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CLASSIFICATION OF POSITIVE SOLUTIONS OF p-LAPLACE EQUATION WITH A GROWTH TERM

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Abstract. We give a structure result for the positive radial solutions of the following equation:

$$\Delta_p u + K(r)u|u|^{q-1} = 0$$

with some monotonicity assumptions on the positive function $K(r)$. Here $r = |x|$, $x \in \mathbb{R}^n$; we consider the case when $n > p > 1$, and $q > p_* = \frac{n(p-1)}{n-p}$.

We continue the discussion started by Kawano et al. in [11], refining the estimates on the asymptotic behavior of Ground States with slow decay and we state the existence of S.G.S., giving also for them estimates on the asymptotic behavior, both as $r \to 0$ and as $r \to \infty$.

We make use of a Emden-Fowler transform which allow us to give a geometrical interpretation to the functions used in [11] and related to the Pohozaev identity. Moreover we manage to use techniques taken from dynamical systems theory, in particular the ones developed in [10] for the problems obtained by substituting the ordinary Laplacian Δ for the p-Laplacian Δ_p in the preceding equations.

1. Introduction

Let $\Delta_p u = \operatorname{div}(|Du|^{p-2}Du)$, $p > 1$, denote the degenerate p-Laplace operator. The aim of this paper is to study the existence and the asymptotic behavior of positive radial solutions of the following quasilinear elliptic equation:

$$\Delta_p u + K(|x|)u|u|^{q-1} = 0 \tag{1.1}$$

where $K(|x|)$ is a radial function which we assume to be as regular as needed, usually C^2. In particular we focus our attention on the existence of radial Ground States (G.S.), Singular Ground States (S.G.S) and crossing solutions in a ball. By G.S. we mean a positive solution $u(x)$ defined in the whole space $\mathbb{R}^n$ such that $\lim_{|x|\to\infty} u(|x|) = 0$, and by a S.G.S. we mean a G.S. which is not defined at the

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origin and satisfies $\lim_{|x|\to 0} u(|x|) = +\infty$. By crossing solution we mean a solution $u(x)$ such that $u(x) > 0$ if $|x| < R$ and $u(x) = 0$ if $|x| = R$, therefore such a solution can also be regarded as a solution of the Dirichlet problem in a ball of radius R.
We will use the term "singular solution" to refer only to a solution $v(x)$ such that $\lim_{|x|\to 0} v(|x|) = +\infty$.
We are only able to deal with radial solutions, so we shall consider the following O.D.E.

$$
\begin{aligned}
(u'|u'|^{p-2})' + \frac{n-1}{r}u'|u'|^{p-2} + K(r)u|u|^{q-1} = 0 \\
u(0) = A > 0 \quad u'(0) = 0
\end{aligned}
\tag{1.2}
$$

where $|x| = r$, n is the dimension of the space and " $'$ " denotes derivation with respect to r. A general assumption in this paper is that $n > p$ and $q > p - 1$. We will denote with $p^* = \frac{np}{n-p} - 1$ the Sobolev critical exponent and with $p_* = \frac{n(p-1)}{n-p}$ another constant which plays a critical role in this context. We will usually assume $q > p_*$.

In recent years this equation has been studied by many authors: the situation for the autonomous case is almost completely understood, see in particular the survey given in [6]. The purpose of this paper is to refine the results obtained by Kawano et al. in [11]. We combine some elements of that approach with others taken from dynamical systems theory, in particular the techniques developed by Johnson, Battelli, Pan and Yi in [1] and in [10], for the corresponding problem with the usual Laplacian. We make use of a new transform of Fowler type, introduced in [5], which enables us to give a geometrical interpretation, from the point of view of dynamical systems, to the function $J(r)$ used in [11], closely related to the Pohozaev identity. Exploiting these techniques we are able to refine the estimates on the asymptotic behavior of the solutions and to state the existence of S.G.S. Furthermore we give a non existence result which allows us to classify all the possible S.G.S. In particular we complete the analysis of the problem of the existence of S.G.S when $q > p_*$ for the autonomous equation, presented in [6]. We are able to show that, under rather general assumptions, we can only have two kind of behavior as $r \to 0$ for positive solutions $u(r)$: the regular, that is $0 < u(0) < \infty$, and the singular, that is $u(r) \sim r^{\frac{-p}{q-p+1}}$, if for example we assume $0 < K(0) < \infty$. Moreover we have only two kinds of behavior as $r \to \infty$ for positive solutions: fast decay, that is always $u(r) \sim r^{-\frac{n-p}{p-1}}$, and slow decay, that is $u(r) \sim r^{\frac{-p}{q-p+1}}$ if $K(r)$ is strictly positive and bounded for r large.

With the notation $u(r) \sim r^{-\alpha}$ as $r \to c$ we mean that both the limits

$$\limsup_{r\to c} u(r)r^{\alpha} \quad \text{and} \quad \liminf_{r\to c} u(r)r^{\alpha}$$

are positive and finite.

We recall now some classical definitions which will be useful in the following sections. Given a system of the form

$$\dot{x} = f(x, t)$$

and a solution $x(t)$, the α-limit set of $x(t)$ is the set

$$A = \left\{P : \exists t_n \to -\infty \quad \text{such that } \lim_{n\to\infty} x(t_n) = P\right\},$$

while the ω-limit set is the set

$$W = \left\{P : \exists t_n \to +\infty \quad \text{such that } \lim_{n\to\infty} x(t_n) = P\right\}.$$

One can show that, if $x(t)$ is bounded on $\mathbb{R}$, then those sets are compact. Moreover if the system is autonomous these sets are invariant for the flow generated by the system. If the system is non-autonomous they are no longer invariant; however we will see that they are still useful for the present purposes.

2. Autonomous problem

We begin by introducing a transform of Fowler type which establishes a bijective relationship between the solutions of (1.2) and the ones of a two-dimensional dynamical system, thus allowing us to reach a geometrical understanding of the behavior of the solutions. In particular we define

$$\begin{aligned} x_l = u(r)r^{\alpha_l} \qquad y_l = u'(r)|u'(r)|^{p-2}r^{\beta_l} \qquad r = e^t \\ \delta_l = \alpha_l(l-q) \qquad \phi(t) = K(e^t) = K(r) \qquad h_l(t) = \phi(t)e^{\delta t} \end{aligned} \tag{2.1}$$

where

$$\alpha_l = \frac{p}{l-p+1}, \qquad \beta_l = \frac{pl}{l-p+1} - 1, \qquad \gamma_l = \beta_l - (n-1), \qquad p \neq l-1$$

so that equation (1.2) can be written as the following dynamical system

$$\begin{pmatrix} \dot{x}_l \\ \dot{y}_l \end{pmatrix} = \begin{pmatrix} \alpha_l & 0 \\ 0 & \gamma_l \end{pmatrix} \begin{pmatrix} x_l \\ y_l \end{pmatrix} + \begin{pmatrix} y_l|y_l|^{\frac{2-p}{p-1}} \\ -h_l(t)x|x|^{q-1} \end{pmatrix} \tag{2.2}$$

where " $\cdot$ " denotes derivation with respect to t. We will often set $l = p^*$ and in this case we will leave unsaid the subscript l. Sometimes it will be useful to set $l = q$ in order to have $h_q(t) = K(r)$. We point out that choosing $p = 2$ and $q = p^*$ our transformation coincides with the one used in [9].

2.1. Remark (Regularity Hypothesis)**.** It is important to observe that system (2.2) is C^1 if and only if $q \geq 1$ and $1 \leq p \leq 2$.

If this hypothesis is not satisfied the dynamical system is not even Lipschitz so that local uniqueness of the solutions near the x and y axis is not anymore ensured, thus our use of the term "dynamical system" is not quite rigorous.

2.2. Remark. Note that

$$\alpha_l + \gamma_l < 0 \iff l > p^* \quad \text{and} \quad \alpha_l + \gamma_l > 0 \iff l < p^*$$

and $l = p^*$ gives $\alpha_l + \gamma_l = 0$. Note also that if $l > p_*$ we have $0 < \alpha_l < \frac{n-p}{p-1}$. Observe that δ_l increases when l increases and $\lim_{l\to\infty} \delta_l = p$.

