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ON NATURAL METRICS ON TANGENT BUNDLES OF RIEMANNIAN MANIFOLDS

MOHAMED TAHAR KADAOUI ABBASSI AND MAÂTI SARIH

Abstract. There is a class of metrics on the tangent bundle TM of a Riemannian manifold (M, g) (oriented , or non-oriented, respectively), which are 'naturally constructed' from the base metric g [15]. We call them "g-natural metrics" on TM. To our knowledge, the geometric properties of these general metrics have not been studied yet. In this paper, generalizing a process of Musso-Tricerri (cf. [18]) of finding Riemannian metrics on TM from some quadratic forms on $OM \times \mathbb{R}^m$ to find metrics (not necessary Riemannian) on TM, we prove that all g-natural metrics on TM can be obtained by Musso-Tricerri's generalized scheme. We calculate also the Levi-Civita connection of Riemannian g-natural metrics on TM. As application, we sort out all Riemannian g-natural metrics with the following properties, respectively: 1) The fibers of TM are totally geodesic. 2) The geodesic flow on TM is incompressible. We shall limit ourselves to the non-oriented situation.

Introduction

Geometry of the tangent bundle TM of an m-dimensional Riemannian manifold (M, g) with Sasaki metric has been extensively studied since the 60's. Nevertheless, the rigidity of this metric (cf. [3], [18] and [22]) has incited some geometers to tackle the problem of the construction and the study of other metrics on TM. The Cheeger-Gromoll metric (cf. [8]) has appeared as a nicely fitted one to overcome this rigidity, and has been, thus, studied by many authors (see [2], [3] and [22]). Using the concept of naturality, O. Kowalski and M. Sekizawa [15] have given a full classification of metrics which are 'naturally constructed' from a metric g on the base M, supposing that M is oriented. Other presentations of the basic results from [15] (involving also the non-oriented case and something more) can be found in [13] or [17]. We call these metrics *g-natural metrics on TM*. To our knowledge, the geometric properties of these general metrics on TM have not been studied yet (see also [16]).

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In this paper, we deal with g-natural metrics on TM in the case when the orientation of M is not taken into account. In fact, in the non-oriented case we only lose some special g-natural metrics over Riemannian manifolds of dimensions 2 and 3; in dimensions $m > 3$, the oriented case and the non-oriented case coincide. In § 2, we sort out from g-natural metrics on TM (which may even be degenerate) those which are regular and those which are Riemannian.

In § 3, we generalize, to the non Riemannian case, the process of construction of Riemannian metrics on TM from symmetric basic tensor fields of type $(2, 0)$ on $OM \times \mathbb{R}^m$, presented in [18] by E. Musso and F. Tricerri, where OM is the bundle of orthonormal frames. We show then that all g-natural metrics can be obtained by the generalized Musso-Tricerri's process.

In § 4, we give explicit formulas of the Levi-Civita connection $\bar{\nabla}$ of a g-natural metric on TM and we provide necessary and sufficient conditions on G to have the fibers of TM totally geodesic.

On the other hand, it is well known that with respect to Sasaki metric and Cheeger-Gromoll metric on TM, the geodesic flow of TM is incompressible (cf. [2] and [21]). In § 5, we give necessary and sufficient conditions on Riemannian g-natural metrics which let the geodesic flow of TM incompressible. As a consequence, it is particularly worth mentioning that some g-natural metrics present a kind of rigidity related to the geodesic flow. For instance, Let $\mathcal{R}^3$ denote the vector space of all g-natural metrics of the form $G = a \cdot g^s + b \cdot g^h + c \cdot g^v$ (i.e., linear combinations with constant coefficients of the three classical lifts g^s, g^h and g^v of g). Define $\mathcal{C}$ as the 2-dimensional cone in $\mathcal{R}^3$ characterized by the inequalities $a > 0$, $c > 0$ and $b^2 - a(a + c) < 0$. Then $\mathcal{C}$ is just the subset of all Riemannian metrics in $\mathcal{R}^3$. Now, we can prove that, for every G from $\mathcal{C}$ with $b \neq 0$, the geodesic flow on TM is incompressible, with respect to G, if and only if (M, g) is an Einstein space with vanishing scalar curvature.

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1. Preliminaries

Let ∇ be the Levi-Civita connection of g. Then the tangent space of TM at any point $(x, u) \in TM$ splits into the horizontal and vertical subspaces with respect to ∇:

$$(TM)_{(x,u)} = H_{(x,u)} \oplus V_{(x,u)} .$$

If $(x, u) \in TM$ is given then, for any vector $X \in M_x$, there exists a unique vector $X^h \in H_{(x,u)}$ such that $p_* X^h = X$, where $p : TM \to M$ is the natural projection. We call X^h the *horizontal lift* of X to the point $(x, u) \in TM$. The *vertical lift* of a vector $X \in M_x$ to $(x, u) \in TM$ is a vector $X^v \in V_{(x,u)}$ such that $X^v(df) = Xf$, for all functions f on M. Here we consider 1-forms df on M as functions on TM (i.e. $(df)(x, u) = uf$). Note that the map $X \to X^h$ is an isomorphism between the vector spaces M_x and $H_{(x,u)}$. Similarly, the map $X \to X^v$ is an isomorphism between the vector spaces M_x and $V_{(x,u)}$. Obviously, each tangent vector $\tilde{Z} \in (TM)_{(x,u)}$ can be written in the form $\tilde{Z} = X^h + Y^v$, where $X, Y \in M_x$ are uniquely determined vectors.

If φ is a smooth function on M, then

$$X^h(\varphi \circ p) = (X\varphi) \circ p \quad \text{and} \quad X^v(\varphi \circ p) = 0 \tag{1.1}$$

hold for every vector field X on M.

A system of local coordinates $\{(U; x^i, i = 1, \dots, m)\}$ in M induces on TM a system of local coordinates $\{(p^{-1}(U); x^i, u^i, i = 1, \dots, m)\}$. Let $X = \sum_i X^i \frac{\partial}{\partial x^i}$ be the local expression in U of a vector field X on M. Then, the horizontal lift X^h and the vectical lift X^v of X are given, with respect to the induced coordinates, by:

$$X^h = \sum X^i \frac{\partial}{\partial x^i} - \sum \Gamma^i_{jk} u^j X^k \frac{\partial}{\partial u^i}\,, \tag{1.2}$$

and

$$X^v = \sum X^i \frac{\partial}{\partial u^i}\,, \tag{1.3}$$

where (Γ^i_{jk}) denote the Christoffel's symbols of g.

Now, let r be the norm of a vector u. Then, for any function f of $\mathbb{R}$ to $\mathbb{R}$, we get

$$X^h_{(x,u)}(f(r^2)) = 0, \tag{1.4}$$

$$X^v_{(x,u)}(f(r^2)) = 2f'(r^2) g_x(X_x, u), \tag{1.5}$$

and in particular, we have

$$X^h_{(x,u)}(r^2) = 0\,, \tag{1.6}$$

and

$$X^v_{(x,u)}(r^2) = 2g_x(X_x, u)\,. \tag{1.7}$$

Let X, Y and Z be any vector fields on M. If F_Y is the function on TM defined by $F_Y(x,u) = g_x(Y_x, u)$, for all $(x,u) \in TM$, then we have

$$X^h_{(x,u)}(F_Y) = g_x((\nabla_X Y)_x, u) = F_{\nabla_X Y}(x,u)\,, \tag{1.8}$$

$$X^v_{(x,u)}(F_Y) = g_x(X, Y), \tag{1.9}$$

$$X^h_{(x,u)}(g(Y,Z) \circ p) = X_x(g(Y,Z))\,, \tag{1.10}$$

$$X^v_{(x,u)}(g(Y,Z) \circ p) = 0\,. \tag{1.11}$$

The formulas (1.4)–(1.9) follow from (1.1) and

$$X^h u^i = -\sum X^\lambda u^\mu \Gamma^i_{\lambda\mu} \quad \text{and} \quad X^v u^i = X^i\,, \tag{1.12}$$

and the relations (1.10) and (1.11) follow easily from (1.1).

