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OSCILLATORY PROPERTIES OF FOURTH ORDER SELF-ADJOINT DIFFERENTIAL EQUATIONS

SIMONA FIŠNAROVÁ

Abstract. Oscillation and nonoscillation criteria for the self-adjoint linear differential equation

$$(t^\alpha y'')'' - \frac{\gamma_{2,\alpha}}{t^{4-\alpha}} y = q(t)y, \quad \alpha \notin \{1,3\}\,,$$

where

$$\gamma_{2,\alpha} = \frac{(\alpha-1)^2(\alpha-3)^2}{16}$$

and q is a real and continuous function, are established. It is proved, using these criteria, that the equation

$$(t^\alpha y'')'' - \left(\frac{\gamma_{2,\alpha}}{t^{4-\alpha}} + \frac{\gamma}{t^{4-\alpha}\ln^2 t}\right) y = 0$$

is nonoscillatory if and only if $\gamma \le \frac{\alpha^2-4\alpha+5}{8}$.

1. Introduction

In this paper we investigate oscillatory and asymptotic properties of the fourth order differential equation

$$(t^\alpha y'')'' - \frac{\gamma_{2,\alpha}}{t^{4-\alpha}} y = q(t)y, \tag{1}$$

where

$$\gamma_{2,\alpha} = \frac{(\alpha-1)^2(\alpha-3)^2}{16}, \quad \alpha \notin \{1,3\}$$

and q is a real and continuous function. Recently, several papers dealing with oscillatory properties of $2n$-th order two-terms differential equation

$$(-1)^n (t^\alpha y^{(n)})^{(n)} = q(t)y \tag{2}$$

appeared, where (2) is viewed as a perturbation of the Euler differential equation

$$(-1)^n (t^\alpha y^{(n)})^{(n)} - \frac{\gamma_{n,\alpha}}{t^{2n-\alpha}} y = 0$$

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and where

$$\gamma_{n,\alpha} := (-1)^n \prod_{j=0}^{n-1} (\lambda - j)(\lambda + \alpha - j - n)|_{\lambda = \frac{2n-1-\alpha}{2}}, \quad \alpha \notin \{1, 3, \dots, 2n-1\},$$

see [4] and the references given therein.

Our paper can be regarded as a continuation of the investigation of [3], where the case $\alpha = 0$ in (1) has been studied. We show that the results of [3] can be extended to the general case $\alpha \notin \{1, 3\}$ and we also show that the sign restriction on q assumed in that paper can be removed.

The paper is arranged as follows. In the next section we recall necessary definitions and we present some auxiliary results. Section 3 contains the main results of the paper – oscillation and nonoscillation criteria for (1) and the last section is devoted to some remarks concerning open problems and possibilities of extension of our results.

2. Preliminary results

In this section we recall some basic concepts of the theory of fourth order self-adjoint differential equations. These concepts can be formulated for arbitrary even order self-adjoint equations, but in order to simplify formulations, we present them here for fourth order equations only.

Consider the fourth order equation

$$L(y) := (r_2(t)y'')'' - (r_1(t)y')' + r_0(t)y = 0, \quad r_2(t) > 0, \tag{3}$$

where r_0, r_1, r_2 are continuous functions. We use the relationship between (3) and the linear Hamiltonian system (further referred to as LHS)

$$x' = A(t)x + B(t)u, \quad u' = C(t)x - A^T(t)u, \tag{4}$$

where A, B, C are 2×2 matrices, B, C symmetric. If y is a solution of (3) and if we set

$$x = \begin{pmatrix} y \\ y' \end{pmatrix}, \quad u = \begin{pmatrix} -(r_2(t)y'')' + r_1(t)y' \\ r_2(t)y'' \end{pmatrix},$$

then (x, u) is a solution of (4) with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 & 0 \\ 0 & r_2^{-1}(t) \end{pmatrix}, \quad C(t) = \begin{pmatrix} r_0(t) & 0 \\ 0 & r_1(t) \end{pmatrix}$$

and we say that the solution (x, u) of (4) is *generated* by the solution y of (3). Moreover, if the columns of the matrix solution (X, U) of (4) are generated by the solutions y_1, y_2 of (3), then (X, U) is said to be *generated* by y_1, y_2.

Oscillatory properties of (3) and (4) are defined using the concept of conjugate points. We say, that two different points t_1, t_2 are *conjugate* relative to LHS (4), if there exists a nontrivial solution (x, u) of (4) such that $x(t_1) = 0 = x(t_2)$ and $x(t) \not\equiv 0$ on $[t_1, t_2]$. According to the substitution which converts (3) into (4), points t_1, t_2 are conjugate relative to (3) if there exists a nontrivial solution y of (3) such that $y^{(i)}(t_1) = 0 = y^{(i)}(t_2)$ for $i = 0, 1$. Equation (3) (system (4)) is said to be *oscillatory* if for every $T \in \mathbb{R}$ there exist points $t_1, t_2 \in [T, \infty)$ which are conjugate relative to equation (3) (system (4)). Otherwise we say that equation (3) (system (4)) is *nonoscillatory*.