We will see that for $q > p^*$ we will obtain G.S. with decay rate $\sim r^{-\alpha_q}$ therefore, for any given ϵ, we can choose q large enough in order to have G.S. with decay rate slower than $r^{-\epsilon}$.
Moreover it will be possible to control the asymptotic behavior of functions $K(r)$ that tend to 0 as $r \to \infty$, by choosing the correct value of l. But if we are dealing with a $K(r) = o(r^p)$ we will always obtain $h_l(t) \to 0$ as $t \to \infty$. In this case positive solution cannot tend to 0.

2.3. Remark. The solutions $u(r)$ of equation (1.2) corresponds to the trajectories $(x(t), y(t))$ of system (2.2) having the origin as α-limit point. Moreover if $u(r) > 0$ then $x(t) > 0$ and $u'(r) > 0$ implies $y(t) > 0$.

2.4. Remark. It is well known that $u'(r) < 0$ for $r > 0$ small, thus the trajectories $(x(t), y(t))$ corresponding to $u(r)$ lie in the 4^{th} quadrant as $t \to -\infty$.

2.5. Remark. Crossing solutions $u(r)$ correspond to trajectories of system (2.2) departing from the origin and getting into the 4^{th} quadrant, until they cross the y negative semiaxis.

2.6. Observation. Consider system (2.2), when $h_l(t) \equiv h > 0$ is a constant. Then we have exactly 3 critical points: the origin $O \equiv (0,0)$, $P \equiv (P_x, P_y)$ and $-P$ where $P_x > 0$ and $P_y < 0$.

Assume that the limit $\lim_{t\to-\infty} h_l(t)$ is finite and positive, then the same statement holds for system (3.3) with $\xi > 0$, which will be introduced later on. Analogously we have exactly three critical points also for system (3.3) with $\xi < 0$, when the limit $\lim_{t\to\infty} h_l(t)$ is finite and positive.

From now on we restrict our attention to the halfplane defined by $x \geq 0$, since trajectories corresponding to positive $u(r)$ have to stay there.

We define now two functions which were introduced in [11], which are closely related to the Pohozaev identity. Let $u(r)$ be a solution of (1.2), then:

$$P_u(r) = \frac{n-p}{p} r^{n-1} u(r) u'(r) |u'(r)|^{p-2} + r^n \frac{p-1}{p} |u'(r)|^p + \frac{K(r)}{p} r^n \frac{|u(s)|^{q+1}}{q+1}\,;$$

here P is defined in the domain of definition of u, and

$$J(r) := \int_0^r \frac{dK(s)}{ds} \frac{s^{\alpha(q+1)}}{q+1}\, ds = \int_{-\infty}^t \frac{dh(s)}{ds} \frac{e^{\alpha(q+1)s}}{q+1}\, ds\,.$$

The function $J(r)$ is the one which plays a discriminating role in the analysis derived in [11], even if we have rewritten it in a form which seems to us to be simpler. Now we repeat one of the key observation of [11]: observe that for any given $u(r)$, regular solution of (1.2), we have

$$P_u(r) = J(r)|u(r)|^{q+1} - \int_0^r J(r)|u(s)|^q u'(s)\, ds\,. \tag{2.3}$$

moreover note that, for a singular solution $v(r)$, we have

$$P_v(r) = J(r)|v(r)|^{q+1} - \int_0^r J(r)|v(s)|^q v'(s)\, ds - \lim_{r\to 0} P_v(r)\,. \tag{2.4}$$

In our analysis we will also need the following function similar to $J(r)$

$$G(r) := \int_r^\infty \frac{dK(s)}{ds} \frac{s^{\alpha(q+1)}}{q+1}\, ds = \int_t^\infty \frac{dh(s)}{ds} \frac{e^{\alpha(q+1)s}}{q+1}\, ds$$

especially to analyze positive solutions with fast decay. In fact we have:

$$P_v(r) = -\left(G(r)|v(r)|^{q+1} - \int_r^\infty G(s)|v(s)|^q v'(s)\, ds \right) + \lim_{r\to\infty} P_v(r)\,. \tag{2.5}$$

2.7. Remark. Note that, if $\dot{h}(t) \geq 0$ for any t and the inequality is strict for some t, we have that both $J(r)$ and $G(r)$ are positive for any r, while, if $\dot{h}(t) \leq 0$ for any t and the inequality is strict for some t, we have that $J(r)$ and $G(r)$ are negative.

2.8. Remark. Consider a solution $u(r)$, recalling Remark 2.4 we have that if $J(r) < 0$ for any r we have $P_u(r) < 0$ for any r, while if $J(r) > 0$ we have $P_u(r) > 0$.

We introduce a function which will play a crucial role in the following analysis. Let us consider system (2.2); we define

$$\begin{aligned} H_l(x_l(t), y_l(t), t) : &= P_u(e^t)e^{(\alpha_l+\gamma_l)t} \\ &= \frac{n-p}{p} x_l y_l + \frac{p-1}{p}|y_l|^{\frac{p}{p-1}} + h_l(t)\frac{|x_l|^{q+1}}{q+1}\,. \end{aligned} \tag{2.6}$$

Observe that if we set $l = p^*$ we obtain $\alpha_l + \gamma_l = 0$ and $H(t)$ becomes an energy function, in fact differentiating we get:

$$\frac{d}{dt} H_{p^*}(x_{p^*}(t), y_{p^*}(t), t) = \frac{d}{dt} h_{p^*}(t) \frac{|x_{p^*}|^{q+1}}{q+1}\,, \tag{2.7}$$

thus the monotonicity of $h(t)$ implies the monotonicity of $H(t)$. Now we give a lemma that describes the level sets of this function.

2.9. Lemma. *Consider any T such that $0 < h_l(T) < \infty$. Then the equation $H_l(x_l, y_l, T) = 0$, restricted to the halfplane $x \geq 0$, defines a closed bounded curve containing the origin and which is contained in the closed 4^{th} quadrant. The equation $H_l(x_l, y_l, T) = -b < 0$, where $H_l(P(T), T) = -b^*(T) < -b < 0$, defines a closed bounded curve in the halfplane $x \geq 0$. Finally, the equation $H(x_l, y_l, T) = b > 0$ defines a closed bounded curve in the whole plane which contains the origin in its interior.*

Now we give some information about the asymptotic behavior of the solutions, both as $r \to 0$ and as $r \to \infty$.

2.10. Lemma. *Consider a solution $v(r)$ of Eq. (1.2) defined and positive in a right neighborhood of $r = 0$. Fix l in order to have that $\lim_{t\to-\infty} h_l(t) < \infty$. Suppose that the corresponding trajectory of (2.2) admits the origin as α-limit point, then we have that $v(0) < \infty$. Consider a solution $u(r)$ of Eq. (1.2) defined and positive in a right neighborhood of $r = \infty$. Fix l in order to have that*

$\lim_{t\to-\infty} h_l(t) < \infty$. *Suppose that the corresponding trajectory of* (2.2) *admits the origin as ω-limit point, then we have that* $u(r) \sim r^{-\frac{n-p}{p-1}}$ *as* $r \to \infty$.

Proof. The proof of Lemma 2.9 is completely analogous to the one of Lemma 2.6 in [5]. The proof of Lemma 2.10 can be easily obtained reasoning as in the proofs of Observation 5.4 and Observation 5.5 in [5], for the equation with two growth term. □

We will see that the solutions of Eq. (1.2) can exhibit only two kind of behavior as $r \to 0$, that is the regular, just described, and the singular. They correspond respectively to the case in which the trajectory has the origin as α-limit point or when it is bounded and bounded away from the x axis, as $t \to \infty$. Analogously we also have only two kinds of decay as $r \to \infty$: the slow one, which depends on the asymptotic behavior of $K(r)$ and on q, and the fast one that is always $\sim r^{-\frac{n-p}{p-1}}$. Once again they correspond to trajectories bounded and bounded away form the x axis or converging to the origin.