Next, we shall introduce some notations which will be used describing vectors getting from lifted vectors by basic operations on TM. Let T be a tensor field of type $(1,s)$ on M. If $X_1, X_2, \dots, X_{s-1} \in M_x$, then $h\{T(X_1, \dots, u, \dots, X_{s-1})\}$

(resp.$v\{T(X_1,\dots,u,\dots,X_{s-1})\}$) is a horizontal (resp.vertical) vector at (x,u) which is introduced by the formula

$$h\{T(X_1,\dots,u,\dots,X_{s-1})\}=\sum u^\lambda(T(X_1,\dots,(\frac{\partial}{\partial x^\lambda})_x,\dots,X_{s-1}))^h$$

$$\Big(\text{resp. } v\{T(X_1,\dots,u,\dots,X_{s-1})\}=\sum u^\lambda(T(X_1,\dots,(\frac{\partial}{\partial x^\lambda})_x,\dots,X_{s-1}))^v\Big).$$

In particular, if T is the identity tensor of type $(1,1)$, then we obtain the geodesic flow vector field at (x,u), $\xi_{(x,u)}=\sum u^\lambda(\frac{\partial}{\partial x^\lambda})^h_{(x,u)}$, and the canonical vertical vector at (x,u), $U_{(x,u)}=\sum u^\lambda(\frac{\partial}{\partial x^\lambda})^v_{(x,u)}$. Moreover $h\{T(X_1,\dots,u,\dots,u,\dots,X_{s-1})\}$ and $v\{T(X_1,\dots,u,\dots,u,\dots,X_{s-1})\}$ are introduced by similar way. Also we make the conventions $h\{T(X_1,\dots,X_{s-1})\}=(T(X_1,\dots,X_{s-1}))^h$ and $v\{T(X_1,\dots,X_{s-1})\}=(T(X_1,\dots,X_{s-1}))^v$. Thus $h\{X\}=X^h$ and $v\{X\}=X^v$, for each vector field X on M.

The bracket operation of vector fields on the tangent bundle is given by

$$[X^h,Y^h]_{(x,u)}=[X,Y]^h_{(x,u)}\,-\,v\{R(X_x,Y_x)u\}\,, \tag{1.13}$$

$$[X^h,Y^v]_{(x,u)}=(\nabla_XY)^v_{(x,u)}\,, \tag{1.14}$$

$$[X^v,Y^v]_{(x,u)}=0\,, \tag{1.15}$$

for all vector fields X and Y on M, where R is the Riemannian curvature of g defined by

$$R(X,Y)=[\nabla_X,\nabla_Y]\,-\,\nabla_{[X,Y]}\,.$$

Finally, the following Koszul formula holds

$$\begin{aligned} g(\nabla_XY,Z)&=X(g(Y,Z))+Y(g(Z,X))-Z(g(X,Y))\\ &\quad+g([X,Y],Z)+g([Z,X],Y)-g([Y,Z],X)\,, \end{aligned} \tag{1.16}$$

for all vector fields X, Y and Z on M.

Now, if we write $p_M:TM\to M$ for the natural projection and F for the natural bundle with $FM=p_M^*(T^*\otimes T^*)M\to M$, $Ff(X_x,g_x)=(Tf\cdot X_x,(T^*\otimes T^*)f\cdot g_x)$ for all manifolds M, local diffeomorphisms f of M, $X_x\in T_xM$ and $g_x\in(T^*\otimes T^*)_xM$. The sections of the canonical projection $FM\to M$ are called *F-metrics* in literature. So, if we denote by $\oplus$ the fibered product of fibered manifolds, then the F-metrics are mappings $TM\oplus TM\oplus TM\to\mathbb{R}$ which are linear in the second and the third argument.

As generalization of the notion of F-metrics, we can define the notion of F-tensor fields of any type on a manifold. For $(p,q)\in\mathbb{N}^2$, we write $p_M:TM\to M$ for the natural projection and F for the natural bundle with $FM=p_M^*(\underbrace{T^*\otimes\cdots\otimes T^*}_{p-\text{times}}\otimes\underbrace{T\otimes\cdots\otimes T}_{q-\text{times}})M\to M$, $Ff(X_x,S_x)=(Tf\cdot X_x,(T^*\otimes\cdots\otimes T^*\otimes T\otimes\cdots\otimes T)f\cdot S_x)$ for all manifolds M, local diffeomorphisms f of M, $X_x\in T_xM$ and $S_x\in(T^*\otimes\cdots\otimes T^*\otimes T\otimes\cdots\otimes T)_xM$. We call the sections of the canonical projection $FM\to M$ *F-tensor fields of type* (p,q). So F-tensor fields are mappings A :

$TM \oplus \underbrace{TM \oplus \cdots \oplus TM}_{q-\text{times}} \to \bigsqcup_{x\in M} \otimes^p M_x$ which are linear in the last q summands such that $\pi_2 \circ A = \pi_1$, where π_1 and π_2 are the natural projections of the source and target fiber bundles of A respectively. For $p = 0$ and $q = 2$, we obtain the classical notion of F-metrics.

If we fix an F-metric δ on M, then there are three distinguished constructions of metrics on the tangent bundle TM, which are given as follows [15]:

(a) If we suppose that δ is symmetric, then the *Sasaki lift* δ^s of δ is defined as follows:

$$\begin{cases} \delta^s_{(x,u)}(X^h, Y^h) = \delta(u; X, Y)\,, & \delta^s_{(x,u)}(X^h, Y^v) = 0\,, \\ \delta^s_{(x,u)}(X^v, Y^h) = 0\,, & \delta^s_{(x,u)}(X^v, Y^v) = \delta(u; X, Y)\,, \end{cases}$$

for all X, $Y \in M_x$. If δ is non degenerate and positive definite, then the same holds for δ^s.

(b) The *horizontal lift* δ^h of δ is a pseudo-Riemannian metric on TM which is given by:

$$\begin{cases} \delta^h_{(x,u)}(X^h, Y^h) = 0\,, & \delta^h_{(x,u)}(X^h, Y^v) = \delta(u; X, Y)\,, \\ \delta^h_{(x,u)}(X^v, Y^h) = \delta(u; X, Y)\,, & \delta^h_{(x,u)}(X^v, Y^v) = 0\,, \end{cases}$$

for all X, $Y \in M_x$. If δ is positive definite, then δ^s is of signature (m, m).

(c) The *vertical lift* δ^v of δ is a degenerate metric on TM which is given by:

$$\begin{cases} \delta^v_{(x,u)}(X^h, Y^h) = \delta(u; X, Y)\,, & \delta^v_{(x,u)}(X^h, Y^v) = 0\,, \\ \delta^v_{(x,u)}(X^v, Y^h) = 0\,, & \delta^v_{(x,u)}(X^v, Y^v) = 0\,, \end{cases}$$

for all X, $Y \in M_x$. The rank of δ^v is exactly that of δ. If $\delta = g$ is a Riemannian metric on M, then the three lifts of δ just constructed coincide with the three well-known classical lifts of the metric g to TM.

2. Natural metrics on tangent bundles

Now, we shall describe all first order natural operators $D : S^2_+T^* \rightsquigarrow (S^2T^*)T$ transforming Riemannian metrics on manifolds into metrics on their tangent bundles, where $S^2_+T^*$ and S^2T^* denote the bundle functors of all Riemannian metrics and all symmetric two-forms over m-manifolds respectively. For the concept of naturality and related notions, see [13] for more details.

Let us call every section $G : TM \to (S^2T^*)TM$ a (possibly degenerate) *metric*. Then we can assert:

Proposition 2.1 ([15])**.** *There is a bijective correspondence between the triples of natural F-metrics $(\zeta_1, \zeta_2, \zeta_3)$, where ζ_1 and ζ_3 are symmetric, and natural* (*possibly degenerate*) *metrics G on the tangent bundles given by*

$$G = \zeta_1^s + \zeta_2^h + \zeta_3^v\,.$$

Therefore, to find all first order natural operators $S^2_+T^* \rightsquigarrow (S^2T^*)T$ transforming Riemannian metrics on manifolds into metrics on their tangent bundles, it suffices to describe all natural F-metrics, i.e. natural operators $S^2_+T^* \rightsquigarrow (T, F)$. In this sense, we can state

Proposition 2.2 ([1]). *All first order natural F-metrics ζ in dimension $m > 1$ form a family parametrized by two arbitrary smooth functions α_0, $\beta_0 : [0, \infty) \to \mathbb{R}$ in the following way: For every Riemannian manifold (M, g) and tangent vectors u, X, $Y \in M_x$*

$$\zeta_{(M,g)}(u)(X, Y) = \alpha_0(g(u, u))g(X, Y) + \beta_0(g(u, u))g(u, X)g(u, Y)\,. \tag{2.1}$$

If $m = 1$, then the same assertion holds, but we can always choose $\beta_0 = 0$. In particular, all first order natural F-metrics are symmetric.