The equation

$$L(y) = w(t)y\,, \tag{5}$$

where w is a positive and continuous function and L is a nonoscillatory operator given by (3), is said to be *conditionally oscillatory*, if there exists $\lambda_0 > 0$ (the so called *oscillation constant* of (5)) such that the equation $L(y) = \lambda w(t)y$ is oscillatory for $\lambda > \lambda_0$ and nonoscillatory for $\lambda < \lambda_0$.

A matrix solution (X, U) of (4) is said to be a *conjoined basis* of (4) if it satisfies $X^T(t)U(t) = U^T(t)X(t)$ and $\operatorname{rank}(X^T, U^T)^T = 2$. We say that a conjoined basis (X, U) of (4) is the *principal solution* of (4) if $X(t)$ is nonsingular for large t and for any other conjoined basis $(\bar X, \bar U)$ such that the matrix $\bar X^T U - \bar U^T X$ is nonsingular $\lim_{t\to\infty} \bar X^{-1}(t)X(t) = 0$ holds. This limit equals zero if and only if

$$\lim_{t\to\infty} \left(\int^t X^{-1}(s)B(s)X^{T^{-1}}(s)\,ds\right)^{-1} = 0\,.$$

A principal solution of (4) is determined uniquely up to a right multiple by a constant nonsingular 2×2 matrix. If (X, U) is the principal solution, any conjoined basis $(\bar X, \bar U)$ such that the (constant) matrix $\bar X^T U - \bar U^T X$ is nonsingular is said to be a *nonprincipal solution* of (4). Solutions of y_1, y_2 of (3) are said to form a *principal* (*nonprincipal*) *system of solutions* if the solution (X, U) of (4) generated by these solutions is principal (nonprincipal).

Now we formulate some auxiliary statements which we use in the proofs of our main results.

Lemma 1 ([6])**.** *Equation* (3) *is nonoscillatory if and only if there exists* $T \in \mathbb{R}$ *such that*

$$\mathcal{F}(y; T, \infty) := \int_T^\infty \left[r_2(t)y''^2(t) + r_1(t)y'^2(t) + r_0(t)y^2(t)\right]\,dt > 0$$

for any nontrivial $y \in W^{2,2}(T, \infty)$ *with compact support in* (T, ∞)*.*

Lemma 2 ([7] (Wirtinger inequality))**.** *Let* $y \in W^{1,2}(T, \infty)$ *have compact support in* (T, ∞) *and let* M *be a positive differentiable function such that* $M'(t) \neq 0$ *for* $t \in [T, \infty)$*. Then*

$$\int_T^\infty |M'(t)|y^2(t)\,dt \le 4\int_T^\infty \frac{M^2(t)}{|M'(t)|}y'^2(t)\,dt\,. \tag{6}$$

Now we recall some properties of the general n-order linear differential equation.

The linear n-order differential equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0\,, \tag{7}$$

is said to be *disconjugate* on an interval I if every nontrivial solution of (7) has less than n zeros on I, multiple zeros being counted according to their multiplicities.

The functions $y_1, \dots, y_n \in C^n$ are said to form a *Markov system* on interval I if the Wronskians

$$W(y_1, \dots, y_k) = \begin{vmatrix} y_1 & \cdots & y_k \\ \vdots & \vdots & \vdots \\ y_1^{(k-1)} & \cdots & y_k^{(k-1)} \end{vmatrix}, \quad k = 1, \dots, n,$$

are positive throughout I.

Note, that if $y_1, \dots, y_n \in C^n$ and $y_i = o(y_{i+1})$ as $t \to \infty$, then there exists $T \in \mathbb{R}$ such that $y_1, \dots, y_n$ is the Markov system on $[T, \infty)$, see [1]. Moreover, the following statement holds.

Lemma 3. ([1]) *The equation* (7) *has a Markov fundamental system of solutions on an interval I, if and only if, it is disconjugate on I.*

Lemma 4 ([1])**.** *Suppose that equation* (3) *is disconjugate on an interval $I \subseteq \mathbb{R}$ and let y_1, y_2, y_3, y_4 be a fundamental system of solutions of this equation. Then the operator L given by* (3) *admits the factorization*

$$L(y) = \frac{1}{a_0(t)} \left(\frac{1}{a_1(t)} \left(\frac{r_2(t)}{a_2^2(t)} \left(\frac{1}{a_1(t)} \left(\frac{y}{a_0(t)} \right)' \right)' \right)' \right)'$$

on I, where

$$a_0 = y_1, \quad a_1 = \left(\frac{y_2}{y_1} \right)', \quad a_2 = (a_0 a_1)^{-1}.$$

Lemma 5. *For any y sufficiently smooth*

$$(8) \qquad (t^\alpha y'')'' - \frac{\gamma_{2,\alpha}}{t^{4-\alpha}} y = \frac{1}{t^{\frac{3-\alpha}{2}}} \left\{ t^{1+\frac{\sqrt{2\beta}}{2}} \left[t^{1-\sqrt{2\beta}} \left(t^{1+\frac{\sqrt{2\beta}}{2}} \left(\frac{y}{t^{\frac{3-\alpha}{2}}} \right)' \right)' \right]' \right\}'$$

and for any $y \in W^{2,2}(T, \infty)$, $T \in \mathbb{R}$, with compact support in (T, ∞)

$$(9) \quad \int_T^\infty \left[t^\alpha y''^2(t) - \frac{\gamma_{2,\alpha}}{t^{4-\alpha}} y^2(t) \right] dt = \int_T^\infty t^{1-\sqrt{2\beta}} \left\{ \left[t^{1+\frac{\sqrt{2\beta}}{2}} \left(\frac{y(t)}{t^{\frac{3-\alpha}{2}}} \right)' \right]' \right\}^2 dt,$$

where $\beta = \alpha^2 - 4\alpha + 5$.