2.11. Proposition. *Consider Eq.* (1.2) *and assume* $K(r) \equiv \text{const.} > 0$ *and* $q = p^*$. *Consider the corresponding autonomous system of the form* (2.2) *with* $l = q = p^*$. *Then the following holds.*

- A *All the trajectories corresponding to positive values* $H(x,y) = b > 0$ *represent periodic trajectories which cross the axis. They correspond to singular solutions* $u(r)$ *of* (1.2) *with infinitely many positive maxima and negative minima; moreover there exists* $a > 0$ *such that* $-ar^{-\alpha} \le u(r) \le ar^{-\alpha} \quad \forall r > 0$.
- B *The trajectory corresponding to* $H(x,y) = 0$ *is homoclinic to the origin; this means that all the solutions* $u(r)$ *of* (1.2) *are monotone decreasing G.S., with decay rate* $\sim r^{-\frac{n-p}{p-1}}$ *at* ∞ *(fast decay).*
- C *All the trajectories corresponding to some negative value* $H(x_1,x_2) = -b > H(P)$ *represent periodic trajectories which belong to the* $x \ge 0$ *halfplane. They represent monotone decreasing S.G.S.* $u(r)$ *of Eq.* (1.2) *with rate of decay and growth* $\sim r^{-\frac{n-p}{p}}$ *respectively at* ∞ *and at* 0.
- D *For the value* $H = H(P)$ *we have one fixed point* P, *which corresponds to a monotone decreasing S.G.S of* (1.2) *of the form* $u(r) = P_x r^{-\frac{n-p}{p}}$ *where we recall that* P_x *depends only on the value of* K.

All the solutions $u(r)$ *regular at the origin are G.S with fast decay, therefore no crossing solutions can exist. Moreover no other S.G.S can exist but the ones described.*

2.12. Remark. The preceding proposition can be trivially generalized to the case in which $q \ne p^*$, but $h(t) \equiv \text{const.} > 0$, that is $K(r) = Ar^{-\frac{n-p}{p}(p^*-q)}$ where $A > 0$ is a constant.

Proof. To prove the claim is enough to observe that the system (2.2) corresponding to Eq. (1.2) is autonomous, with these assumptions, and admits H_{p^*} as a first integral; then using Lemma 2.9 and Lemma 2.10 we get the thesis. □

2.13. Remark. Assume that the regularity hypothesis is satisfied. Then all the solutions $u(r)$ corresponding to the homoclinic trajectory are such that $u(0) > 0$ and $u'(0) = 0$. Therefore no other solutions $u(r)$ positive in a right neighborhood of $r = 0$ can exist, but the ones described in the Proposition.

A priori we could find solution $u(r)$ of (1.2), corresponding to trajectories of (2.2) having the origin as α-limit point. In Observation 3.17 we show that, if $q < \frac{p}{p-1}$, this case can be excluded.

2.14. Observation. Suppose that the regularity hypothesis is satisfied and assume that system (2.2) is autonomous. Then it admits periodic solutions if and only if $l = p^*$. Moreover if $l \neq p^*$ and the regularity hypothesis is not satisfied, the periodic trajectories if they exist, must have the origin in their interior or cross it.

Proof. This fact easily follows applying the Poincarè-Bendixson criterion that affirms that a necessary condition for the existence of periodic solutions in an autonomous system of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} \tag{2.8}$$

is that

$$\frac{df_1(x,y)}{dx} + \frac{df_2(x,y)}{dy} = \alpha_l + \gamma_l = 0\,. \tag{2.9}$$

If we remove the regularity hypothesis we can still apply the criterion to each open quadrant. Observe that the flow is always rotating clockwise on the axes and remember that on the axes and in the origin we lose local uniqueness of the solutions and conclude. $\square$

2.15. Remark. Observe that the homoclinic and the other trajectories of system (2.2) where $h_l(t) \equiv$ const. > 0, correspond to families of solutions, because the system is autonomous, so it is invariant for translation in time. To be more precise, if $u(r)$ is a solution (regular or singular), $u_s(r) = u(\frac{r}{s})s^{-\frac{n-p}{p}}$ is a solution as well. Therefore if we call $u_A(r)$ the solution such that $u(0) = A$ and $u'(0) = 0$, then $u_A(r) = Au_1(A^{\frac{p}{n-p}}r)$, where $u_1(r)$ is $u_B(r)$ where $B = 1$.

We recall that for the autonomous equation (1.2)) with $q = p^*$ and $K(r) \equiv 1$ is already known the exact expression

$$u_A(r) = A\left[1 + D\left[(A^{\frac{p}{n-p}}r)^{\frac{p}{p-1}}\right]\right]^{-\frac{n-p}{p}}$$

where $D = (p-1)(n-p)n^{\frac{1}{p-1}}$ is a constant, see [6].

3. Non-Autonomous problem

We begin with a lemma concerning the phase portrait of the non autonomous system (2.2).

3.1. Lemma. *Consider any trajectory of the non autonomous system* (2.2) *passing through the* 1^{st} *quadrant, which has not the origin as* α*-limit point. Then it comes from the* 2^{nd} *quadrant and goes into the* 4^{th} *quadrant after finite time.*

Proof. Set $l = p^*$ and consider system (2.2). Consider a trajectory $(\breve{x}(t), \breve{y}(t))$ belonging to the 1^{st} quadrant for a certain $t = \breve{t}$; assume that it is bounded away from the origin for $t \to -\infty$. We claim that there exists a $t_1 < \breve{t}$ for which $(\breve{x}(t), \breve{y}(t))$ crosses the y positive semiaxis. Suppose by contradiction that $\breve{x}(t) > 0$ for any $t > T$ where $T \geq -\infty$ is the inf of the maximal interval of continuation of $(\breve{x}(t), \breve{y}(t))$. Suppose that $(\breve{x}(t), \breve{y}(t))$ is unbounded as $t \to T^-$, then $\lim_{t \to T^-} H(\breve{x}(t), \breve{y}(t), t) = +\infty$. We recall that

$$\frac{d}{dt} H(x(t), y(t), t) := \frac{d}{dt} h(t) \frac{|x|^{q+1}}{q+1}\,.$$

Therefore, recalling that $\breve{x}(t)$ is finite we conclude that $\frac{d}{dt} H(\breve{x}(t), \breve{y}(t), t) < \infty$ for any t finite. Therefore we have $T = -\infty$. Now recalling that $\frac{d}{dt}\breve{x}(t) > \epsilon > 0$ we conclude that $(\breve{x}(t), \breve{y}(t))$ crosses the y axis after finite time since the distance from the trajectory and the axis is finite.

Now we follow the trajectory forward in time. Suppose that it does not cross the x axis, then we have $\frac{d}{dt}\breve{x}(t) > \epsilon > 0$ for any $t > \breve{t}$ and for some $\epsilon > 0$. Assume that the sup of the maximal interval of continuation is $\breve{T}$. Suppose that $\breve{T} < \infty$, then there exist A such that $h(t) > A$ for any $t < \breve{T}$. We define the function $H_A(x(t), y(t))$ obtained setting $h(t) = A$ in $H(t)$:

$$H_A(x(t), y(t)) := \frac{n-p}{p} xy + \frac{p-1}{p} |y|^{\frac{p}{p-1}} + A|x|^{q+1}\,. \tag{3.1}$$

Differentiating we get

$$\frac{d}{dt} H_A(x(t), y(t)) = [A - h(t)] x|x|^{q-1} \dot{x}\,,$$

thus $H_A(t)$ is decreasing along $(\breve{x}(t), \breve{y}(t))$. Since the level sets of $H_A(t)$ are bounded, we have that $(\breve{x}(t), \breve{y}(t))$ is bounded, so it can be continued also for $t > \breve{T}$. Thus $\breve{T} = \infty$; now observing that $\frac{d}{dt}\breve{y}(t) < -\epsilon < 0$ for any t we have that $(\breve{x}(t), \breve{y}(t))$ must cross the x axis. $\square$

Now we need to introduce a new transform in order to deal with an autonomous system. Applying to Eq. (1.2) the change of variables (2.1) and setting $z_l = t$ we obtain the following system:

$$\begin{pmatrix} \dot{x}_l \\ \dot{y}_l \\ \dot{z}_l \end{pmatrix} = \begin{pmatrix} \alpha_l & 0 & 0 \\ 0 & \gamma_l & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_l \\ y_l \\ z_l \end{pmatrix} + \begin{pmatrix} \psi_{p^*}(y_l) \\ -h_l(z_l)\psi_q(x_l) \\ 1 \end{pmatrix} \tag{3.2}$$

where $\psi_m(s) = s|s|^{m-2}$. We will also consider the system obtained setting $z = e^{\xi t}$ in order to investigate the behavior as $t \to -\infty$, setting $\xi > 0$, and as $t \to \infty$, setting $\xi < 0$:

$$\begin{pmatrix} \dot{x}_l \\ \dot{y}_l \\ \dot{z}_l \end{pmatrix} = \begin{pmatrix} \alpha_l & 0 & 0 \\ 0 & \gamma_l & 0 \\ 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} x_l \\ y_l \\ z_l \end{pmatrix} + \begin{pmatrix} \psi_{p^*}(y_l) \\ -h_l(z_l)\psi_q(x_l) \\ 0 \end{pmatrix}. \tag{3.3}$$