Definition 2.3. Let (M, g) be a Riemannian manifold. We shall call a metric G on TM which comes from g by a first order natural operator $S^2_+T^* \rightsquigarrow (S^2T^*)T$ a *g-natural metric.*

Thus, all g-natural metrics on the tangent bundle of a Riemannian manifold (M, g) are completely determined by Propositions 2.1 and 2.2, as follows:

Corollary 2.4. *Let (M, g) be a Riemannian manifold and G be a g-natural metric on TM. Then there are functions α_i, $\beta_i : [0, \infty) \to \mathbb{R}$, $i = 1, 2, 3$, such that for every u, X, $Y \in M_x$, we have*

$$\begin{cases} G_{(x,u)}(X^h, Y^h) = (\alpha_1 + \alpha_3)(r^2)g_x(X, Y) \\ \qquad\qquad\qquad\quad +(\beta_1 + \beta_3)(r^2)g_x(X, u)g_x(Y, u)\,, \\ G_{(x,u)}(X^h, Y^v) = \alpha_2(r^2)g_x(X, Y) + \beta_2(r^2)g_x(X, u)g_x(Y, u)\,, \\ G_{(x,u)}(X^v, Y^h) = \alpha_2(r^2)g_x(X, Y) + \beta_2(r^2)g_x(X, u)g_x(Y, u)\,, \\ G_{(x,u)}(X^v, Y^v) = \alpha_1(r^2)g_x(X, Y) + \beta_1(r^2)g_x(X, u)g_x(Y, u)\,, \end{cases} \tag{2.2}$$

where $r^2 = g_x(u, u)$. For $m = 1$, the same holds with $\beta_i = 0$, $i = 1, 2, 3$.

Remark 2.5. In [15], the last problem of classification of metrics on TM, was stated differently, i.e. the question was to find all second order natural transformations of Riemannian metrics on manifolds to metrics on tangent bundles. Nevertheless, by virtue of Proposition 18.19 in [13], the two problems are equivalent.

Notations 2.6. In the sequel, we shall use the following notations:

- $\phi_i(t) = \alpha_i(t) + t\beta_i(t)$,
- $\alpha(t) = \alpha_1(t)(\alpha_1 + \alpha_3)(t) - \alpha_2^2(t)$,
- $\phi(t) = \phi_1(t)(\phi_1 + \phi_3)(t) - \phi_2^2(t)$,

for all $t \in [0, \infty)$.

Now, similar arguments as in [4] enables us to specify regular (i.e. non degenerate) and Riemannian g-natural metrics as follows:

Proposition 2.7. *The necessary and sufficient conditions for a g-natural metric G on the tangent bundle of a Riemannian manifold (M, g) to be regular are that the functions of Proposition 2.4, defining G, satisfy $\alpha(t) \neq 0$ and $\phi(t) \neq 0$, for* *all* $t \in [0,\infty)$. *For* $m = 1$ *the two conditions reduce to one, i.e.* $\alpha(t) \neq 0$, *for all* $t \in [0,\infty)$.

Proof. Let $x \in M$, $u \in M_x \setminus \{0_x\}$ and $X_1 = \frac{1}{r}.u$, where $r = \|u\|$.
For $m = 1$, the determinant of the matrix of $G_{(x,u)}$, with respect to the basis $\{X_1^h, X_1^v\}$ of $(TM)_{(x,u)}$, is given by $\alpha(t).\phi(t)$.
For $m > 1$, choosing vectors $X_2, \dots, X_m$ of M_x such that $\{X_1, X_2, \dots, X_m\}$ is an orthonormal basis of (M_x, g_x), then the matrix of $G_{(x,u)}$, with respect to the basis $\{X_1^h, X_2^h, \dots, X_m^h, X_1^v, X_2^v, \dots, X_m^v\}$ of $(TM)_{(x,u)}$, is given by $P_m(r^2)$, where P_m is the $(2m, 2m)$-matrix-valued real function

$$P_m(t) = \begin{pmatrix} (\phi_1+\phi_3)(t) & 0\cdots 0 & \phi_2(t) & 0\cdots 0 \\ 0 & & 0 & \\ \vdots & (\alpha_1+\alpha_3)(t)\cdot I_{m-1} & \vdots & \alpha_2(t)\cdot I_{m-1} \\ 0 & & 0 & \\ \phi_2(t) & 0\cdots 0 & \phi_1(t) & 0\cdots 0 \\ 0 & & 0 & \\ \vdots & \alpha_2(t)\cdot I_{m-1} & \vdots & \alpha_1(t)\cdot I_{m-1} \\ 0 & & 0 & \end{pmatrix}, \tag{2.3}$$

I_{m-1} being the identity matrix of $GL(m)$, and we can prove by induction on m that the determinant of the last matrix is equal to $\phi(t)\cdot\alpha^{m-1}(t)$.

On the other hand, if $u = 0$ then the determinant of the matrix of $G_{(x,0)}$ with respect to any basis $\{X_1^h, X_2^h, \dots, X_m^h, X_1^v, X_2^v, \dots, X_m^v\}$ of $(TM)_{(x,u)}$, where $\{X_1, X_2, \dots, X_m\}$ is any orthonormal basis of (M_x, g_x), is given by $\alpha^m(0) = \phi(0)\cdot\alpha^{m-1}(0)$. Now the result follows easily. □

Similarly, we can prove the following:

Proposition 2.8 ([5])**.** *The necessary and sufficient conditions for a g-natural metric G on the tangent bundle of a Riemannian manifold* (M,g) *to be Riemannian are that the functions of Proposition 2.4, defining G, satisfy the inequalities*

$$\begin{cases} \alpha_1(t) > 0\,, & \phi_1(t) > 0\,, \\ \alpha(t) > 0\,, & \phi(t) > 0\,, \end{cases} \tag{2.4}$$

for all $t \in [0,\infty)$. *For* $m = 1$ *the system reduces to* $\alpha_1(t) > 0$ *and* $\alpha(t) > 0$, *for all* $t \in [0,\infty)$.

Important conventions:

1) In the sequel, when we consider an arbitrary Riemannian g-natural metric G on TM, we implicitly suppose that it is defined by the functions α_i, $\beta_i : [0,\infty) \to \mathbb{R}$, $i = 1,2,3$, given in Corollary 2.4 and satisfying (2.4).

2) Unless otherwise stated, all real functions α_i, β_i, ϕ_i, α and ϕ and their derivatives are evaluated at $r^2 := g_x(u,u)$.

3. g-natural metrics by the scheme of Musso-Tricerri

Considering TM as a vector bundle associated with the bundle of orthonormal frames OM, E. Musso and F. Tricerri have constructed an interesting class of *Riemannian* natural metrics on TM [18]. This construction is not a classification *per se*, but it is a construction process of *Riemannian* metrics on TM from symmetric, positive semi-definite tensor fields Q of type $(2,0)$ and rank $2m$ on $OM \times \mathbb{R}^m$, which are basic for the natural submersion $\Phi : OM \times \mathbb{R}^m \to TM$, $\Phi(v,\varepsilon) = (x, \sum_i \varepsilon^i v_i)$, for $v = (x; v_1, \ldots, v_m) \in OM$ and $\varepsilon = (\varepsilon^1, \ldots, \varepsilon^m) \in \mathbb{R}^m$. Recall that Q is *basic* means that Q is $O(m)$-invariant and $Q(X,Y) = 0$, if X is tangent to a fiber of Φ.

Given such a Q, there is a unique *Riemannian* metric G^Q on TM such that $\Phi^*(G^Q) = Q$. This metric is determined by the formula

$$G^Q_{(x,u)}(X,Y) = Q_{(v,\varepsilon)}(X',Y'), \tag{3.1}$$

where (v,ε) belongs to the fiber $\Phi^{-1}(x,u)$, X, Y are elements of $(TM)_{(x,u)}$, X', Y' are tangent vectors to $OM \times \mathbb{R}^m$ at (v,ε) with $d\Phi(X') = X$ and $d\Phi(Y') = Y$.

Now, we can check that this process can be generalized to construct also metrics on TM which are not necessarily Riemannian (even degenerate ones). Precisely, we have:

Proposition 3.1. *Let Q be a symmetric tensor field of type $(2,0)$ on $OM \times \mathbb{R}^m$, which is basic for the natural submersion $\Phi : OM \times \mathbb{R}^m \to TM$. Then there is a unique metric G^Q on TM such that $\Phi^*(G^Q) = Q$. It is given by (3.1). Furthermore, we have:*

1. *The rank of G^Q is equal to that of Q.*
2. *G^Q is Riemannian if and only if Q is positive semi-definite of rank $2m$.*

Remark that the rank of Q is less than or equal to $2m$, since Q is basic, and that the second assertion of the previous proposition corresponds exactly to the original process of E. Musso and F. Tricerri given in [18]. Note also that such a generalization is possible due to the fact that the process of identification of TM as an associated bundle is natural.