Proof. Since the fundamental system of solutions of

$$(10) \qquad (t^\alpha y'')'' - \frac{\gamma_{2,\alpha}}{t^{4-\alpha}} y = 0$$

is

$$y_1(t) = t^{\frac{3-\alpha}{2}}, \; y_2(t) = t^{\frac{3-\alpha-\sqrt{2\beta}}{2}}, \; y_3(t) = t^{\frac{3-\alpha}{2}} \ln t, \; y_4(t) = t^{\frac{3-\alpha+\sqrt{2\beta}}{2}},$$

formula (8) follows from Lemma 4. The relation (9) we prove using (8) and integration by parts

$$\begin{aligned}\int_T^\infty \Big[t^\alpha y''^2(t) - \frac{\gamma_{2,\alpha}}{t^{4-\alpha}} y^2(t)\Big]\, dt &= \int_T^\infty y(t)\Big[\,(t^\alpha y''(t))'' - \frac{\gamma_{2,\alpha}}{t^{4-\alpha}} y(t)\Big]\, dt \\ &= \int_T^\infty y(t)\Bigg(\frac{1}{t^{\frac{3-\alpha}{2}}}\Big\{t^{1+\frac{\sqrt{2\beta}}{2}}\Big[t^{1-\sqrt{2\beta}}\Big(t^{1+\frac{\sqrt{2\beta}}{2}}\Big(\frac{y(t)}{t^{\frac{3-\alpha}{2}}}\Big)'\Big)'\Big]'\Big\}'\Bigg)\, dt \\ &= \int_T^\infty t^{1-\sqrt{2\beta}}\Big\{\Big[t^{1+\frac{\sqrt{2\beta}}{2}}\Big(\frac{y(t)}{t^{\frac{3-\alpha}{2}}}\Big)'\Big]'\Big\}^2\, dt\,.\end{aligned}$$

□

3. Oscillation and nonoscillation criteria for (1)

Theorem 1. *Suppose that*

$$\int^\infty \Big(q(t) - \frac{\alpha^2 - 4\alpha + 5}{8t^{4-\alpha}\ln^2 t}\Big) t^{3-\alpha} \ln t\, dt = \infty\,. \tag{11}$$

Then equation (1) *is oscillatory.*

Proof. Let $T \in \mathbb{R}$ be arbitrary, $T < t_0 < t_1 < t_2 < t_3$. We construct a function $0 \not\equiv y \in W^{2,2}(T,\infty)$, with compact support in (T,∞), as follows

$$y(t) = \begin{cases} 0\,, & t \le t_0\,, \\ f(t)\,, & t_0 \le t \le t_1\,, \\ h(t)\,, & t_1 \le t \le t_2\,, \\ g(t)\,, & t_2 \le t \le t_3\,, \\ 0\,, & t \ge t_3\,, \end{cases}$$

where $f \in C^2[t_0, t_1]$ is any function such that

$$f(t_0) = 0 = f'(t_0)\,, \quad f(t_1) = h(t_1)\,, \quad f'(t_1) = h'(t_1)\,,$$
$$h(t) = t^{\frac{3-\alpha}{2}}\sqrt{\ln t}$$

and g is the solution of (10) satisfying

$$g(t_2) = h(t_2)\,, \quad g'(t_2) = h'(t_2)\,, \quad g(t_3) = 0 = g'(t_3)\,. \tag{12}$$

We show that for y defined in this way

$$\mathcal{F}(y; T, \infty) := \int_T^\infty \Big[t^\alpha y''^2(t) - \Big(\frac{\gamma_{2,\alpha}}{t^{4-\alpha}} + q(t)\Big) y^2(t)\Big]\, dt \le 0\,,$$

if t_2, t_3 are sufficiently large and thus equation (1) is oscillatory according to Lemma 1.