Now we give the definitions of three sets of system (2.2) for a generic value of the parameter l:

$$U^+ := \{(x, y, z) \mid x \le 0,\ y \le 0 \quad \text{and} \quad \dot{x} > 0\}$$

$$U^- := \{(x, y, z) \mid x \le 0,\ y \le 0 \quad \text{and} \quad \dot{x} < 0\}$$

$$c := \{(x, y, z) \mid x \le 0,\ y \le 0 \quad \text{and} \quad \dot{x} = 0\}$$

$$S_l := \{(x_l, y_l, z_l) \mid x_l \le 0,\ y_l \le 0 \quad \text{and} \quad H_l(x_l, y_l, z_l) = 0\}.$$

3.2. Remark. We will sometimes focus our attention on the set $S_l \bigcap \{z_l = 0\}$ of system (3.3). Note that $P_u(r) < 0$ for any $r \ge 0$ implies that, for the corresponding trajectory, we have $H_{p^*}(x_{p^*}(t), y_{p^*}(t), t) < 0$ for any t and also as $t \to \pm\infty$. Note also that $P_u(r) < 0$ implies $H_l(x_l(t), y_l(t), t) < 0$ for any t finite, but letting $t \to \pm\infty$ we can only say that $H_l(t) \le 0$.

3.3. Theorem. *Assume that $J(r) \le 0$ for any $r > 0$, but $J(r) \not\equiv 0$ and that*

$$0 < \liminf_{t\to\infty} h(t) \le \limsup_{t\to\infty} h(t) < \infty.$$

Then all the solutions $u(r)$ of Eq. (1.2) are G.S. with decay of order $\sim r^{-\alpha_{p^*}}$. Moreover assume that $h(t)$ is monotone for t large and that $0 < \lim_{t\to\infty} h(t) = A < \infty$. Then for each G.S. $u(r)$ there exists a S.G.S. $v(r)$ of the frozen Eq. (1.2) where $K(r) = Ar^{-\alpha_{p^*}(p^*-q)}$ such that

$$lim_{r\to\infty}(u(r) - v(r))r^{\alpha_{p^*}} = 0.$$

Proof. We recall that the S.G.S. $v(r)$ have already been described in Proposition 2.11. Set $l = p^*$; consider any solution $u(r)$, then for the corresponding trajectory we have $H(x(t), y(t), t) < 0$, see Remark 2.8, so it lies inside S_{p^*}. Since S_{p^*} is a surface homeomorphic to a cylinder and bounded in the (x, y) variables, we have that x_{p^*} is bounded and positive. Thus the corresponding $u(r)$ is a G.S. with slow decay, that is $u(r) \sim r^{-\alpha}$.

Let us consider the trajectory $(x(t), y(t), z(t))$ of the system (3.3), corresponding to $u(r)$. If we assume that $h(t)$ is monotone for t large, we can conclude that $H(x(t), y(t), z(t))$ is monotone. Assume at first that $\frac{dh}{dz} = 0$, so that local uniqueness of the solution is ensured. Observe that the system (3.3) with $\xi < 0$ admits a critical point $P_\infty = (x_P, y_P, 0)$ where $y_P < 0 < x_P$. Note that the value of H is negative and bounded below by the value of the function at P_∞. Thus the limit for $t \to \infty$ of $H(x(t), y(t), t)$ exists and is negative. Now observe that the Ω-limit set of the trajectory has to belong to the $z = 0$ plane. Note that, if we restrict our attention to this plane, we obtain a system analogous to (2.2) where $h(t) \equiv \lim_{t\to\infty} h(t)$. Recalling that, from Proposition 2.11, we know that each negative value of H characterizes a closed trajectory of (2.2), we have the thesis.

If the hypothesis is not satisfied we have that the system is only continuous in the plane $z = 0$. So, in principle, we could lose local uniqueness of the solutions. Note that $H(x(t), y(t), z(t))$ is monotone along the solutions, so we can assume, for example, that it is increasing. Consider a trajectory $(x(t_n), y(t_n), z(t_n))$ having $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$ in its ω-limit set. Since H is monotone and continuous we have $H(x_1, y_1, 0) = H(x_2, y_2, 0)$, so the thesis is proved. $\square$

The existence of the G.S. was already proved in [11], using different arguments; anyway our approach allow us to refine the estimate on the asymptotic behavior.

3.4. Remark. To satisfy the hypothesis of the theorem it is enough to take $h(t)$ monotone decreasing and strictly positive. For example, we can set $q = p^*$ and choose a function $K(r)$ which is strictly positive and monotone decreasing.

Now we want to show which are the possible asymptotic behaviors of positive solutions as $r \to 0$ and as $r \to \infty$. We need to introduce the following function:

$$j_l(r) = K(r) r^{\frac{p}{l-p+1}(l-q)} = h_l(t) \quad \text{where} \quad t = log(r).$$

This function is in fact $K(r)$ multiplied by some power of r.

3.5. Proposition. *Consider a solution $v(r)$ of (1.2), defined in a neighborhood of $r = \infty$. Assume that $j_{p^*}(r)$ is monotone for r large.*

- *Assume that there exists $l > p_*$ such that $0 < \lim_{r\to\infty} j_l(r) < \infty$ and suppose that $\lim_{r\to\infty} |\frac{dj_l}{dr}(r) r^{1+\delta}| = 0$, for some $\delta > 0$ small. Then*

$$v(r) \sim r^{-\frac{p}{l-p+1}} \quad or \quad v(r) \sim r^{-\frac{n-p}{p-1}},$$

that is $v(r)$ has slow decay or fast decay, respectively.
- *Assume that there exist $l_2 \geq l_1 > p_*$ and $\delta > 0$ such that*

$$\limsup_{r\to\infty} j_{l_1}(r) < \infty, \qquad \liminf_{r\to\infty} j_{l_2}(r) > 0, \quad and \quad \lim_{r\to\infty} \frac{dj_{l_1}}{dr}(r) r^{1+\delta} = 0.$$

Then, for any $\epsilon > 0$ we have

$$\frac{1}{\epsilon} r^{-\epsilon-\frac{p}{l_1-p+1}} < v(r) \leq C r^{-\frac{p}{l_2-p+1}} \quad (slow\ decay),$$

where $C > 0$ is a given positive constant, or

$$v(r) \sim r^{-\frac{n-p}{p-1}} \quad (fast\ decay).$$

Analogously consider a solution $v(r)$ of (1.2), defined in a right neighborhood of $r = 0$, and assume that $j_{p^*}(r)$ is monotone for $t \to 0$. Then $v(r)$ can have only two kind of behavior as $r \to 0$: the regular behavior, that is $0 < v(0) < \infty$, and the singular behavior.

Assume that there exists $l > p_*$ and $\delta > 0$ such that

$$0 < \lim_{r\to 0} j_l(r) < \infty \quad and \quad \lim_{r\to 0} \frac{dj_l}{dr}(r) r^{1-\delta} = 0.$$

Then the singular behavior is $v(r) \sim r^{-\frac{p}{l-p+1}}$.

Assume that there exist $l_2 \geq l_1 > p_*$ and $\delta > 0$ such that

$$\limsup_{r\to 0} j_{l_2}(r) < \infty \quad \textit{and} \quad \liminf_{r\to 0} j_{l_1}(r) > 0\,, \quad \textit{and} \quad \lim_{r\to 0}\left|\frac{dj_{l_2}}{dr}(r)r^{1-\delta}\right| = 0\,;$$

then the singular behavior is $\frac{1}{\epsilon}r^{\epsilon-\frac{p}{l_2-p+1}} < v(r) \leq Dr^{-\frac{p}{l_1-p+1}}$.

Proof. We begin with the first claim. Consider system (3.3) with $\xi < 0$. Observe that, if the system is Lipschitz, the Ω-limit set of any bounded trajectory must belong to the $z = 0$ plane. The dynamics in this plane is that of the autonomous system (2.2) where $h_l(t) \equiv h_l(z)\lfloor_{z=0}$.