Let $(e_1, \ldots, e_m)$ be an orthonormal frame field defined on an open set $U \subset M$, and let $(x^1, \ldots, x^m)$ be a local coordinate system on U. We define a local coordinate system $(x^1, \ldots, x^m, u^1, \ldots, u^m)$ on $p^{-1}(U)$ as follows:

$$x^i(x,u) = x^i(x),\ u^i(x,u) = u^i,\ (x,u) \in p^{-1}(U), \quad \text{where} \quad u = \sum_i u^i e_i(x).$$

We denote with Γ^i_j the local 1-forms defined by

$$\nabla_X e_i = \sum_j \Gamma^j_i(X) e_j.$$

Let e^i be the 1-forms on $p^{-1}(U)$ defined by $e^i(e_k) = \delta^i_k$, and

$$Du^i = du^i + \sum_j u^j p^*(\Gamma^i_j).$$

Then $(e_1^h, \dots, e_m^h, e_1^v, \dots, e_m^v)$ is a frame field on $p^{-1}(U)$, whose dual coframe is given by

$$p^* e^1, \dots, p^* e^m, Du^1, \dots, Du^m . \tag{3.2}$$

Let $\theta = (\theta^1, \dots, \theta^m)$ denote the canonical 1-form on OM, and let π denote the natural projection $OM \xrightarrow{\pi} M$. Then

$$d\pi_v(X) = \sum_i \theta^i(X) v_i\,, \quad v = (x; v_1, \dots, v_m)\,.$$

If we denote with $\omega = (\omega_j^i)$ the $so(m)$-valued differential form defined by the Levi-Civita connection of g, then we find that

$$\theta^i,\ i = 1, \dots, m;\ \omega_j^i,\ 1 \le i \le j \le m;\ d\varepsilon^i,\ i = 1, \dots, m\,,$$

is an absolute parallelism on $OM \times \mathbb{R}^m$. Note that we use here (and in the sequel) the abuse of notation $\theta = \pi_1^* \theta$, where $\pi_1 : OM \times \mathbb{R}^m \to OM$ is the natural first projection. We put

$$D\varepsilon^i = d\varepsilon^i + \sum_j \varepsilon^j \omega_j^i\,.$$

On the other hand, we have [18]:

Lemma 3.2. *Any basic symmetric quadratic form Q on $OM \times \mathbb{R}^m$ is a second order polynomial in θ^i and $D\varepsilon^i$ whose coefficients yield Q invariant under the $O(m)$-action.*

As an application of Proposition 3.1, we consider the two following symmetric quadratic forms on $OM \times \mathbb{R}^m$,

$$Q^h = \sum_i \theta^i D\varepsilon^i \quad \text{and} \quad Q^v = \sum_i (\theta^i)^2\,,$$

which are basic by Lemma 3.2. They give rise, via the scheme of Proposition 3.1, to the classical lifts g^h and g^v, respectively. It is clear that Q^h, as g^h, is of signature (m, m), and that Q^v is degenerate of rank m as the metric g^v.

Generally, we can assert the following:

Proposition 3.3. *Every g-natural metric on the tangent bundle TM of a Riemannian manifold (M, g) can be constructed by the Musso-Tricerri's generalized scheme, given by Proposition 3.1.*

Proof. Let G be a g-natural metric on the tangent bundle TM of a Riemannian manifold (M, g). With respect to the coframe (3.2), G can be written as follows:

$$\begin{aligned} G = {} & (\alpha_1 + \alpha_3)(r^2) \textstyle\sum_i (p^* e^i)^2 + (\beta_1 + \beta_3)(r^2) (\sum_i u^i (p^* e^i))^2 \\ & + \alpha_1(r^2) \textstyle\sum_i (Du^i)^2 + \beta_1(r^2) (\sum_i u^i Du^i)^2 \\ & + \alpha_2(r^2) \textstyle\sum_i (p^* e^i) Du^i + \beta_2(r^2) (\sum_i u^i (p^* e^i)) (\sum_i u^i Du^i)\,, \end{aligned} \tag{3.3}$$

where $r^2 = \sum_i (u^i)^2$ and α_i, β_i, $i = 1, 2, 3$, are functions from $[0, \infty)$ to $\mathbb{R}$.

Consider the symmetric tensor field Q of type $(2,0)$ on $OM \times \mathbb{R}^m$

$$
\begin{aligned}
Q = {} & (\alpha_1+\alpha_3)(r^2)\textstyle\sum_i(\theta^i)^2 + (\beta_1+\beta_3)(r^2)(\sum_i \varepsilon^i\theta^i)^2 \\
& +\alpha_1(r^2)\textstyle\sum_i(D\varepsilon^i)^2 + \beta_1(r^2)(\sum_i \varepsilon^i D\varepsilon^i)^2 \\
& +\alpha_2(r^2)\textstyle\sum_i \theta^i D\varepsilon^i + \beta_2(r^2)(\sum_i \varepsilon^i\theta^i)(\sum_i \varepsilon^i D\varepsilon^i)\,,
\end{aligned}
\tag{3.4}
$$

where $r^2 = \sum_i(\varepsilon^i)^2$. It is easy to see [18] that:

$$
\begin{aligned}
R_a^*(\theta^i) \quad &= \textstyle\sum_j (a^{-1})^i_j \theta^j\,, \\
R_a^*(\omega^i_j) \quad &= \textstyle\sum_{k,l} (a^{-1})^i_k \omega^k_l a^l_j\,, \\
R_a^*(D\varepsilon^i) &= \textstyle\sum_j (a^{-1})^i_j D\varepsilon^j\,,
\end{aligned}
$$

for all $a \in O(m)$, where R_a is the natural translation by a. Then Q is $O(m)$-invariant.

On the other hand, Q is basic by Lemma 3.2. Therefore, by virtue of Proposition 3.1, Q induces a unique metric on TM. Furthermore, we claim that Q induces G. Indeed, if we denote by Φ_U the $O(m)$-valued function on $\pi^{-1}(U)$ given by

$$(\Phi_U)^i_j(v) = g(e_i(\pi(v)), v_j)\,,$$

then the forms ω^i_j are related to the local 1-forms Γ^i_j as follows

$$\omega^i_j = \sum_k (\Phi_U^{-1})^i_k d(\Phi_U)^k_j + \sum_{k,l}(\Phi_U^{-1})^i_k (p^*\Gamma^k_l)(\Phi_U)^l_j\,. \tag{3.5}$$

We can also check easily that

$$\Phi^*(u^i) = \sum_j (\Phi_U)^i_j \varepsilon^j\,. \tag{3.6}$$

Using formulas (3.5) and (3.6) and the following commutative diagram:

$$
\begin{tikzcd}
OM \times \mathbb{R}^m \arrow[r, "\Phi"] \arrow[d, "\pi_1"'] & TM \arrow[d, "p"] \\
OM \arrow[r, "\pi"] & M,
\end{tikzcd}
$$

we get

$$\Phi^*(p^*e^i) = \sum_j (\Phi_U)^i_j \theta^j\,, \tag{3.7}$$

and

$$\Phi^*(Dv^i) = \sum_j (\Phi_U)^i_j D\varepsilon^j\,. \tag{3.8}$$

Note that formula (4.10) in [18] should read our formula (3.7), i.e. $\Phi^*(p^*e^i)$ instead of p^*e^i. Since Φ_U is $O(m)$-valued, we have by virtue of formulas (3.3), (3.4), (3.6)–(3.8) and the $O(m)$-invariance of Q, the identity $\Phi_U^*(G) = Q$. $\square$

4. The Levi-Civita connection of (TM, G)

In this section, we shall calculate the Levi-Civita connection of a Riemannian g-natural metric G on the tangent bundle of a Riemannian manifold (M, g). We can assert the following:

Proposition 4.1. *Let (M, g) be a Riemannian manifold, ∇ its Levi-Civita connection and R its curvature tensor. Let G be a Riemannian g-natural metric on TM. Then the Levi-Civita connection $\bar{\nabla}$ of (TM, G) is characterized by*

$$
\begin{array}{ll}
\text{(i)} & (\bar{\nabla}_{X^h} Y^h)_{(x,u)} = (\nabla_X Y)^h_{(x,u)} + h\{A(u; X_x, Y_x)\} + v\{B(u; X_x, Y_x)\}\,, \\
\text{(ii)} & (\bar{\nabla}_{X^h} Y^v)_{(x,u)} = (\nabla_X Y)^v_{(x,u)} + h\{C(u; X_x, Y_x)\} + v\{D(u; X_x, Y_x)\}\,, \\
\text{(iii)} & (\bar{\nabla}_{X^v} Y^h)_{(x,u)} = h\{C(u; Y_x, X_x)\} + v\{D(u; Y_x, X_x)\}\,, \\
\text{(iv)} & (\bar{\nabla}_{X^v} Y^v)_{(x,u)} = h\{E(u; X_x, Y_x)\} + v\{F(u; X_x, Y_x)\}\,,
\end{array}
$$

for all vector fields X, Y on M and $(x, u) \in TM$, where A, B, C, D, E and F are the F-tensor fields of type $(2,1)$ on M defined, for all u, X, $Y \in M_x$, $x \in M$, by:

$$
\begin{aligned}
A(u; X, Y) = & -\tfrac{\alpha_1\alpha_2}{2\alpha}[R(X,u)Y + R(Y,u)X] \\
& + \tfrac{\alpha_2(\beta_1+\beta_3)}{2\alpha}[g_x(Y,u)X + g_x(X,u)Y] \\
& + \tfrac{1}{\alpha\phi}\{\alpha_2[\alpha_1(\phi_1(\beta_1+\beta_3) - \phi_2\beta_2) + \alpha_2(\beta_1\alpha_2 \\
& - \beta_2\alpha_1)]g_x(R(X,u)Y,u) + \phi_2\alpha(\alpha_1+\alpha_3)' g_x(X,Y) \\
& + [\alpha\phi_2(\beta_1+\beta_3)' + (\beta_1+\beta_3)[\alpha_2(\phi_2\beta_2 - \phi_1(\beta_1+\beta_3)) \\
& + (\alpha_1+\alpha_3)(\alpha_1\beta_2 - \alpha_2\beta_1)]]g_x(X,u)g_x(Y,u)\}u\,,
\end{aligned}
$$

$$
\begin{aligned}
B(u; X, Y) = & \tfrac{\alpha_2^2}{\alpha}R(X,u)Y - \tfrac{\alpha_1(\alpha_1+\alpha_3)}{2\alpha}R(X,Y)u \\
& - \tfrac{(\alpha_1+\alpha_3)(\beta_1+\beta_3)}{2\alpha}[g_x(Y,u)X + g_x(X,u)Y] \\
& + \tfrac{1}{\alpha\phi}\{\alpha_2[\alpha_2(\phi_2\beta_2 - \phi_1(\beta_1+\beta_3)) + (\alpha_1+\alpha_3)(\beta_2\alpha_1 \\
& - \beta_1\alpha_2)]g_x(R(X,u)Y,u) - \alpha(\phi_1+\phi_3)(\alpha_1+\alpha_3)' g_x(X,Y) \\
& + [-\alpha(\phi_1+\phi_3)(\beta_1+\beta_3)' \\
& + (\beta_1+\beta_3)[(\alpha_1+\alpha_3)[(\phi_1+\phi_3)\beta_1 - \phi_2\beta_2] \\
& + \alpha_2[\alpha_2(\beta_1+\beta_3) - (\alpha_1+\alpha_3)\beta_2]]g_x(X,u)g_x(Y,u)\}u\,,
\end{aligned}
$$

$$\begin{aligned}
C(u;X,Y) &= -\tfrac{\alpha_1^2}{2\alpha}R(Y,u)X - \tfrac{\alpha_1(\beta_1+\beta_3)}{2\alpha}g_x(X,u)Y \\
&+ \tfrac{1}{\alpha}[\alpha_1(\alpha_1+\alpha_3)' - \alpha_2(\alpha_2' - \tfrac{\beta_2}{2})]g_x(Y,u)X \\
&+ \tfrac{1}{\alpha\phi}\{\frac{\alpha_1}{2}[\alpha_2(\alpha_2\beta_1 - \alpha_1\beta_2) + \alpha_1(\phi_1(\beta_1+\beta_3) \\
&- \phi_2\beta_2)]g_x(R(X,u)Y,u) + \alpha[\tfrac{\phi_1}{2}(\beta_1+\beta_3) + \phi_2(\alpha_2' - \tfrac{\beta_2}{2})]g_x(X,Y) \\
&+ [\alpha\phi_1(\beta_1+\beta_3)' + [\alpha_2(\alpha_1\beta_2 - \alpha_2\beta_1) \\
&+ \alpha_1(\phi_2\beta_2 - (\beta_1+\beta_3)\phi_1)][(\alpha_1+\alpha_3)' + \tfrac{\beta_1+\beta_3}{2}] \\
&+ [\alpha_2(\beta_1(\phi_1+\phi_3) - \beta_2\phi_2) + \alpha_1(\beta_2(\alpha_1+\alpha_3) \\
&- \alpha_2(\beta_1+\beta_3)](\alpha_2' - \tfrac{\beta_2}{2})]g_x(X,u)g_x(Y,u)\}u\,,
\end{aligned}$$

$$\begin{aligned}
D(u;X,Y) &= \tfrac{1}{\alpha}\{\tfrac{\alpha_1\alpha_2}{2}R(Y,u)X - \tfrac{\alpha_2(\beta_1+\beta_3)}{2}g_x(X,u)Y \\
&+ [-\alpha_2(\alpha_1+\alpha_3)' + (\alpha_1+\alpha_3)(\alpha_2' - \tfrac{\beta_2}{2})]g_x(Y,u)X\} \\
&+ \tfrac{1}{\alpha\phi}\{\tfrac{\alpha_1}{2}[(\alpha_1+\alpha_3)(\alpha_1\beta_2 - \alpha_2\beta_1) \\
&+ \alpha_2(\phi_2\beta_2 - \phi_1(\beta_1+\beta_3))]g_x(R(X,u)Y,u) \\
&- \alpha[\tfrac{\phi_2}{2}(\beta_1+\beta_3) + (\phi_1+\phi_3)(\alpha_2' - \tfrac{\beta_2}{2})]g_x(X,Y) \\
&+ [\alpha\phi_2(\beta_1+\beta_3)' + [(\alpha_1+\alpha_3)(\alpha_2\beta_1 - \alpha_1\beta_2) \\
&+ \alpha_2(\phi_1(\beta_1+\beta_3) - \phi_2\beta_2)][(\alpha_1+\alpha_3)' + \tfrac{\beta_1+\beta_3}{2}] \\
&+ [(\alpha_1+\alpha_3)(\beta_2\phi_2 - \beta_1(\phi_1+\phi_3)) + \alpha_2(\beta_2(\alpha_1+\alpha_3) \\
&- \alpha_2(\beta_1+\beta_3)](\alpha_2' - \tfrac{\beta_2}{2})]g_x(X,u)g_x(Y,u)\}u\,,
\end{aligned}$$

$$\begin{aligned}
E(u;X,Y) &= \tfrac{1}{\alpha}[\alpha_1(\alpha_2' + \tfrac{\beta_2}{2}) - \alpha_2\alpha_1'][g_x(Y,u)X + g_x(X,u)Y] \\
&+ \tfrac{1}{\alpha\phi}\{\alpha[\phi_1\beta_2 - \phi_2(\beta_1 - \alpha_1')]g_x(X,Y) \\
&+ [\alpha(2\phi_1\beta_2' - \phi_2\beta_1') + 2\alpha_1'[\alpha_1(\alpha_2(\beta_1+\beta_3) \\
&- \beta_2(\alpha_1+\alpha_3)) + \alpha_2(\beta_1(\phi_1+\phi_3) - \beta_2\phi_2)] \\
&+ (2\alpha_2' + \beta_2)[\alpha_1(\phi_2\beta_2 - \phi_1(\beta_1+\beta_3)) \\
&+ \alpha_2(\alpha_1\beta_2 - \alpha_2\beta_1)]]g_x(X,u)g_x(Y,u)\}u\,,
\end{aligned}$$

$$\begin{aligned}
F(u;X,Y) &= \tfrac{1}{\alpha}[-\alpha_2(\alpha_2' + \tfrac{\beta_2}{2}) + (\alpha_1+\alpha_3)\alpha_1'][g_x(Y,u)X + g_x(X,u)Y] \\
&+ \tfrac{1}{\alpha\phi}\{\alpha[(\phi_1+\phi_3)(\beta_1 - \alpha_1') - \phi_2\beta_2]g_x(X,Y) \\
&+ [\alpha((\phi_1+\phi_3)\beta_1' - 2\phi_2\beta_2') + 2\alpha_1'[\alpha_2(\beta_2(\alpha_1+\alpha_3) \\
&- \alpha_2(\beta_1+\beta_3)) + (\alpha_1+\alpha_3)(\beta_2\phi_2 - \beta_1(\phi_1+\phi_3))] \\
&+ (2\alpha_2' + \beta_2)[\alpha_2(\phi_1(\beta_1+\beta_3) - \phi_2\beta_2) \\
&+ (\alpha_1+\alpha_3)(\alpha_2\beta_1 - \alpha_1\beta_2)]]g_x(X,u)g_x(Y,u)\}u\,.
\end{aligned}$$

For $m = 1$ the same holds with $\beta_i = 0$, $i = 1, 2, 3$.