Denote

$$K := \int_{t_0}^{t_1} \Big[t^\alpha f''^2(t) - \Big(\frac{\gamma_{2,\alpha}}{t^{4-\alpha}} + q(t)\Big) f^2(t)\Big]\, dt$$

and consider the interval $[t_1, t_2]$. Since

$$h''^2(t) = t^{-1-\alpha}\Big[\gamma_{2,\alpha}\ln t + \frac{(1-\alpha)(2-\alpha)(3-\alpha)}{4} + \frac{\beta}{8\ln t} - \frac{2-\alpha}{4\ln^2 t} + \frac{1}{16\ln^3 t}\Big]\,,$$

by a direct computation we have

$$\begin{aligned}\int_{t_1}^{t_2}\Big(t^{\alpha}h''^2(t) & -\frac{\gamma_{2,\alpha}}{t^{4-\alpha}}h^2(t)\Big)\,dt = \frac{(1-\alpha)(2-\alpha)(3-\alpha)}{4}\,(\ln t_2-\ln t_1)\\ &+\frac{\beta}{8}\int_{t_1}^{t_2}\frac{dt}{t\ln t}+\int_{t_1}^{t_2}\left(\frac{1}{16t\ln^3 t}-\frac{2-\alpha}{4t\ln^2 t}\right)dt\\ &=\frac{(1-\alpha)(2-\alpha)(3-\alpha)}{4}\ln t_2\\ &+\frac{\beta}{8}\int_{t_1}^{t_2}\frac{dt}{t\ln t}+L+o(1)\,,\end{aligned}$$

where L is a real constant and $\beta=\alpha^2-4\alpha+5$. By the symbol $o(F(t))$ we mean any function $G(t)$ satisfying $\lim_{t\to\infty}\frac{G(t)}{F(t)}=0$.
Concerning the last interval $[t_2,t_3]$, if we denote

$$x=\begin{pmatrix} g\\ g'\end{pmatrix},\quad u=\begin{pmatrix}-(t^{\alpha}g'')'\\ t^{\alpha}g''\end{pmatrix},\quad \tilde h=\begin{pmatrix} h\\ h'\end{pmatrix},$$

then system (4) with

$$A=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad B(t)=\begin{pmatrix}0&0\\0&t^{-\alpha}\end{pmatrix},\quad C(t)=\begin{pmatrix}-\frac{\gamma_{2,\alpha}}{t^{4-\alpha}}&0\\0&0\end{pmatrix}$$

is the LHS associated with equation (10). Using this relationship between (4) and (10) and conditions (12) we have

$$\begin{aligned}\int_{t_2}^{t_3}\Big[t^{\alpha}g''^2(t)-\frac{\gamma_{2,\alpha}}{t^{4-\alpha}}g^2(t)\Big]\,dt &= \int_{t_2}^{t_3}[u^T(t)B(t)u(t)+x^T(t)C(t)x(t)]\,dt\\ &=\int_{t_2}^{t_3}[u^T(t)(x'(t)-Ax(t))+x^T(t)C(t)x(t)]\,dt\\ &=u^T(t)x(t)|_{t_2}^{t_3}+\int_{t_2}^{t_3}x^T(t)[-u'(t)-A^Tu(t)+C(t)x(t)]\,dt\\ &=-\,u^T(t_2)x(t_2)\,.\end{aligned}$$

Let (X,U) be the principal solution of the LHS associated with (10). Since $(\bar X,\bar U)$ defined by

$$\begin{aligned}\bar X(t)&=X(t)\int_t^{t_3}X^{-1}(s)B(s)X^{T-1}(s)\,ds,\\ \bar U(t)&=U(t)\int_t^{t_3}X^{-1}(s)B(s)X^{T-1}(s)\,ds-X^{T-1}(t)\end{aligned}$$

is also a conjoined basis of this LHS, by using (12) we get

$$
\begin{aligned}
x(t) &= X(t)\int_t^{t_3} X^{-1}(s)B(s)X^{T-1}(s)\,ds \\
&\quad\times\left(\int_{t_2}^{t_3} X^{-1}(s)B(s)X^{T-1}(s)\,ds\right)^{-1} X^{-1}(t_2)\tilde{h}(t_2)\,,\\
u(t) &= \left(U(t)\int_t^{t_3} X^{-1}(s)B(s)X^{T-1}(s)\,ds - X^{T-1}(t)\right)\\
&\quad\times\left(\int_{t_2}^{t_3} X^{-1}(s)B(s)X^{T-1}(s)\,ds\right)^{-1} X^{-1}(t_2)\tilde{h}(t_2)
\end{aligned}
$$

and hence

$$
\begin{aligned}
-u^T(t_2)x(t_2) &= \tilde{h}^T(t_2)X^{T-1}(t_2)\left(\int_{t_2}^{t_3} X^{-1}(s)B(s)X^{T-1}(s)\,ds\right)^{-1} X^{-1}(t_2)\tilde{h}(t_2)\\
&\quad - \tilde{h}^T(t_2)U(t_2)X^{-1}(t_2)\tilde{h}(t_2)\,.
\end{aligned}
$$