Assume at first that there exists l such that $\lim_{t\to\infty} h_l(t)$ exists and $0 < \lim_{t\to\infty} h_l(t) < \infty$. Then system (3.3), with this choice for l, admits exactly three critical points which are the origin, $P = (x_P, y_P, 0)$ and $-P$. Observe that

$$\lim_{z\to 0}\frac{dh_l(z)}{dz} = \lim_{z\to 0}\frac{dj_l(r)}{dr}\frac{dr}{dz} = \lim_{r\to\infty}\frac{1}{\xi}\frac{d_l(r)}{dr}r^{1-\xi} = 0\,,$$

if we choose $-\xi < \delta$. Therefore, in the subset where $x > 0$ and $y < 0$, the system is Lipschitz.

We have already described the case in which $l = p^*$, in the preceding theorem, so we assume $l \neq p^*$. We recall that, for any trajectory defined in a neighborhood of $t = \infty$, we have that there exists the limit $\lim_{t\to\infty} H_{p^*}(x_{p^*}(t), y_{p^*}(t), t)$. According to Observation 2.14 we cannot have periodic trajectories in the $x \geq 0$ subset. Thus bounded trajectories corresponding to positive $u(r)$, can only have the origin or P as Ω-limit set. Now recalling Lemma 2.10 the corresponding $u(r)$ can only have fast decay or slow decay, respectively.

We examine now the general case: consider at first a trajectory $(\bar{x}(t), \bar{y}(t), \bar{z}(t))$ of system (3.3) with $l = p^*$, such that $\lim_{t\to\infty} H(\bar{x}(t), \bar{y}(t), \bar{z}(t)) \leq 0$. Let us set now $l = l_2$ and consider system (3.3). Observe that the set $S_{l_2} \bigcap \{z_{l_2} = 0\}$ is bounded for any $M > 0$ and call D_{l_2} its interior. Note that the ω-limit set of $(\bar{x}_{l_2}(t), \bar{y}_{l_2}(t), \bar{z}_{l_2}(t))$ belongs to $\overline{D_{l_2}}$. Therefore for the corresponding $v(r)$ we have $v(r) \leq Dr^{\frac{-p}{l_2-p+1}}$ for some given $D > 0$. Set now $l = l_1 - \epsilon = l_0$: observe that $\lim_{t\to\infty} h_{l_0}(t) = 0$, thus the only critical point of system (3.3) is the origin.

Once more the hypothesis on $\frac{dj_l(r)}{dr}$ ensure that $\frac{dh_l(z)}{dz}\lfloor_{z=0}= 0$. Thus system (3.3) restricted to $x > 0$ and $y < 0$ is Lipschitz. If $(x_{l_0}(t), y_{l_0}(t), z_{l_0}(t))$ converges to the origin as $t \to \infty$, it must correspond to a solution $u(r)$ with fast decay, see Lemma 3.5. Otherwise it is unbounded, therefore, if it has slow decay, we have that, for any $\epsilon > 0$, $u(r) > \frac{1}{\epsilon}r^{-\epsilon-\frac{p}{l_1-p+1}}$. Thus we can have solutions with fast decay, corresponding to trajectory converging to the origin, and with slow decay, which are the ones described in the thesis. Note that for the trajectory described we have $\lim_{t\to\infty} H_l(x_l(t), y_l(t), t) \leq 0$.

Now we claim that any trajectory $(\bar{x}_{p^*}(t), \bar{y}_{p^*}(t), \bar{z}_{p^*}(t))$ such that

$$\lim_{t\to\infty} H_{p^*}(\bar{x}_{p^*}(t), \bar{y}_{p^*}(t), t) > 0\,,$$

has to get into the set $x < 0$ in finite time. Note that this limit exists because of the assumptions regarding the monotonicity of $h(t)$ for t large. We recall that we are considering trajectories which can be continued in the future for any t. Suppose by contradiction that $\bar{x}_{p^*}(t) > 0$ for any t; first of all note that there exists T such that $H_{p^*}(\bar{x}_{p^*}(t), \bar{y}_{p^*}(t), t) > 0$, for any $t > T$.

Consider system 2.2: since $H_{p^*}(\bar{x}_{p^*}(t), \bar{y}_{p^*}(t), t) > 0$, when the trajectory is in U^+ it is bounded away from the isocline $\dot{y} = 0$, while in U^- it is bounded away from the isocline $\dot{x} = 0$. If it is in U^+ for some t it will reach the isocline $\dot{x} = 0$ and get into U^- in finite time, since $\dot{y} < -\epsilon < 0$ for some $\epsilon > 0$. Analogously, if it is in U^-, it will reach the y axis in finite time, since $\dot{\bar{x}} < -\epsilon < 0$. This proves the claim.

We have already examined bounded trajectories: consider now a trajectory $(\hat{x}_{l_2}(t), \hat{y}_{l_2}(t), \hat{z}_{l_2}(t))$ that is unbounded as $t \to \infty$. Then we have $\lim_{t\to\infty} H_{l_2}(\hat{x}_{l_2}(t), \hat{y}_{l_2}(t), t) = \infty$; thus there exist T such that $H_{l_1}(\hat{x}_{l_2}(t), \hat{y}_{l_2}(t), t)$, and hence $H(\hat{x}(t), \hat{y}(t), t)$, are positive for any $t > T$. Therefore there exist a $T_1 > T$ such that $\hat{x}(T_1) < 0$, thus it cannot represent a positive solution $u(r)$. Reasoning in the same way we can conclude that, if $\lim_{t\to-\infty} H_{p^*}(x_{p^*}(t), y_{p^*}(t), t) > 0$, then the trajectory of (2.2) must cross the positive y semiaxis, thus it cannot represent a positive solution $u(r)$.

The proof of the claim regarding the asymptotic behavior of solutions as $r \to 0$ is completely analogous, so it will be skipped. □

Now we give a corollary to make clearer which could be the applications of the theorem. In particular we want to emphasize that, if $K(r)$ is uniformly positive and bounded, then we can set $\bar{l} = s = q$ in the theorem.

3.6. Corollary. *Assume that $K(r)$ is strictly positive and bounded and that it is monotone as $r \to 0$ and as $r \to \infty$. Moreover assume that there exists $\delta > 0$ small, so that $\lim_{r\to 0} K'(r) r^{1-\delta} = 0 = \lim_{r\to\infty} K'(r) r^{1+\delta}$ and consider a solution $u(r)$ defined and positive for any $r > 0$. Then as $r \to 0$ we have*

$$u(r) < \infty \quad (\textit{regular behavior}) \quad u(r) \sim r^{\frac{-p}{q-p+1}} \quad (\textit{singular behavior}),$$

while as $r \to \infty$ we have

$$u(r) \sim r^{-\frac{n-p}{p-1}} \quad (\textit{fast decay}) \quad u(r) \sim r^{\frac{-p}{q-p+1}} \quad (\textit{slow decay}).$$

3.7. Remark. Note that we can drop the technical assumption on $K'(r)$ (and on $j_l'(r)$) of Proposition 3.5, here and in Theorems 3.8 and 3.10, but we loose something on the precision of the estimate on the asymptotic behavior. To be more precise we would have

$cr^{-\frac{n-p}{p}} < u(r) < cr^{\frac{-p}{s_2-p+1}}$ as $r \to 0$ for singular solutions and
$cr^{-\frac{n-p}{p}} < u(r) < cr^{\frac{-p}{l_1-p+1}}$ as $r \to \infty$ for slow decaying solutions.

We are ready now to state one of the main theorem of the paper.

3.8. Theorem. *Assume that $J(r) \le 0$ for any $r > 0$, but $J(r) \not\equiv 0$, then all the solutions $u(r)$ of Eq. (1.2) can be continued for any $r > 0$ and are always positive.*

A_1 *Moreover assume that there exist $\bar{l}$ and $\delta > 0$ such that*

$$0 < \lim_{r\to\infty} j_{\bar{l}}(r) < \infty \quad \textit{and} \quad \lim_{r\to\infty} \left| \frac{dj_{\bar{l}}}{dr}(r) r^{1+\delta} \right| = 0 \,.$$

Then all the regular solutions $u(r)$ of Eq. (1.2) are G.S. with decay rate $\sim r^{\frac{-p}{l-p+1}}$ as $r \to \infty$ (slow decay).