Proof. At first, we can easily check the following formulas which relate the metric G to the base metric g. Let X and Y be vector fields on M and $(x,u) \in TM$, then, according to (2.2), we have

$$g_x(X,u) = \frac{1}{\phi_1+\phi_3} G_{(x,u)}(X^h, h\{u\}) , \tag{4.1}$$

$$g_x(X,Y) = \frac{1}{\alpha_1+\alpha_3}\{G_{(x,u)}(X^h,Y^h) - (\beta_1+\beta_3)g_x(X_x,u)g_x(Y_x,u)\} , \tag{4.2}$$

and similarly, with respect to vertical lifts, we have

$$g_x(X,u) = \frac{1}{\phi_1} G_{(x,u)}(X^v, v\{u\}) , \tag{4.3}$$

$$g_x(X,Y) = \frac{1}{\alpha_1}\{G_{(x,u)}(X^v,Y^v) - \beta_1 g_x(X_x,u)g_x(Y_x,u)\} . \tag{4.4}$$

Using Koszul formula (1.16), and then (1.4), (1.8), (1.10) and (1.13), we can write for each vector field Z on M,

$$2G_{(x,u)}(\bar{\nabla}_{X^h}Y^h, Z^h) = 2G_{(x,u)}((\nabla_X Y)^h, Z^h) - 2\alpha_2 g_x(R(X_x,u)Y_x, Z_x) ,$$

and by virtue of (4.1) and (4.2), we have

$$2G_{(x,u)}(\bar{\nabla}_{X^h}Y^h, Z^h) = 2G_{(x,u)}(h\{T_{11}\}, Z^h(x,u)) ,$$

where T_{11} is given by

$$T_{11} = (\nabla_X Y)_x + \frac{\alpha_2}{\alpha_1+\alpha_3}\{-R(X_x,u)Y_x + \frac{\beta_1+\beta_3}{\phi_1+\phi_3} g_x(R(X_x,u)Y_x,u)u\} . \tag{4.5}$$

By similar way, using Koszul formula (1.16), and then (1.4), (1.5), (1.8)-(1.10), (1.13), (1.14), (4.3) and (4.4), we can write for each vector field Z on M,

$$2G_{(x,u)}(\bar{\nabla}_{X^h}Y^h, Z^v) = 2G_{(x,u)}(v\{T_{12}\}, Z^v(x,u)) ,$$

where T_{12} is given by

$$\begin{aligned} T_{12} = {} & \frac{1}{\alpha_1}\{\alpha_2(\nabla_X Y)_x - \frac{\beta_1+\beta_3}{2}[g_x(Y_x,u)X_x + g_x(X_x,u)Y_x]\} \\ & - \frac{1}{2}R(X_x,Y_x)u + \frac{1}{\phi_1}\{(\beta_2 - \frac{\alpha_2}{\alpha_1}\beta_1)g_x((\nabla_X Y)_x,u) \\ & - (\alpha_1+\alpha_3)' g_x(X_x,Y_x) + [\frac{\beta_1(\beta_1+\beta_3)}{\alpha_1} \\ & - (\beta_1+\beta_3)']g_x(X_x,u)g_x(Y_x,u)\}u . \end{aligned} \tag{4.6}$$

Similar formulas can be obtained using the same formulas and some formulas from § 1, i.e.

$$\begin{aligned} 2G_{(x,u)}(\bar{\nabla}_{X^v}Y^h, Z^h) &= 2G_{(x,u)}(h\{T_{21}\}, Z^h(x,u)) , \\ 2G_{(x,u)}(\bar{\nabla}_{X^v}Y^h, Z^v) &= 2G_{(x,u)}(v\{T_{22}\}, Z^v(x,u)) , \\ 2G_{(x,u)}(\bar{\nabla}_{X^v}Y^v, Z^h) &= 2G_{(x,u)}(h\{T_{31}\}, Z^h(x,u)) , \\ 2G_{(x,u)}(\bar{\nabla}_{X^v}Y^v, Z^v) &= 2G_{(x,u)}(v\{T_{32}\}, Z^v(x,u)) , \end{aligned}$$

where T_{21}, T_{22}, T_{31} and T_{32} are given by

$$\begin{aligned}T_{21} = {} & \tfrac{1}{\alpha_1+\alpha_3}\{(\alpha_1+\alpha_3)' g_x(X_x,u)Y_x + \tfrac{\beta_1+\beta_3}{2} g_x(Y_x,u)X_x \\ & - \tfrac{\alpha_1}{2} R(X_x,u)Y_x + \tfrac{1}{2(\phi_1+\phi_3)}\{\alpha_1(\beta_1+\beta_3) g_x(R(X_x,u)Y_x,u) \\ & + (\alpha_1+\alpha_3)(\beta_1+\beta_3) g_x(X_x,Y_x) + [2(\alpha_1+\alpha_3)(\beta_1+\beta_3)' \\ & - (2(\alpha_1+\alpha_3)' + (\beta_1+\beta_3))(\beta_1+\beta_3)] g_x(X_x,u) g_x(Y_x,u)\}u\},\end{aligned} \tag{4.7}$$

$$T_{22} = \tfrac{1}{\alpha_1}(\alpha_2' - \tfrac{\beta_2}{2})[g_x(X_x,u)Y_x - \tfrac{1}{\phi_1} G_{(x,u)}(X^v,Y^v)u], \tag{4.8}$$

$$\begin{aligned}T_{31} = {} & \tfrac{1}{\alpha_1+\alpha_3}\{(\alpha_2' + \tfrac{\beta_2}{2})[g_x(Y_x,u)X_x + g_x(X_x,u)Y_x] \\ & + \tfrac{1}{\phi_1+\phi_3}[\beta_2(\alpha_1+\alpha_3) g_x(X_x,Y_x) + (2(\alpha_1+\alpha_3)\beta_2' \\ & - (\beta_1+\beta_3)(\beta_2+2\alpha_2')) g_x(X_x,u) g_x(Y_x,u)]u\},\end{aligned} \tag{4.9}$$

$$\begin{aligned}T_{32} = {} & \tfrac{1}{\alpha_1}\{\alpha_1'[g_x(Y_x,u)X_x + g_x(X_x,u)Y_x] + \tfrac{1}{\phi_1}[\alpha_1(\beta_1 \\ & - \alpha_1') g_x(X_x,Y_x) + (\alpha_1\beta_1' - 2\alpha_1'\beta_1) g_x(X_x,u) g_x(Y_x,u)]u\}.\end{aligned} \tag{4.10}$$

If we put $Q_1 = \bar{\nabla}_{X^h}Y^h$, $Q_2 = \bar{\nabla}_{X^v}Y^h$ and $Q_3 = \bar{\nabla}_{X^v}Y^v$, then we can write

$$Q_i = h\{T_{i1}\} + v\{T_{i2}\} + h\{A_i\} + v\{B_i\}; \quad i = 1,2,3.$$

From the equalities

$$G_{(x,u)}(Q_i, Z^h(x,u)) = G_{(x,u)}(h\{T_{i1}\}, Z^h(x,u)), \tag{4.11}$$

$$G_{(x,u)}(Q_i, Z^v(x,u)) = G_{(x,u)}(v\{T_{i2}\}, Z^v(x,u)), \tag{4.12}$$

we obtain the following identities

$$\begin{aligned}(\alpha_1+\alpha_3)A_i + \alpha_2 B_i = {} & -\alpha_2 T_{i2} - [(\beta_1+\beta_3) g_x(A_i,u) \\ & + \beta_2(g_x(B_i,u) + g_x(T_{i2},u))]u,\end{aligned} \tag{4.13}$$

$$\begin{aligned}\alpha_2 A_i + \alpha_1 B_i = {} & -\alpha_2 T_{i1} - [\beta_2(g_x(A_i,u) + g_x(T_{i1},u)) \\ & + \beta_1 g_x(B_i,u)]u.\end{aligned} \tag{4.14}$$

Letting $Z_x = u$ into the equations (4.11) and (4.12), we obtain

$$\begin{aligned}(\phi_1+\phi_3) g_x(A_i,u) + \phi_2 g_x(B_i,u) &= -\phi_2 g_x(T_{i2},u) \\ \phi_2 g_x(A_i,u) + \phi_1 g_x(B_i,u) &= -\phi_2 g_x(T_{i1},u).\end{aligned}$$

Consequently, we can write

$$g_x(A_i,u) = \frac{\phi_2}{\phi}[\phi_2 g_x(T_{i1},u) - \phi_1 g_x(T_{i2},u)], \tag{4.15}$$

$$g_x(B_i,u) = \frac{\phi_2}{\phi}[-(\phi_1+\phi_3) g_x(T_{i1},u) + \phi_2 g_x(T_{i2},u)]. \tag{4.16}$$

Substituting from (4.15)–(4.16) into (4.11) and (4.12), we obtain

$$\begin{cases}(\alpha_1+\alpha_3)A_i + \alpha_2 B_i = D_{i1}, \\ \alpha_2 A_i + \alpha_1 B_i = D_{i2}.\end{cases} \tag{4.17}$$

where D_{i1} and D_{i2} are given by

$$D_{i1} = -\alpha_2 T_{i2} - \frac{\alpha_2(\beta_1+\beta_3) - \beta_2(\alpha_1+\alpha_3)}{\phi}[\phi_2 g_x(T_{i1},u) - \phi_1 g_x(T_{i2},u)]u\,,$$

$$D_{i2} = -\alpha_2 T_{i1} - \frac{\alpha_1\beta_2 - \beta_1\alpha_2}{\phi}[(\phi_1+\phi_3) g_x(T_{i1},u) - \phi_2 g_x(T_{i2},u)]u\,.$$

The resolution of the system (4.17) gives by routine calculations the result. □

Remark 4.2. Note that when we take into account the orientation of M, general formulas of g-natural metrics on TM become larger (precisely for the dimensions 2 and 3 of M), as given explicitly in [15]. This yields very complicated formulas calculating the Levi-Civita connection of an arbitrary Riemannian g-natural metric. This question has been treated in detail in [4].