Since the principal solution of LHS associated with (10) is generated by $y_1(t) = t^{\frac{3-\alpha-\sqrt{2\beta}}{2}}$, $y_2(t) = t^{\frac{3-\alpha}{2}}$, we have

$$
\begin{aligned}
X(t) &= \begin{pmatrix} t^{\frac{3-\alpha-\sqrt{2\beta}}{2}} & t^{\frac{3-\alpha}{2}} \\ \frac{3-\alpha-\sqrt{2\beta}}{2}t^{\frac{1-\alpha-\sqrt{2\beta}}{2}} & \frac{3-\alpha}{2}t^{\frac{1-\alpha}{2}} \end{pmatrix},\\
U(t) &= \begin{pmatrix} \frac{(3-\alpha-\sqrt{2\beta})(1-\alpha-\sqrt{2\beta})(1-\alpha+\sqrt{2\beta})}{8}t^{\frac{\alpha-3-\sqrt{2\beta}}{2}} & \frac{(3-\alpha)(1-\alpha)^2}{8}t^{\frac{\alpha-3}{2}} \\ \frac{(3-\alpha-\sqrt{2\beta})(1-\alpha-\sqrt{2\beta})}{4}t^{\frac{\alpha-1-\sqrt{2\beta}}{2}} & \frac{(3-\alpha)(1-\alpha)}{4}t^{\frac{\alpha-1}{2}} \end{pmatrix},\\
\tilde{h}(t) &= \begin{pmatrix} t^{\frac{3-\alpha}{2}}\sqrt{\ln t} \\ \frac{1}{2}t^{\frac{1-\alpha}{2}}\left[(3-\alpha)\sqrt{\ln t} + \frac{1}{\sqrt{\ln t}}\right] \end{pmatrix}
\end{aligned}
$$

and by a direct computation we obtain

$$
\begin{aligned}
\tilde{h}^T(t)U(t)\;X^{-1}(t)\tilde{h}(t) &= \frac{(3-\alpha)(2-\alpha)(1-\alpha)}{4}\ln t\\
&\quad + \frac{\alpha^2-4\alpha+3}{4} + \frac{4-2\alpha-\sqrt{2\beta}}{8\ln t}\,.
\end{aligned}
$$

Next we show that the function $\frac{g}{h}$ is decreasing on interval $[t_2, t_3]$. To show this fact we proceed similarly as in [2]. The system of functions

$$
\begin{aligned}
&y_1(t) = t^{\frac{3-\alpha-\sqrt{2\beta}}{2}}\,, \quad y_2(t) = t^{\frac{3-\alpha}{2}}\,, \quad h(t) = t^{\frac{3-\alpha}{2}}\sqrt{\ln t}\,,\\
&y_3(t) = t^{\frac{3-\alpha}{2}}\ln t\,, \quad y_4(t) = t^{\frac{3-\alpha+\sqrt{2\beta}}{2}}
\end{aligned}
$$

is a Markov system as can be verified by a direct computation, and using the rules for computations of Wronskians or again directly, it is possible to show that the system of functions $-\left(\frac{y_1}{h}\right)', -\left(\frac{y_2}{h}\right)', \left(\frac{y_3}{h}\right)', \left(\frac{y_4}{h}\right)'$ is also a Markov system, hence a fundamental system of solutions of a disconjugate fourth order linear differential equation. Consequently, if $t_2 < t_3$ are sufficiently large, any solution of this equation, particularly $\left(\frac{g}{h}\right)'$, has at most three zeros on $[t_2, t_3]$. Therefore, to prove that $\frac{g}{h}$ is decreasing on $[t_2, t_3]$ it suffices to show that

$$\left(\frac{g}{h}\right)''(t_2) < 0 \quad \text{and} \quad \left(\frac{g}{h}\right)''(t_3) > 0 \,. \tag{13}$$

Indeed, if (13) holds and since $\left(\frac{g}{h}\right)'(t_2) = 0 = \left(\frac{g}{h}\right)'(t_3)$, then $\left(\frac{g}{h}\right)'$ has even number of zeros on (t_2, t_3). This, together with the fact that $\left(\frac{g}{h}\right)'$ can have at most three zeros on $[t_2, t_3]$ implies that $\left(\frac{g}{h}\right)'(t) < 0$, $t \in (t_2, t_3)$. To show that (13) really holds we proceed as follows. Using (12), it is easy to verify that

$$\left(\frac{g}{h}\right)''(t_2) = \frac{g'' - h''}{h}(t_2) \quad \text{and} \quad \left(\frac{g}{h}\right)''(t_3) = \frac{g''}{h}(t_3)$$

and since $h(t) > 0$, we have to show that $g''(t_2) - h''(t_2) < 0$ and $g''(t_3) > 0$.
We have

$$\begin{aligned} u(t_2) &= U(t_2)X^{-1}(t_2)\tilde{h}(t_2) - X^{T-1}(t_2) \\ &\quad \times \left(\int_{t_2}^{t_3} X^{-1}(s)B(s)X^{T-1}(s)\,ds\right)^{-1} X^{-1}(t_2)\tilde{h}(t_2)\,. \end{aligned}$$

