1$), (H$_2$), (H$_3$) *and* (H$_5$) *hold. In addition suppose that*

$$\text{(H}_6\text{)} \begin{cases} \text{there exists a nonincreasing continuous function} \\ g_1 : (0,\infty) \to (0,\infty), \text{ and a continuous function} \\ h_1 : [0,\infty) \to (0,\infty) \text{ with } \frac{h_1}{g_1} \text{ nondecreasing on } (0,\infty) \\ \text{and with } f(t,u) \geq g_1(u) + h_1(u) \text{ for } (t,u) \in N \times (0,\infty); \end{cases} \tag{3.20}$$

(H$_7$) *there exists $R > r$ with*

$$\frac{R}{\phi^{-1}\Big(g_1(R)\big[1 + \frac{h_1(\frac{R}{T+1})}{g_1(\frac{R}{T+1})}\big]\Big)} < \|v\|, \tag{3.21}$$

where v satisfies

$$\begin{cases} \Delta\big(\phi(\Delta v(t-1))\big) + q(t) = 0\,, & t \in N\,, \\ v(0) = v(T+1) = 0\,. \end{cases} \tag{3.22}$$

Then (3.1) *has a solution $u \in C(N^+, \mathbf{R})$ with $u > 0$ on N and $r < \|u\| \leq R$.*

Proof. To show the existence of the solution described in the statement of Theorem 3.2, we will apply Lemma 3.2. First we choose $\epsilon > 0 (\epsilon < r)$ with

$$\frac{1}{\phi^{-1}\Big(1 + \frac{h(r)}{g(r)}\Big)} \int_\epsilon^r \frac{dy}{\phi^{-1}(g(y))} > b_0\,. \tag{3.23}$$

Let $n_0 \in \{1, 2, \cdots\}$ be chosen so that $\frac{1}{n_0} < \frac{\epsilon}{2}$ and $\frac{1}{n_0} < \frac{r}{T+1}$ and let $Z^+ = \{n_0, n_0+1, \dots\}$.

First we will show that

$$\begin{cases} \Delta\big(\phi(\Delta u(t-1))\big) + q(t) f(t, u(t)) = 0\,, & t \in N\,, \\ u(0) = \frac{1}{n}\,, \quad u(T+1) = \frac{1}{n}\,, & n \in Z^+\,, \end{cases} \tag{3.24^n}$$

has a solution u_n for each $n \in Z^+$ with $u_n(t) > \frac{1}{n}$ on N and $r < \|u_n\| \leq R$. To show $(3.24)^n$ has such a solution for each $n \in Z^+$, we will deal with the modified boundary value problem

$$\begin{cases} \Delta\big(\phi(\Delta u(t-1))\big) + q(t) f^*(t, u(t)) = 0\,, & t \in N\,, \\ u(0) = \frac{1}{n}\,, \quad u(T+1) = \frac{1}{n}\,, & n \in Z^+\,, \end{cases} \tag{3.25^n}$$

with

$$f^*(t, u(t)) = \begin{cases} f(t, u(t))\,, & u \geq \frac{1}{n}\,, \\ f(t, \frac{1}{n})\,, & 0 \leq u \leq \frac{1}{n}\,. \end{cases}$$

Fix $n \in Z^+$. Let $\Phi : K \to C(N^+, \mathbf{R})$ be defined by

$$w(t) := (\Phi u)(t) = \begin{cases} \frac{1}{n}, & t = 0 \quad \text{or} \quad t = T+1, \\ \frac{1}{n} + \sum_{s=t}^{T} \phi^{-1}\Big(\tau + \sum_{z=1}^{s} f^*(z, u(z))\Big), & t \in N, \end{cases} \tag{3.26}$$

where τ is a solution of the equation

$$\phi^{-1}(\tau) + \sum_{s=1}^{T} \phi^{-1}\Big(\tau + \sum_{z=1}^{s} f^*(z, u(z))\Big) = 0. \tag{3.27}$$

From section 2, $\Phi : K \to C(N^+, \mathbf{R})$ is completely continuous. Moreover, we have

$$\begin{cases} \Delta\big(\phi(\Delta w(t-1))\big) + q(t) f^*\big(t, u(t)\big) = 0, & t \in N, \\ w(0) = \frac{1}{n}, \quad w(T+1) = \frac{1}{n}, & n \in Z^+. \end{cases} \tag{3.28}$$

This implies that $\Delta\big(\phi(\Delta w(t-1))\big) \le 0$, $t \in N$. Thus $\Delta^2 w(t-1) \le 0$, $t \in N$, and $w(t) \ge \frac{1}{n}$. Consequently, $w(t) - \frac{1}{n} \ge \mu(t)\|w - \frac{1}{n}\|$ (from Lemma 3.1), thus $w(t) \ge \frac{1}{n} + \mu(t)(\|w\| - \frac{1}{n}) \ge \mu(t)\|w\|$, $t \in N^+$, and so $\Phi : K \to K$.