A_2 *Assume that there exist $l_1 \geq l_2 \geq p_*$ and $\delta > 0$ such that*

$$\limsup_{r\to\infty} j_{l_1}(r) < \infty, \qquad \liminf_{r\to\infty} j_{l_2}(r) > 0 \quad \textit{and} \quad \lim_{r\to\infty} \frac{dj_{l_1}}{dr}(r) r^{1+\delta} = 0 .$$

Then all regular solutions $u(r)$ of Eq. (1.2) are G.S. such that for any given $\epsilon > 0$ we have

$$\frac{1}{\epsilon} r^{-\epsilon - \frac{p}{l_1-p+1}} < u(r) \leq C r^{-\frac{p}{l_2-p+1}} \quad \textit{(slow decay)},$$

where $C > 0$ is a given positive constant.

Assume that $G(r) \leq 0$ for any $r > 0$ and that $G(r) \not\equiv 0$.

B_1 *Assume that there exist s and $\delta > 0$ such that*

$$\lim_{r\to 0} j_s(r) = D > 0 \quad \textit{and} \quad \lim_{r\to 0} \frac{dj_s}{dr}(r) r^{1-\delta} = 0 .$$

Then there exists at least one S.G.S. $v(r)$ with slow decay, that is $v(r) \sim r^{\frac{-p}{s-p+1}}$ as $r \to 0$ and has the same rate of decay as the G.S., for r large. If $s > p^*$ this is the only S.G.S. admissible, while if $s = p^*$ we could also have other S.G.S., with the same behavior as the one described, both as $r \to 0$ and as $r \to \infty$.

B_2 *Assume that there exist $s_2 \geq s_1 > p_*$ and $\delta > 0$ such that*

$$0 < \liminf_{r\to 0} j_{s_2}(r) \leq \limsup_{r\to 0} j_{s_1}(r) < \infty \quad \textit{and} \quad \lim_{r\to 0} \frac{dj_{s_2}}{dr}(r) r^{1-\delta} = 0 .$$

Assume that there exists a S.G.S. $v(r)$; then it must have the same decay of the G.S. as $r \to \infty$ and for any $\epsilon > 0$ we have $\frac{1}{\epsilon} r^{\epsilon - \frac{p}{l_2-p+1}} < v(r) \leq C r^{-\frac{p}{l_1-p+1}}$ as $r \to 0$, where $C > 0$ is a given constant.

C *Assume that $n > p$ and $K(r) = o(r^{-p})$, then all the solutions $u(r)$ of (1.2), can be continued for any r and are always positive and have positive finite limit. No S.G.S can exist: if hypothesis B_1 is satisfied then there exists a singular solution which behaves like $\sim r^{\frac{-p}{s-p+1}}$ as $r \to 0$, is monotone decreasing, is well defined and positive for any $r > 0$ and has positive finite limit.*

Proof. Set $l = p^*$ in (2.1); we recall that the trajectory $(x(t), y(t))$ of (2.2) corresponding to a regular solution $u(r)$ of (1.2) have the origin as α-limit point. Observe that, due to the assumption on $J(r)$ we have $H_{p^*}(x_{p^*}(t), y_{p^*}(t), t) < 0$ for any t and also as $t \to \infty$, see (2.8). Thus $(x_{p^*}(t), y_{p^*}(t))$ cannot converge to the origin, hence $u(r)$ cannot have fast decay.

Let us assume that $\bar{l} > p^*$ since the case $\bar{l} = p^*$ has already been described in Theorem 3.3. Consider system (3.3) with $\xi < 0$ and $l = \bar{l}$. Note that the level sets of (2.2) defined by $H_{\bar{l}}(x_{\bar{l}}, y_{\bar{l}}, t) < 0$ are bounded for any t, therefore we deduce the continuability of the trajectory. Observe that the system admits three critical points which are the origin, $P = (x_P, y_P, 0)$ and $-P$, where $y_P < 0 < x_P$.

From the assumption on $\frac{dj_{\bar{l}}}{dr}$ we know that the system, restricted to $x_{\bar{l}} > 0$ and $y_{\bar{l}} < 0$, is Lipschitz. Moreover, if A_1 is satisfied, $S_{\bar{l}} \bigcap \{z = 0\}$ is bounded, thus the trajectories considered must have P or the origin as Ω-limit point. But, according to Lemma 2.10, in the latter case the corresponding $u(r)$ would have fast decay. But this is impossible, so $(x_{\bar{l}}(t), y_{\bar{l}}(t), z_{\bar{l}}(t))$ must converge to P and the claim is proved.

Assume that A_2 is satisfied, then the trajectory belongs to S_{l_2}, thus it is positive and decaying, but cannot have fast decay. Using Proposition 3.5 we have the thesis. Note that we are not assuming that $h(t)$ is monotone, but we already know that $H(x(t), y(t), t) < 0$, thus the proof still works.

Now assume that hypothesis B_1 is satisfied and consider system (3.3) with $\xi > 0$ and $l = s$. Recall that the system admits three critical points: P, $-P$ and the origin. We recall that the hypothesis guarantees that $\frac{dh_s}{dz}(z)\lfloor_{z=0}= 0$. Thus P admits a center unstable manifold CU which is transversal to the $z = 0$ plane. Therefore the matrix of the linearized system has an eigenvector parallel to the z direction corresponding to the eigenvalue ξ. Note also that if $s > p^*$, CU is one-dimensional; in fact it is a trajectory $(\bar{x}_s(t), \bar{y}_s(t), \bar{z}_s(t))$. Note that $\lim_{t\to-\infty} H_s(\bar{x}_s(t), \bar{y}_s(t), t) < 0$, therefore $\limsup_{t\to-\infty} H_{p^*}(\bar{x}_{p^*}(t), \bar{y}_{p^*}(t), t) \leq 0$; from the assumption on $J(r)$ we have $H_{p^*}(\bar{x}_{p^*}(t), \bar{y}_{p^*}(t), t) < 0$ for any t, unless we have some t for which $y(t) > 0$. But this case can be excluded since the flow on the y axis is always going downwards. Thus we can repeat the proof done for the regular solution and find the same behavior at ∞.

We want to prove that any S.G.S. $v(r)$ corresponds always to a trajectory belonging to CU. First of all observe that, for the corresponding trajectory, $\lim_{t\to-\infty} H(x(t), y(t), t) \leq 0$.
In fact, assume by contradiction that $\lim_{t\to-\infty} H(x(t), y(t), t) > 0$, then there exists $T > 0$ such that $H(x(t), y(t), t) > 0$ for any $t < -T$. Then following the trajectory backwards and reasoning as done in the proof of Proposition 3.5, we conclude that the trajectories must have $y(t) > 0$ for some t. Then, recalling Lemma 3.1, we conclude that such a trajectory cannot represent a positive solution. Thus $\lim_{t\to-\infty} H(x(t), y(t), t) \leq 0$.

Moreover, from Lemma 3.1, we know that $v'(r) \leq 0$ for any r. Thus we have $H(x(t), y(t), t) \leq 0$ for any t, see equation (2.4). Then we can repeat the proof of Proposition 3.5 and conclude that singular solutions, as $r \to 0$ can only have the behavior described in the thesis. If B_2 is satisfied, the non existence reasoning continue to apply, but we cannot use anymore invariant manifold theory to conclude the existence of CU. Thus we lose the existence result.

Suppose that C is satisfied, then we cannot find any l in order to make $\lim_{t\to\infty} h_l(t) > 0$. In [11], pages 738-739, it is proved that decaying solutions can only have fast decay. Therefore, if we set $l = p^*$, we find that decaying solutions $v(r)$ must correspond to trajectories $(\bar{x}_{p^*}(t), \bar{y}_{p^*}(t))$ of (2.2) converging to the origin, as $t \to \infty$; therefore we have $\lim_{t\to\infty} H_{p^*}(\bar{x}_{p^*}(t), \bar{y}_{p^*}(t), t) = 0$. Let us call $u(r)$ a generic solution, regular or singular, which is defined and positive for any $r > 0$. We have seen that $u(r)$ corresponds to a trajectory for which $H_{p^*}(t)$ is negative for any t. Moreover it is easy to prove that $H_{p^*}(t)$ is negative also letting

$t \to \infty$. Thus $u(r)$ cannot be decaying. The continuability and the positiveness of a generic $u(r)$, defined in a neighborhood of $r = 0$, follows from the fact that the corresponding trajectory is forced to stay in the set defined by $H(x, y, t) < 0$, which is bounded for any t finite. Moreover, from this observation, we also deduce that they are in the 4^{th} quadrant, thus $u'(r) \le 0$. Thus $u(r)$ is monotone decreasing and must have positive lower bound, thus the thesis is proved. The same kind of argument apply also to the trajectory belonging to CU, thus the claim regarding the singular solution is proved as well. □

Once again we restrict to a simple situation in order to make clearer which could be the applications of the theorem.