Since (X, U) is the principal solution of LHS generated by y_1, y_2, the second term of this relation tends to zero as $t_3 \to \infty$, t_2 being fixed, hence

$$u(t_2) = \begin{pmatrix} -(t_2^\alpha g''(t_2))' \\ t_2^\alpha g''(t_2) \end{pmatrix} \sim U(t_2)X^{-1}(t_2)\tilde{h}(t_2)$$

and consequently

$$g''(t_2) \sim \frac{(y_1''y_2' - y_2''y_1')h + (y_2''y_1 - y_1''y_2)h'}{W(y_1, y_2)}(t_2)\,,$$

where $W(y_1, y_2)$ is the Wronskian of y_1, y_2. By a direct computation

$$g''(t_2) \sim \frac{(3-\alpha)(1-\alpha)}{4} t_2^{\frac{-1-\alpha}{2}} \sqrt{\ln t_2} + \frac{4 - 2\alpha - \sqrt{2\beta}}{4} t_2^{\frac{-1-\alpha}{2}} \frac{1}{\sqrt{\ln t_2}}$$

and

$$h''(t_2) = \frac{(3-\alpha)(1-\alpha)}{4} t_2^{\frac{-1-\alpha}{2}} \sqrt{\ln t_2} + \frac{2-\alpha}{2} t_2^{\frac{-1-\alpha}{2}} \frac{1}{\sqrt{\ln t_2}} - \frac{1}{4} t_2^{\frac{-1-\alpha}{2}} \frac{1}{\sqrt{\ln^3 t_2}}\,.$$

It follows from the last two equalities that

$$g''(t_2) - h''(t_2) \sim -\frac{\sqrt{2\beta}}{4} t_2^{\frac{-1-\alpha}{2}} \frac{1}{\sqrt{\ln t_2}} + \frac{1}{4} t_2^{\frac{-1-\alpha}{2}} \frac{1}{\sqrt{\ln^3 t_2}} < 0$$

for t_2 sufficiently large. Next we have

$$u(t_3) = -X^{T-1}(t_3)\left(\int_{t_2}^{t_3} X^{-1}(s)B(s)X^{T-1}(s)ds\right)^{-1} X^{-1}(t_2)\tilde{h}(t_2)\,,$$

which implies

$$g''(t_3) = -\frac{t_3^{-\alpha}}{W(y_1,y_2)(t_3)} \begin{pmatrix} -y_2(t_3) \\ y_1(t_3) \end{pmatrix}^T \times \left(\int_{t_2}^{t_3} X^{-1}(s)B(s)X^{T-1}(s)\,ds \right)^{-1} X^{-1}(t_2)\tilde{h}(t_2)\,.$$

One can verify directly that

$$\Big(\int_{t_2}^{t_3} X^{-1}(s)B(s)X^{T-1}(s)\,ds \Big)^{-1} = \frac{\beta}{2d} \begin{pmatrix} \ln \frac{t_3}{t_2} & \frac{2}{\sqrt{2\beta}}\left(t_3^{\frac{\sqrt{2\beta}}{2}} - t_2^{\frac{\sqrt{2\beta}}{2}} \right) \\ \frac{2}{\sqrt{2\beta}}\left(t_3^{\frac{\sqrt{2\beta}}{2}} - t_2^{\frac{\sqrt{2\beta}}{2}} \right) & \frac{1}{\sqrt{2\beta}}\left(t_3^{\sqrt{2\beta}} - t_2^{\sqrt{2\beta}} \right) \end{pmatrix},$$

where

$$\begin{aligned} d &= \frac{\beta^2}{4} \det \left(\int_{t_2}^{t_3} X^{-1}(s)B(s)X^{T-1}(s)ds \right) \\ &= \frac{1}{\sqrt{2\beta}} \left(t_3^{\frac{\sqrt{2\beta}}{2}} - t_2^{\frac{\sqrt{2\beta}}{2}} \right) \left[\left(t_3^{\frac{\sqrt{2\beta}}{2}} + t_2^{\frac{\sqrt{2\beta}}{2}} \right) \ln \frac{t_3}{t_2} - \frac{4}{\sqrt{2\beta}} \left(t_3^{\frac{\sqrt{2\beta}}{2}} - t_2^{\frac{\sqrt{2\beta}}{2}} \right) \right] > 0 \end{aligned}$$

and

$$X^{-1}(t_2)\tilde{h}(t_2) = \begin{pmatrix} -\frac{1}{\sqrt{2\beta \ln t_2}} t_2^{\frac{\sqrt{2\beta}}{2}} \\ \sqrt{\ln t_2} + \frac{1}{\sqrt{2\beta \ln t_2}} \end{pmatrix}.$$

Thus, again by a direct computation we get

$$g''(t_3) = \frac{\frac{1}{2}\left(\sqrt{\ln t_2} + \frac{1}{\sqrt{2\beta \ln t_2}} \right) t_3^{\frac{2\sqrt{2\beta}-\alpha-1}{2}} + o\left(t_3^{\frac{2\sqrt{2\beta}-\alpha-1}{2}} \right)}{d} > 0\,.$$

Now, since $\frac{g}{h}$ is monotone on $[t_2, t_3]$, using the second mean value theorem of integral calculus there exists $\xi \in [t_2, t_3]$ such that

$$\begin{aligned} \int_{t_2}^{t_3} q(t)g^2(t)\,dt &= \int_{t_2}^{t_3} q(t)h^2(t) \left(\frac{g}{h} \right)^2 (t)\,dt \\ &= \left(\frac{g}{h} \right)^2 (t_2) \int_{t_2}^{\xi} q(t)h^2(t)\,dt + \left(\frac{g}{h} \right)^2 (t_3) \int_{\xi}^{t_3} q(t)h^2(t)\,dt \end{aligned}$$

and consequently, according to conditions (12)

$$\int_{t_1}^{t_2} q(t)h^2(t)\,dt + \int_{t_2}^{t_3} q(t)g^2(t)\,dt = \int_{t_1}^{\xi} q(t)h^2(t)\,dt\,.$$