We first show

$$u \ne \lambda \Phi u \quad \text{for} \quad \lambda \in [0,1), \ u \in \partial\Omega_r \cap K, \tag{3.29}$$

where Ω_r is defined above.

Suppose this is false i.e., suppose there exists $u \in \partial\Omega_r$ and $\lambda \in [0,1)$ with $u = \lambda\Phi u$. We can assume $\lambda \ne 0$. Now since $u = \lambda \Phi u$ we have

$$\begin{cases} -\Delta\big(\phi(\Delta u(t-1))\big) = \lambda^{p-1} q(t) f^*\big(t, u(t)\big), & t \in N, \\ u(0) = \frac{\lambda}{n}, \quad u(T+1) = \frac{\lambda}{n}, & n \in Z^+. \end{cases} \tag{3.30}$$

Clearly there exists $t_0 \in N$ with $\Delta u(t) \ge 0$ on $[0, t_0) = \{0, 1, \ldots, t_0 - 1\}$, $\Delta u(t) \le 0$ on $[t_0, T+1) = \{t_0, t_0+1, \ldots, T\}$ and $u(t_0) = \|u\| = r$ (note $u \in \partial\Omega_r \cap K$). Also notice $u(t) \ge \mu(t)\|u\| = \mu(t) r \ge \frac{r}{T+1} > \frac{1}{n_0}$ for $t \in N$, and so

$$f^*\big(t, u(t)\big) = f\big(t, u(t)\big) \le g\big(u(t)\big) + h\big(u(t)\big), \quad t \in N.$$

Fix $z \in N$, and we have

$$-\Delta(\phi(\Delta u(z-1))) \le g(u(z)) \left\{ 1 + \frac{h(u(t_0))}{g(u(t_0))} \right\} q(z). \tag{3.31}$$

The reasoning used to obtain (3.9) and (3.10) in Theorem 3.1, yield:

$$\int_{\frac{\lambda}{n}}^{u(t_0)} \frac{du}{\phi^{-1}(g(u))} \le \phi^{-1}\left(1 + \frac{h(u(t_0))}{g(u(t_0))}\right) \sum_{s=1}^{t_0} \phi^{-1}\Big(\sum_{z=s}^{t_0} q(z)\Big), \tag{3.32}$$

and

$$\int_{\frac{\lambda}{n}}^{u(t_0)} \frac{du}{\phi^{-1}(g(u))} \le \phi^{-1}\left(1 + \frac{h(u(t_0))}{g(u(t_0))}\right) \sum_{s=t_0}^{T} \phi^{-1}\Big(\sum_{z=t_0}^{s} q(z)\Big). \tag{3.33}$$

Now (3.32) and (3.33) imply

$$\int_{\varepsilon}^{r} \frac{du}{\phi^{-1}(g(u))} \le b_0 \phi^{-1}\left(1 + \frac{h(u(r))}{g(u(r))}\right). \tag{3.34}$$

This contradicts (3.23) and consequently (3.29) is true.

Next we show

$$\|w\| = \|\Phi u\| > \|u\|, \quad \forall\, u \in \partial\Omega_R \cap K.$$

To see this let $u \in \partial\Omega_R \cap K$ such that $\|u\| = R$. Also, since $u \in K$ then $u(t) \ge \mu(t)R \ge \frac{R}{T+1} > \frac{1}{n_0}$ for $t \in N$. Thus, $f^*(t,u(t)) = f(t,u(t)) \ge g_1(u) + h_1(u)$ for $t \in N$, so we have

$$\begin{aligned} -\Delta\big(\phi(\Delta w(t-1))\big) &= q(t) f^*\big(t,u(t)\big) = q(t) f\big(t,u(t)\big) \\ &\ge g_1\big(u(t)\big)\left(1 + \frac{h_1\big(u(t)\big)}{g_1\big(u(t)\big)}\right) q(t) \\ &\ge g_1(R)\left(1 + \frac{h_1(\frac{R}{T+1})}{g_1(\frac{R}{T+1})}\right) q(t) := C(R) q(t). \end{aligned} \tag{3.35}$$

Then we obtain

$$-\Delta\left(\phi\left(\Delta \frac{w(t-1)}{\phi^{-1}(C(R))}\right)\right) \ge q(t), \quad w(0) = w(T+1) = \frac{\lambda}{n} \ge 0. \tag{3.36}$$