3.9. Corollary. *Set $q > p^*$. Assume $J(r) \le 0$ and that the function $K(r)$ is strictly positive and bounded and that the limit $\lim_{r\to\infty} K(r) = A > 0$ exists. Moreover assume that there exists $\delta > 0$ such that there exist the limits $\lim_{r\to 0} K'(r) r^{1-\delta} = 0$ and $\lim_{r\to\infty} K'(r) r^{1+\delta}$. Then any solution $u(r)$ of* (1.2) *is a monotone decreasing G.S. such that $u(r) \sim r^{\frac{-p}{q-p+1}}$, as $r \to \infty$. Moreover there exist a S.G.S. $v(r)$ with slow decay, that is $u(r) \sim r^{\frac{-p}{q-p+1}}$ both as $r \to 0$ and as $r \to \infty$. No other S.G.S. can exist. Moreover if $1 < q < \frac{p}{p-1}$ and $1 < p \le 2$, these are the only positive solutions of the problem.*

We recall that, if $q > p^*$ and $K(r)$ is monotone decreasing, then $J(r) < 0$.

The corollary is an immediate consequence of the preceding theorem. We only have to remark that, if $1 \le q < \frac{p}{p-1}$ and $1 < p \le 2$, the center stable manifold departing from the origin is made up only of trajectories corresponding to regular solutions $u(r)$. In fact, with these hypothesis we can apply the Observation 3.17. Otherwise we could have also solutions $w(r)$ such that $w(0) = A > 0$ and $w'(0) < 0$.

3.10. Theorem. *Assume that $G(r) \ge 0$ for any $r > 0$, but $G(r) \not\equiv 0$, and that there exist $s_2 \ge s_1 > p_*$, $\bar{l} \ge p^*$ such that*

$$\liminf_{r\to 0}\; j_{s_2}(r) > 0\,, \quad \lim_{r\to 0} j_{s_1}(r) < \infty\,, \quad 0 \le \lim_{r\to\infty} j_{\bar{l}}(r) = L < \infty\,,$$

$$\lim_{r\to 0} \frac{dj_{s_2}}{dr}(r) r^{1-\delta} = 0\,, \qquad \lim_{r\to\infty} \frac{dj_{\bar{l}}}{dr}(r) r^{1+\delta} = 0$$

for some $\delta > 0$ small.

- *A Assume that $L > 0$, then there exist a S.G.S $v(r)$ with slow decay, that is $c r^{-\frac{p}{s_2-p+1}} \le v(r) \le C r^{-\frac{p}{s_1-p+1}}$ as $r \to 0$ and $v(r) \sim r^{-\frac{p}{\bar{l}-p+1}}$ as $r \to \infty$. Moreover, if $\bar{l} \ne p^*$, this is the only S.G.S. with this behavior.*
- *B Assume that the regularity hypothesis is satisfied. Then there exist infinitely many S.G.S. $w(r)$ with fast decay. To be more explicit any solution $w(r)$ has the same behavior as $v(r)$ as $r \to 0$, but we have $w(r) \sim r^{-\frac{n-p}{p-1}}$ as $r \to \infty$.*
- *C Assume that A holds, then any S.G.S. must belong to one of the families of solutions described at the points A and B.*
- *D Assume that $J(r) \ge 0$ for any $r > 0$, but $J(r) \not\equiv 0$, then all solutions $u(r)$ of Eq.* (1.2) *are crossing solutions.*

Proof. We begin by proving D, recalling that Kawanida et al in [11] have already given a proof of this result. Consider system (2.2) where $l = p^*$ and a trajectory $(x(t), y(t))$ corresponding to a solution $u(r)$ of (1.2). First of all from the assumptions on $J(r)$ we have that $H(x(t), y(t), t) > 0$. Therefore, reasoning as in the proof of Proposition 3.5, we conclude that the trajectory starts from the origin, gets into U^+ and then crosses c and gets into U^- in finite time. Then it crosses the y negative semiaxis: thus $u(r)$ is a crossing solution.

Now assume that A is satisfied and consider system (3.3) where $\xi < 0$ and $l = \bar{l}$. Observe that it admits only three critical points O, P and $-P$, belonging to the $z = 0$ plane. Moreover P admits a center stable manifold CS, transversal to the $z = 0$ plane. We recall that the hypothesis guarantees that $\frac{dh_{\bar{l}}}{dz}(z)\lfloor_{z=0} = 0$. Note also that the ω-limit set of any bounded trajectory has to belong to this plane. Furthermore, if $\bar{l} \neq p^*$, in this plane there are no periodic trajectories and CS is one-dimensional.
Let us call $(\grave{x}(t), \grave{y}(t), \grave{z}(t))$ a trajectory belonging to CS and $v(r)$ the corresponding solution of (1.2). Then

$$\lim_{t\to\infty} H_{\bar{l}}(\grave{x}_{\bar{l}}(t), \grave{y}_{\bar{l}}(t), t) < 0 \quad \text{therefore} \quad \lim_{t\to\infty} H_{p^*}(\grave{x}_{p^*}(t), \grave{y}_{p^*}(t), t) = -M \leq 0\,.$$

Then it follows that

$$P_v(r) = -M - \left(G(r)\frac{|v(r)|^{q+1}}{q+1} - \int_r^\infty G(s)|v(s)|^q v'(s)ds \right) < 0\,.$$

Hence $H_{p^*}(\grave{x}_{p^*}(t), \grave{y}_{p^*}(t), t) < 0$ for any t; thus using Proposition 3.5 we can conclude.

Now assume that the regularity hypothesis is satisfied, and consider again system (3.3) where $\xi < 0$ and $l = \bar{l}$. Note that the origin admits a center stable manifold CS_0, which has at least dimension 2 and is transversal to the $z = 0$ plane. Consider a generic trajectory $(\tilde{x}_{\bar{l}}(t), \tilde{y}_{\bar{l}}(t), \tilde{z}_{\bar{l}}(t))$ belonging to CS_0 and the corresponding solution $w(r)$ of (1.2). Recalling Lemma 2.10 we can conclude that $w(r)$ has fast decay.

Moreover

$$\lim_{t\to\infty} H_{\bar{l}}(\tilde{x}_{\bar{l}}(t), \tilde{y}_{\bar{l}}(t), t) = 0 \quad \text{hence} \quad H_{p^*}(\tilde{x}_{p^*}(t), \tilde{y}_{p^*}(t), t) = 0\,.$$

Repeating the reasoning done for $v(r)$ we find that $w(r)$ is a S.G.S. with fast decay.

Now assume by contradiction that there exists a S.G.S. $a(r)$ different from the ones described. Consider again system (3.3) where $\xi < 0$ and $l = \bar{l}$. Observe that any trajectory, bounded in the future and belonging to the $x \geq 0$ subset, must have the origin or P as ω-limit set, if $\bar{l} \neq p^*$. Therefore their behavior has already been described. If $\bar{l} = p^*$ the ω-limit set could also be made up of union of periodic trajectories; anyway the corresponding value of H would be negative, therefore we could repeat the analysis just done and find S.G.S. with slow decay. Then $a(r)$ must correspond to an unbounded trajectory $(\breve{x}_{\bar{l}}(t), \breve{y}_{\bar{l}}(t), \breve{z}_{\bar{l}}(t))$. Thus $\lim_{t\to\infty} H_{\bar{l}}(\breve{x}_{\bar{l}}(t), \breve{y}_{\bar{l}}(t), t) = \infty$; therefore there exist $T > 0$ such that $H_{\bar{l}}(\breve{x}_{\bar{l}}(t), \breve{y}_{\bar{l}}(t), t) > 0$ for any $t > T$. Hence $H_{p^*}(\breve{x}_{p^*}(t), \breve{y}_{p^*}(t), t) > 0$ for any $t > T$; therefore, following the proof of Proposition 3.5 we deduce that there exists $T_1 > T$ such that $\breve{y}_{p^*}(T_1) > 0$. Thus we have found a contradiction and the thesis is proved. $\square$

3.11. Corollary. *Set $p_* < q < p^*$. Assume that $K(r)$ is strictly positive and bounded and that $J(r)$ and $G(r)$ are nonnegative for any r, and $\lim_{r\to\infty} K(r) = A > 0$. Moreover assume that there exists $\delta > 0$ small so that*

$$\lim_{r\to 0} K'(r) r^{1-\delta} = 0 = \lim_{r\to\infty} K'(r) r^{1+\delta}.$$

Then any solution $u(r)$ of (1.2) is a crossing solution. Moreover there exists exactly one S.G.S. $v(r)$ with slow decay, that is $u(r) \sim r^{\frac{-p}{q-p+1}}$ both as $r \to 0$ and as $r \to \infty$. Furthermore assume that the regularity hypothesis is satisfied, then there exist infinitely many S.G.S. with fast decay $w(r)$, that is $w(r) \sim r^{\frac{-p}{q-p+1}}$ as $r \to 0$ and $w(r) \sim r^{-\frac{n-p}{p-1}}$ as $r \to \infty$ No other S.G.S. can exist. Moreover if $1 < q < \frac{p}{p-1}$ and $1 < p \le 2$, these are the only positive solutions of the problem.