Thus, we may summarize

$$\begin{aligned}\mathcal{F}(y;t_0,t_3) = K &+ \frac{(1-\alpha)(2-\alpha)(3-\alpha)}{4}\ln t_2 + \frac{\beta}{8}\int_{t_1}^{t_2}\frac{dt}{t\ln t} + L \\ &+ o(1) - \int_{t_1}^{\xi} q(t)h^2(t)\,dt + \tilde{h}^T(t_2)X^{T-1}(t_2) \\ &\times \left(\int_{t_2}^{t_3} X^{-1}(s)B(s)X^{T-1}(s)\,ds\right)^{-1} X^{-1}(t_2)\tilde{h}(t_2) \\ &- \frac{(1-\alpha)(2-\alpha)(3-\alpha)}{4}\ln t_2 - \frac{(\alpha-1)(\alpha-3)}{4} - o(1)\,.\end{aligned}$$

According to (11), $t_2 > t_1$ be such that

$$\begin{aligned}\int_{t_1}^{t_2}\frac{\beta}{8t\ln t}\,dt - \int_{t_1}^{\xi} q(t)h^2(t)\,dt &= \int_{t_1}^{\xi}\left(\frac{\alpha^2-4\alpha+5}{8t^{4-\alpha}\ln^2 t} - q(t)\right)t^{3-\alpha}\ln t\,dt \\ &- \int_{t_2}^{\xi}\frac{\alpha^2-4\alpha+5}{8t\ln t}\,dt < -(K+L+2)\end{aligned}$$

and the sum of all terms $o(1)$ in the previous computation is less than 1. Using the fact that (X,U) is the principal solution,

$$\tilde{h}^T(t_2)X^{T-1}(t_2)\left(\int_{t_2}^{t_3} X^{-1}(s)B(s)X^{T-1}(s)ds\right)^{-1} X^{-1}(t_2)\tilde{h}(t_2) \to 0$$

as $t_3 \to \infty$, t_2 being fixed. This enables to choose $t_3 > t_2$ such that

$$\begin{aligned}&\tilde{h}^T(t_2)X^{T-1}(t_2) \\ &\quad\times \left(\int_{t_2}^{t_3} X^{-1}(s)B(s)X^{T-1}(s)ds\right)^{-1} X^{-1}(t_2)\tilde{h}(t_2) < \frac{(\alpha-1)(\alpha-3)}{4} + 1\,.\end{aligned}$$

For t_2, t_3 chosen in this way we have

$$\begin{aligned}&\mathcal{F}(y;T,\infty) \\ &\quad < K - (K+L+2) + L + 1 + \frac{(\alpha-1)(\alpha-3)}{4} + 1 - \frac{(\alpha-1)(\alpha-3)}{4} = 0\,.\end{aligned}$$

And it means that equation (1) is oscillatory. $\square$

Note that when $\alpha = 0$ this statement was proved in [3], where the additional condition $q(t) \geq 0$ was assumed. In our proof, since we have shown the monotony of $\frac{q}{h}$, the term $\int_{t_2}^{t_3} q(t)g^2(t)dt$ can be removed from the computations and the sing restriction on the function q may be relaxed.

Theorem 2. *If the second order linear differential equation*

$$(tz')' + \frac{2}{\alpha^2-4\alpha+5}t^{3-\alpha}q(t)z = 0 \tag{14}$$

is nonoscillatory, then equation (1) *is also nonoscillatory.*

Proof. Let $T \in \mathbb{R}$ and $y \in W^{2,2}(T,\infty)$ with compact support in (T,∞) be arbitrary. Using (9) from Lemma 5 and Wirtinger inequality (6) we have

$$\int_T^\infty \Big(t^\alpha y''^2(t) - \frac{\gamma_{2,\alpha}}{t^{4-\alpha}} y^2(t)\Big)\, dt = \int_t^\infty t^{1-\sqrt{2\beta}} \left\{ \left[t^{1+\frac{\sqrt{2\beta}}{2}} \left(\frac{y(t)}{t^{\frac{3-\alpha}{2}}} \right)' \right]' \right\}^2 dt$$

$$\geq \frac{\beta}{2} \int_T^\infty t \left[\left(\frac{y(t)}{t^{\frac{3-\alpha}{2}}} \right)' \right]^2 dt\,,$$

where $\beta = \alpha^2 - 4\alpha + 5$. Denote $z = yt^{\frac{\alpha-3}{2}}$. Since (14) is nonoscillatory, then, using the similar result for the second order equations as in Lemma 1

$$\int_T^\infty \left(tz'^2(t) - \frac{2}{\beta} t^{3-\alpha} q(t) z^2(t) \right) dt > 0\,,$$

if T is sufficiently large and by the above inequality

$$\int_T^\infty \Big[t^\alpha y''^2(t) - \Big(\frac{\gamma_{2,\alpha}}{t^{4-\alpha}} + q(t)\Big) y^2(t)\Big] dt \geq \frac{\beta}{2} \int_T^\infty t \Big[Big(\frac{y(t)}{t^{\frac{3-\alpha}{2}}}\Big)'\Big]^2 dt - \int_T^\infty q(t) y^2(t)\, dt$$

$$= \frac{\beta}{2} \int_T^\infty \left\{ t \left[\left(\frac{y(t)}{t^{\frac{3-\alpha}{2}}} \right)' \right]^2 - \frac{2}{\beta} t^{3-\alpha} q(t) \left(\frac{y(t)}{t^{\frac{3-\alpha}{2}}} \right)^2 \right\} dt > 0\,.$$