The argument used to get (3.11) yields

$$\frac{w(t)}{\phi^{-1}(C(R))} \ge v(t), \qquad t \in N^+. \tag{3.37}$$

Now (3.21) and (3.37) yield

$$\|w\| \ge \|v\| \phi^{-1}(C(R)) > R,$$

i.e.,

$$\|\Phi u\| > \|u\|, \quad \forall\, u \in \partial\Omega_R \cap K.$$

This implies Φ has a fixed point $u_n \in K \cap (\bar{\Omega}_R \setminus \Omega_r)$ i.e., $r < \|u_n\| \le R$. In fact $||u_n|| \ne r$ (note if $||u_n|| = r$ then following essentially the same argument from (3.31)–(3.34) will yield a contradiction). Consequently $(3.25)^n$ (and also $(3.24)^n$) has a solution $u_n(t) \in C(N^+, \mathbf{R})$, $u_n(t) \in K$, with

$$u_n(t) \ge r\mu(t), \quad t \in N, \quad r < \|u_n\| \le R. \tag{3.38}$$

Essentially the same reasoning as before guarantees that there exists a subsequence Z^0 of Z^+, and a function $u \in C(N^+, \mathbf{R})$ with $u_n(t)$ converging to $u(t)$ as $n \to \infty$ through Z^0. It is easy to show that $u(t) \in C(N^+, \mathbf{R})$ is a solution of (3.1) and $r < \|u\| \le R$.

Thus, the proof of Theorem 3.3 is complete. □

Remark 3.3. If in (H_7) we have $R < r$ then (3.1) has a solution $u(t) \in C(N^+, \mathbf{R})$ with $u > 0$ on N and $R \le \|u\| < r$. The argument is similar to that in Theorem 3.2 except here we use Remark 3.2.

Theorem 3.3. *Assume* (H_1)–(H_7) *hold. Then* (3.1) *has two solutions* $u_1, u_2 \in C(N^+, \mathbf{R})$ *with* $u_1 > 0$, $u_2 > 0$ *on* N *and* $0 < \|u_1\| < r < \|u_2\| \le R$.

Proof. The existence of u_1 follows from Theorem 3.1, and the existence of u_2 follows from Theorem 3.2.

Example 3.2. The singular boundary value problem

$$\begin{cases} \Delta\big(\phi(\Delta u(t-1))\big) + \sigma([u(t)]^{-\alpha} + [u(t)]^{\beta} + 1) = 0\,, \quad t \in N\,, \\ u(0) = 0\,, \quad u(T+1) = 0\,, \end{cases} \tag{3.39}$$

has two solutions $u_1, u_2 \in C(N^+, \mathbf{R})$ with $u_1 > 0$, $u_2 > 0$ on N and $\|u_1\| < 1 < \|u_2\|$. Here $\alpha > 0$, $\beta > p-1$, and

$$0 < \sigma < \frac{1}{3}\left(\frac{p}{b_1(p-1+\alpha)}\right)^{p-1},\ b_1 := \sum_{t=1}^{T} t^{\frac{1}{p-1}}\,.$$

To see this we will apply Theorem 3.3 with

$$q(s) = \sigma\,, \quad g(u) = g_1(u) = u^{-\alpha}\,, \quad h(u) = h_1(u) = u^{\beta} + 1\,.$$

Clearly (H_1)–(H_4), (H_6) hold. Also notice (see Example 3.1)

$$b_0 = \max_{t\in N}\left(\sigma^{\frac{1}{p-1}}\sum_{s=1}^{t}(t-s+1)^{\frac{1}{p-1}}\,, \quad \sigma^{\frac{1}{p-1}}\sum_{s=t}^{T}(s-t+1)^{\frac{1}{p-1}}\right) = \sigma^{\frac{1}{p-1}} b_1\,.$$

Consequently (H_5) holds(with $r = 1$), since

$$\begin{aligned} \frac{1}{\phi^{-1}(1+\frac{h(r)}{g(r)})}\int_0^r \frac{dy}{\phi^{-1}(g(y))} &= \frac{p-1}{p-1+\alpha}\phi^{-1}\left(\frac{r^{\alpha+p-1}}{1+r^{\alpha}+r^{\alpha+\beta}}\right) \\ &= (\frac{1}{3})^{\frac{1}{p-1}}\frac{p-1}{p-1+\alpha} > b_0\,. \end{aligned}$$

Finally notice that (since $\beta > p-1$)

$$\begin{aligned} &\lim_{R\to\infty}\frac{R}{\Phi^{-1}(R^{-\alpha}[1+(\frac{R}{T+1})^{\alpha+\beta}+(\frac{R}{T+1})^{\alpha}])} \\ &= \lim_{R\to\infty}\frac{R}{(R^{-\alpha}+(\frac{1}{T+1})^{\alpha+\beta}R^{\beta}+(\frac{1}{T+1})^{\alpha})^{\frac{1}{p-1}}} = 0\,, \end{aligned}$$

so there exists $R > 1$ with (H_7) holding. The result now follows from Theorem 3.3.

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