This corollary is a straightforward consequence of the preceding Theorem. Moreover, exploiting Observation 3.17 we can get also the following corollary.

3.12. Corollary. *Assume that the hypothesis of the Corollary 3.11 are satisfied. Moreover assume $1 < q < \frac{p}{p-1}$ and $1 < p \le 2$, then there are no solutions positive in a right neighborhood of $r = 0$, different from the ones described in Corollary 3.11. If $q \ge \frac{p}{p-1}$, we cannot exclude the existence of positive solutions $u(r)$ such that $u(0) = A > 0$ and $u'(0) \ne 0$*

3.13. Corollary. *Consider the autonomous equation (1.1) where $K(r) \equiv K > 0$ and $p_* < q < p^*$. Then for any given ball of radius R there exists one and only one Dirichlet radial solution.*

Proof. Set $l = q$ in (2.1): the system obtained is autonomous; we recall that the trajectory of system (2.2) containing the regular solutions of (1.2) is invariant for translation in t. So if $u(r)$ is such that $u(R) = 0$ there exists a family of solutions $u_s(r) = u(sr) s^{\frac{q-p+1}{p}}$ such that $u_s(\frac{R}{s}) = 0$, where $s > 0$ can be chosen arbitrarily. $\square$

Note also that if system (2.2) is autonomous we have exactly one S.G.S. with slow decay corresponding to the critical point P. Thus it can be explicitly computed. Using the t invariance property of the trajectories we also deduce the following result.

3.14. Corollary. *Consider the autonomous equation (1.1) where $K(r) \equiv K > 0$. Assume $q > p_*$, then there exist exactly one S.G.S with slow decay $v(r) = x_P r^{\frac{-p}{q-p+1}}$ Assume $p_* < q < p^*$ and that the regularity hypothesis is satisfied, then there exist a family of S.G.S with fast decay $v_s(r)$, with the property $v_s(r) = v(sr) s^{\frac{q-p+1}{p}}$, where $v(r)$ is a member of the family.*
Assume $q > p^*$, then all the regular solutions are G.S. with slow decay; let us denote $u_A(r)$ the solution such that $u_A(0) = A > 0$, then $u_A(r) = u_1(Ar) A^{\frac{q-p+1}{p}}$.

Therefore, knowing a member of the family of G.S. or of S.G.S. we know all of them.

Now we give some examples of application of the Theorems.

3.15. Remark. If $q > p_*$ and $K(r) = r^d log^c(r)$ where $d > -p$, using Theorem 3.8, it is possible to solve completely the problem of the existence and of the asymptotic behavior of G.S. and of S.G.S.

We recall the definition of $j(r) = K(r)r^{\frac{n-p}{p}(p^*-q)}$

3.16. Remark. If $j(r) = r^{-c_1}log^{d_1}(r) + r^{-c_2}log^{d_2}(r)$, where $c < d < p$ are real numbers, each regular solution $u(r)$ is a G.S. and its asymptotic behavior is ruled by the term $r^{-c_2}log^{d_2}(r)$.

This approach is also useful to classify the S.G.S. and to refine the estimate on the asymptotic behavior of G.S., given in [11], of some Matukuma-type equations and of Batt- Faltenbacher-Horst equation.

We state now and proof an observation regarding the correspondence between the solutions of (1.2) and the trajectories of (3.2) belonging to the center-stable manifold.
The claim is already been used, but we give it at the end, since it can be regarded as an appendix

3.17. Observation. Assume that $h_l(t)$ is bounded as $t \to -\infty$. Assume $1 < p \leq 2$ and $1 < l < \frac{p}{p-1}$. Then trajectories $(x_l(t), y_l(t))$ of (2.2) having the origin as α-limit point correspond to regular solutions $u(r)$ of (1.2) and viceversa.

Proof. To simplify the proof we will consider $l = q$ fixed, so we will leave unsaid the subscript. We already know that solutions $u(r)$ of (1.2) correspond to trajectories $(x(t), y(t))$ of (2.2) having the origin as α-limit point. Viceversa we know by Lemma 2.10 that solutions $u(r)$ corresponding to trajectories $(x(t), y(t))$ are such that $u(0)$ is well defined positive and bounded. With this assumption the claim could even be proved simpler, using invariant manifold theory and exponential dichotomy, as done in Theorem 4.1 of [9] for the scalar curvature equation. We only need to prove that $u'(0) = 0$. Exploiting invariant manifold theory, it can be proved that $\lim_{t\to-\infty}(x(t), y(t))e^{-\alpha t}$. Therefore if $\alpha > \beta$ that is $q < \frac{1}{p-1}$ we are done. The idea is to try to weaken this bound by observing that $y(t) \to 0$ faster than $x(t)$ as $t \to -\infty$. We begin by making the following change of variables.

$$W(t) = \frac{y(t)}{x(t)} \quad z(t) = |x|^S \quad \text{where } S = \frac{2-p}{p-1}\frac{1}{m} \tag{3.4}$$

and $m > 0$ will be fixed opportunely later. Applying (3.4) on (2.2) we obtain the following dynamical system:

$$\begin{aligned} \dot{W} = &\quad (\gamma - \alpha)W + \phi(t)Z^{\frac{q-1}{S}} - W^{2+Sm}Z^q \\ \dot{Z} = &\quad S\alpha Z + SW|W|^{Sm}Z^{1+m} \end{aligned} \tag{3.5}$$

We observe now that $(W(t), Z(t)) \to (0, 0)$ as $t \to -\infty$ which is a critical point of (3.5). We impose $l - 1 \geq S$ and $m \geq 1$ in order to linearize near the origin. We can rewrite the first condition in this way: there exist a constant C such that

$$0 \leq C = l - 1 - S = q - 1 - \frac{2-p}{m(p-1)}.$$

Now observe that, linearizing system (3.5) near the origin, we obtain the following matrix:

$$A := \begin{pmatrix} S\alpha & 0 \\ 0 & \gamma - \alpha \end{pmatrix}$$

Recalling that $(W(t), Z(t))$ belongs to the unstable manifold and using again invariant manifold theory we can say that for any given $\epsilon > 0$ we have $W(t) = O(e^{(S\alpha-\epsilon)t})$ as $t \to -\infty$. Therefore we have:

$$|u'|^{p-1} = |y|r^{-\beta} = W(t)x(t)e^{-\beta t} = O(e^{\alpha-\beta+S\alpha-\epsilon}).$$

Observe that $(S+1)\alpha - \beta = 1 - C$. Therefore if we assume $0 \leq C < 1$ we can conclude that $u'(0) = 0$. So if the two conditions $0 < C < 1$ and $m > 1$ are compatible, we have the thesis. These conditions can be rewrite in the following way:

$$\frac{(q-2)(p-1)}{2-p} < \frac{1}{m} < \frac{(q-1)(p-1)}{2-p} \quad \text{and} \quad \frac{1}{m} < 1$$

Thus we can choose m satisfying the conditions if and only if

$$\frac{(l-2)(p-1)}{2-p} < 1 \quad \text{or equivalently} \quad l < \frac{p}{p-1}$$

□

3.18. Remark. Observe that $p^* < \frac{p}{p-1}$, so for the critical case the hypothesis of Observation 3.17 are always satisfied.

3.19. Remark. If we have $l > \frac{p}{p-1}$, we cannot exclude the existence of solutions $u(r)$ such that $u(0) = 0$, but $u'(0) \neq 0$, which would be singular in the origin, but in a different way from the one analyzed in this paper.

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