Therefore, again by Lemma 1, equation (1) is nonoscillatory. □

Corollary 1. *The equation*

$$(t^\alpha y'')'' - \left(\frac{\gamma_{2,\alpha}}{t^{4-\alpha}} + \frac{\gamma}{t^{4-\alpha} \ln^2 t} \right) y = 0 \tag{15}$$

is nonoscillatory if and only if $\gamma \leq \frac{\alpha^2-4\alpha+5}{8}$.

Proof. If $\gamma > \frac{\alpha^2-4\alpha+5}{8}$, then, for $q(t) = \frac{\gamma}{t^{4-\alpha}\ln^2 t}$, condition (11) takes the form

$$\int^\infty \frac{\gamma - \frac{\alpha^2-4\alpha+5}{8}}{t \ln t} = \infty$$

and hence (15) is oscillatory according to Theorem 1. Conversely, since the equation

$$(tz')' + \frac{\mu}{t \ln^2 t} z = 0$$

is nonoscillatory for $\mu \leq \frac{1}{4}$, we have the nonoscillation of

$$(tz')' + \frac{2}{\alpha^2 - 4\alpha + 5} t^{3-\alpha} \frac{\gamma}{t^{4-\alpha} \ln^2 t} z = 0$$

for $\gamma \leq \frac{\alpha^2-4\alpha+5}{8}$ and thus the nonoscillation of (15) follows from Theorem 2. □

4. Notes and remarks

(a) We have excluded from our investigations the case $\alpha \in \{1,3\}$ for the following reason. The solutions of

$$(t^\alpha y'')'' - \frac{\gamma}{t^{4-\alpha}} y = 0 \tag{16}$$

are in the form $y(t) = t^\lambda$, where λ is a root of the polynomial $P_{2,\alpha}(\lambda) - \gamma$ with

$$P_{2,\alpha}(\lambda) = \lambda(\lambda-1)(\lambda+\alpha-2)(\lambda+\alpha-3)\,.$$

There is one positive maximum of $P_{2,\alpha}(\lambda)$ for $\lambda = \frac{3-\alpha}{2}$. Thus, if $\gamma_{2,\alpha} := P_{2,\alpha}(\frac{3-\alpha}{2})$, then the polynomial $P_{2,\alpha}(\lambda) - \gamma_{2,\alpha}$ has four real roots with the double root in $\lambda = \frac{3-\alpha}{2}$. If $\alpha \in \{1,3\}$, then $P_{2,\alpha}(\frac{3-\alpha}{2}) = 0$ and there is no positive constant γ for which (16) is nonoscillatory.

As an analogy of (10) in case $\alpha \in \{1,3\}$ we can take the equation

$$(t^\alpha y'')'' - \frac{\gamma}{t^{4-\alpha}\ln^2 t} y = 0\,,$$

see [5]. However, since we are not able to find solutions of this equation explicitly, we have not succeeded in extending the results to this case, but this problem is a subject of the present investigation.

(b) Oscillation and nonoscillation criteria presented in this paper are closely related to the problem of conditional oscillation of (15). Equation (15) is conditionally oscillatory with

$$L(y) = (t^\alpha y'')'' - \frac{\gamma_{2,\alpha}}{t^{4-\alpha}} y, \quad w(t) = \frac{1}{t^{4-\alpha}\ln^2 t} \tag{17}$$

and with the oscillation constant

$$\lambda_0 = \frac{\alpha^2 - 4\alpha + 5}{8}\,.$$

Conditionally oscillatory of differential equations has applications in the spectral theory of differential operators. In particular, next statement holds, see [6].

Proposition 1. *The spectrum of the operator $\frac{1}{w}L$ in the Hilbert space*

$$\mathcal{L}^2_w = \{y : \int^\infty w(t) y^2(t)dt < \infty\}$$

is discrete and bounded below (the so called property B-D) if and only if the equation $L(y) = \lambda w(t) y$ is nonoscillatory for every $\lambda \in \mathbb{R}$.

With the respect to the previous statement and Corollary 1, the differential operator $\frac{1}{w}L$ given by (17) does not have the property B-D, since for $\lambda > \frac{\alpha^2-4\alpha+5}{8}$ equation (15) is oscillatory and it is nonoscillatory for $\lambda \le \frac{\alpha^2-4\alpha+5}{8}$.

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Department of Mathematics, Masaryk University
Janáčkovo nám. 2a, 662 95 Brno, Czech Republic
E-mail: simona@math.muni.cz