Now, among all Riemannian g-natural metrics on TM, we shall specify those with respect to which all the fibers of TM are totally geodesic.

Theorem 4.3. *Let (M,g) be a Riemannian manifold and G be a Riemannian g-natural metric on TM. The fibers of (TM,G) are totally geodesic if and only if there is a real constant c such that*

$$\left\{\begin{array}{l} \alpha_2(t) = \frac{c}{\sqrt{\phi_1(t)}}(t\cdot\alpha_1'(t) + \alpha_1(t)), \\ \beta_2(t) = \frac{c}{\sqrt{\phi_1(t)}}(\beta_1(t) - \alpha_1'(t))\,, \end{array}\right. \tag{4.18}$$

for all $t \in \mathbb{R}^+$.

Note that $c = 0$, in the system (4.18), corresponds to the case when horizontal and vertical distributions are orthogonal.

Proof. Remark first that the fibers of (TM,G) are totally geodesic if and only if $\bar{\nabla}_{X^v}X^v$ is vertical, for all $X \in \mathfrak{X}(M)$ (cf. [6], p.47). Hence, by virtue of Proposition 4.1, the fibers of (TM,G) are totally geodesic if and only if $E(u;X,X) = 0$, for all $X \in \mathfrak{X}(M)$. Since E is symmetric and linear in the second and third arguments, the last assertion is equivalent to $E(u;u,u) = 0$ and $E(u;X,X) = 0$, for all $u \in TM$ and $X \perp u$.

But, if $X \perp u$ then we have by virtue of Proposition 4.1,

$$E(u;X,X) = \frac{1}{\phi}(\phi_1\beta_2 - \phi_2(\beta_1 - \alpha_1'))\cdot g_x(X,X)\cdot u\,.$$

Hence, $E(u;X,X) = 0$, for all $u \in TM$ and $X \perp u$, is equivalent to

$$\phi_1\beta_2 = \phi_2(\beta_1 - \alpha_1')\,, \tag{4.19}$$

on $\mathbb{R}^{+*}$, and by continuity on $\mathbb{R}^+$.

On the other hand, we have for all $u \in TM$,

$$
\begin{aligned}
E(u;u,u) = {} & \tfrac{r^2}{\phi}\{\phi_1\beta_2 - \phi_2(\beta_1 - \alpha_1') + \tfrac{1}{\alpha}\{2\phi[\alpha_1(\alpha_2' + \tfrac{\beta_2}{2}) - \alpha_2\alpha_1'] \\
& + \alpha[2\phi_1\beta_2' - \phi_2\beta_1'] \cdot r^2 + 2\alpha_1' \cdot r^2[\alpha_1(\alpha_2(\beta_1+\beta_3) - \beta_2(\alpha_1+\alpha_3)) \\
& + \alpha_2(\beta_1(\phi_1+\phi_3) - \beta_2\phi_2)] + (2\alpha_2' + \beta_2)\cdot r^2[\alpha_1(\phi_2\beta_2 - \phi_1(\beta_1+\beta_3)) \\
& + \alpha_2(\alpha_1\beta_2 - \alpha_2\beta_1)]\}\}u \\
= {} & \tfrac{r^2}{\phi}\{\phi_1\beta_2 - \phi_2(\beta_1 - \alpha_1') + \tfrac{1}{\alpha}\{\alpha[2\phi_1\beta_2' - \phi_2\beta_1'] \cdot r^2 \\
& + 2\alpha_1'[\alpha_2(-\phi + \alpha_1(\beta_1+\beta_3)\cdot r^2 + (\phi_1+\phi_3)\beta_1 \cdot r^2 - \phi_2\beta_2 \cdot r^2) \\
& - \alpha_1(\alpha_1+\alpha_3)\beta_2 \cdot r^2] + (2\alpha_2' + \beta_2)[\alpha_1(\phi + (\phi_2\beta_2 \\
& - \phi_1(\beta_1+\beta_3))\cdot r^2) + \alpha_2(\alpha_1\beta_2 - \alpha_2\beta_1)\cdot r^2]\}\}u\,,
\end{aligned}
$$

where $r^2 = g(u,u)$. But

$$
\begin{aligned}
& \alpha_2(-\phi + \alpha_1(\beta_1+\beta_3)\cdot r^2 + (\phi_1+\phi_3)\beta_1 \cdot r^2 - \phi_2\beta_2 \cdot r^2) - \alpha_1(\alpha_1+\alpha_3)\beta_2 \cdot r^2 \\
& \quad = \alpha_2[(\phi_2^2 - \phi_2\beta_2 \cdot r^2) + ((\phi_1+\phi_3)\beta_1 \cdot r^2 - \phi_1(\phi_1+\phi_3)) + \alpha_1(\beta_1+\beta_3)\cdot r^2] \\
& \qquad - \alpha_1(\alpha_1+\alpha_3)\beta_2 \cdot r^2 \\
& \quad = \alpha_2[\phi_2\alpha_2 - \alpha_1(\phi_1+\phi_3) + \alpha_1(\beta_1+\beta_3)\cdot r^2] - \alpha_1(\alpha_1+\alpha_3)\beta_2 \cdot r^2 \\
& \quad = \alpha_2[\phi_2\alpha_2 - \alpha_1(\alpha_1+\alpha_3)] - \alpha_1(\alpha_1+\alpha_3)\beta_2 \cdot r^2 \\
& \quad = \alpha_2^2\phi_2 - \alpha_1(\alpha_1+\alpha_3)(\alpha_2 + \beta_2 \cdot r^2) \\
& \quad = -\alpha \cdot \phi_2\,.
\end{aligned}
$$

By similar way, we find that

$$\alpha_1(\phi + (\phi_2\beta_2 - \phi_1(\beta_1+\beta_3))\cdot r^2) + \alpha_2(\alpha_1\beta_2 - \alpha_2\beta_1)\cdot r^2 = \alpha \cdot \phi_1\,,$$

so that, we obtain

$$
\begin{aligned}
E(u;u,u) &= \tfrac{r^2}{\phi}\{\phi_1\beta_2 - \phi_2(\beta_1 - \alpha_1') + (2\phi_1\beta_2' - \phi_2\beta_1')\cdot r^2 \\
&\qquad - 2\phi_2\alpha_1' + \phi_1(2\alpha_2' + \beta_2)\}u \\
&= \tfrac{r^2}{\phi}\{2\phi_1(\beta_2 + \alpha_2' + \beta_2' \cdot r^2) - \phi_2(\beta_1 + \alpha_1' + \beta_1' \cdot r^2)\}u \\
&= \frac{r^2}{\phi}\{2\phi_1\phi_2' - \phi_2\phi_1'\}u\,.
\end{aligned}
$$

Hence, $E(u;u,u) = 0$, for all $u \in TM$, if and only if $2\phi_1\phi_2' - \phi_2\phi_1' = 0$ on $\mathbb{R}^{+*}$, and by continuity on $\mathbb{R}^+$.

We deduce that the fibers of (TM, G) are totally geodesic if and only if

$$
\begin{cases} \phi_1\beta_2 = \phi_2(\beta_1 - \alpha_1')\,, \\ 2\phi_1\phi_2' = \phi_2\phi_1'\,, \end{cases} \tag{4.20}
$$

on $\mathbb{R}^+$. Now, $(4.20)_2$ is equivalent to

$$
\begin{cases} \alpha_2(t) = \beta_2(t) = 0 \quad \text{whenever} \quad \phi_2(t) = 0\,, \\ \phi_2^2/\phi_1 \quad \text{is constant on each interval where} \quad \phi_2(t) \neq 0 \quad \text{everywhere}. \end{cases} \tag{4.21}
$$

Denote by J the complement of $\phi_2^{-1}(0)$ in $\mathbb{R}^+$. J is an open subset of $\mathbb{R}^+$. We claim that, in the conditions of (4.21), either $J = \emptyset$ or $J = \mathbb{R}^+$. If not, there is $0 < a < b$ such that $]a, b[\, \subset J$ (since J is open) and $a \notin J$. Then there is a constant $d > 0$ such that $\phi_1 = d \cdot \phi_2^2$ on $]a, b[$. When $t \to a$, we have by continuity of ϕ_1 and ϕ_2, $\phi_1(a) = d \cdot \phi_2^2(a)$. Since $a \notin J$, then $\phi_1(a) = 0$, which contradicts the fact that G is Riemannian (Proposition 2.8). We deduce that either $J = \emptyset$ or $J = \mathbb{R}^+$. Hence, (4.21) is equivalent to

$$\begin{cases} \text{either} \quad \alpha_2 = \beta_2 = 0 \quad \text{on} \quad \mathbb{R}^+, \\ \quad \text{or} \quad\ \ \phi_2^2/\phi_1 \quad \text{is constant on} \quad \mathbb{R}^+\,, \end{cases}$$

or equivalently,

$$\phi_2^2/\phi_1 \quad \text{is a constant on} \quad \mathbb{R}^+\,. \tag{4.22}$$

Hence (4.20) holds if and only if

$$\begin{cases} \phi_1\beta_2 = \phi_2(\beta_1 - \alpha_1')\,, \\ \phi_2 = c \cdot \sqrt{\phi_1}\,, \end{cases} \tag{4.23}$$

on $\mathbb{R}^+$, where c is a constant, or equivalently,

$$\begin{cases} \beta_2 = \frac{c}{\sqrt{\phi_1}}(\beta_1 - \alpha_1')\,, \\ \alpha_2 = \phi_2 - t \cdot \beta_2 = \frac{c}{\sqrt{\phi_1}}(t \cdot \alpha_1' + \alpha_1)\,, \end{cases} \tag{4.24}$$

on $\mathbb{R}^+$. This completes the proof. □

Remark 4.4. As other applications of Proposition 4.1, we can check the following assertions:

1) Let (M, g) be a Riemannian manifold and G be a Riemannian g-natural metric on TM. Then M, considered as an embedded submanifold of TM by the zero section, is always totally geodesic.

2) Among all Riemannian g-natural metrics on the tangent bundle TM of a Riemannian manifold (M, g), the only ones with respect to which the vertical lift preserves the parallelism of vector fields on M are those which belong to the 2-dimensional cone $\mathcal{C} = \{a.g^s + b.g^h + c.g^v,\ c > 0, a > 0, b^2 - c(a + c) < 0\}$ in the 3-dimensional real vector space generated by the three classical lifts g^s, g^h and g^v of g.

Note that with respect to any element of $\mathcal{C}$, the horizontal lift also preserves the parallelism of vector fields on M.

5. The geodesic flow in TM and incompressibility

Let (M, g) be a Riemannian manifold, and G be a Riemannian g-natural metric on TM. In this section, we study the situations when the geodesic flow in TM is incompressible with respect to G.

Let $\{(U; x^i, i = 1, \dots, m)\}$ be a local coordinate system in M and $\{(p^{-1}(U); x^i, u^i,\ i = 1, \dots, m)\}$ the induced local coordinate system in TM. Let $\{\Gamma_{ij}^k; i, j, k = 1, \dots, m\}$ and $\{\bar{\Gamma}_{IJ}^K; I, J, K = 1, \dots, 2m\}$ be the Riemann-Christoffel symbols of (M, g) and (TM, G) respectively. If T is an F-tensor field on M of type $(2, 1)$, then we denote by $T(u)^k_{ij}$ $(1 \leq i,j \leq m)$ the components of the $(2,1)$-tensor on M_x determined by the bilinear mapping $T(u;\,.\,,\,.\,) : M_x \times M_x \to M_x$, i.e. $T(u)^k_{ij} = dx^k[T(u; \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})]$, $(1 \leq i,j \leq m)$. Now, the expressions of the identities of Proposition 4.1 in local coordinates yield the following:

Lemma 5.1. *The Riemann-Christoffel symbols of (TM, G) are given by*

$$\begin{aligned}
\bar{\Gamma}^k_{m+i\,m+j}(u) &= E(u)^k_{ij}\,,\\
\bar{\Gamma}^{m+k}_{m+i\,m+j}(u) &= F(u)^k_{ij} - \Gamma^k_{\lambda\mu}(x)E(u)^\lambda_{ij}u^\mu\,,\\
\bar{\Gamma}^k_{i\,m+j}(u) &= \Gamma^\lambda_{i\mu}(x)\bar{\Gamma}^k_{m+\lambda\,m+j}(u)u^\mu + C(u)^k_{ij}\,,\\
\bar{\Gamma}^{m+k}_{i\,m+j}(u) &= \Gamma^\lambda_{i\mu}(x)\bar{\Gamma}^{m+k}_{m+\lambda\,m+j}(u)u^\mu + \Gamma^k_{ij}(x) - \Gamma^k_{l\mu}(x)C(u)^l_{ij}u^\mu\,,\\
\bar{\Gamma}^k_{ij}(u) &= \Gamma^k_{ij}(x) + A(u)^k_{ij} + \Gamma^l_{i\mu}(x)\bar{\Gamma}^k_{m+l\,j}(u)u^\mu\\
&\quad + \Gamma^l_{j\mu}(x)\bar{\Gamma}^k_{m+l\,i}(u)u^\mu - \Gamma^l_{i\mu}(x)\Gamma^t_{j\lambda}(x)\bar{\Gamma}^k_{m+l\,m+t}(u)u^\lambda u^\mu\,,\\
\bar{\Gamma}^{m+k}_{ij}(u) &= -\Gamma^k_{\lambda\mu}(x)\Gamma^\lambda_{ij}(x)u^\mu - \Gamma^l_{i\mu}(x)\Gamma^k_{jl}(x)u^\mu - \Gamma^k_{\lambda\mu}(x)A(u)^\lambda_{ij}u^\mu\\
&\quad + B(u)^k_{ij} + \frac{\partial\Gamma^k_{j\mu}}{\partial x^i}(x)u^\mu + \Gamma^l_{i\mu}(x)\bar{\Gamma}^{m+k}_{m+l\,j}(u)u^\mu\\
&\quad + \Gamma^l_{j\mu}(x)\bar{\Gamma}^{m+k}_{m+l\,i}(u)u^\mu - \Gamma^l_{i\mu}(x)\Gamma^t_{j\lambda}(x)\bar{\Gamma}^{m+k}_{m+l\,m+t}(u)u^\lambda u^\mu\,.
\end{aligned}$$

for all $x \in U$ and $u \in p^{-1}(x)$.

Note that we have been using, and we will be, the so-called Einstein's summation. Now, our main result in this section is:

Theorem 5.2. *Let (M,g) be a Riemannian manifold and G be a g-natural metric on the tangent bundle TM. Then the geodesic flow of (M,g) is incompressible with respect to G if and only if the following conditions are satisfied:*

(i) *$\phi_1 + \phi_3$ is constant on each interval where $\alpha_2 = 0$ and $\beta_2 \neq 0$ everywhere;*

(ii) *$\mathrm{Ric}_x(u,u) = \theta(r^2)g_x(u,u)$ whenever $\alpha_2(r^2) \neq 0$;*

where θ is the function defined from $\mathbb{R}^+ \setminus [(\alpha_2)^{-1}(0)]$ to $\mathbb{R}$ by

$$\theta = \frac{1}{\alpha_1}\{(m-1)(\beta_1+\beta_3) + \frac{2\phi_2\alpha}{\alpha_2\phi}(\phi_1+\phi_3)'\}, \tag{5.1}$$

$r^2 = g_x(u,u)$ and Ric *is the Ricci tensor field on (M,g).*

Proof. Let ξ be the geodesic flow vector of (M,g). It is a vector field on TM which is locally expressed as $\xi^i = u^i$, $\xi^{m+i} = -\Gamma^k_{ij}u^ju^k$. We shall compute the divergence of ξ, $\mathrm{div}_G\xi$, relatively to the metric G on TM. By the definition of the divergence, we have

$$\begin{aligned}
(\mathrm{div}_G\xi)_{(x,u)} &= \tfrac{\partial \xi^i}{\partial x^i} + \tfrac{\partial \xi^{m+i}}{\partial u^i} + \bar{\Gamma}^I_{IJ}\xi^J \\
&= -2\cdot\Gamma^i_{ij} + (\bar{\Gamma}^i_{ij} + \bar{\Gamma}^{m+i}_{m+ij})u^j - (\bar{\Gamma}^i_{im+j} + \bar{\Gamma}^{m+i}_{m+im+j})\Gamma^j_{kl}u^k u^l \\
&= [\Gamma^i_{ij} + A(u)^i_{ji} + \Gamma^l_{i\mu}\Gamma^\lambda_{j\nu}\bar{\Gamma}^i_{m+\lambda m+l}u^\mu u^\nu + \Gamma^l_{i\mu}C(u)^i_{jl}u^\mu \\
&\quad + \Gamma^l_{i\mu}\Gamma^\lambda_{j\nu}\bar{\Gamma}^i_{m+\lambda m+l}u^\mu u^\nu + \Gamma^l_{j\mu}C(u)^i_{il}u^\mu - \Gamma^l_{i\mu}\Gamma^\lambda_{j\nu}\bar{\Gamma}^i_{m+\lambda m+l}u^\mu u^\nu \\
&\quad + \Gamma^l_{j\mu}\Gamma^{m+i}_{m+im+l}u^\mu + \Gamma^i_{ij} - \Gamma^i_{l\mu}C(u)^l_{ji}u^\mu]u^j \\
&\quad - [\Gamma^l_{i\mu}\Gamma^j_{kl}\bar{\Gamma}^i_{m+lm+j}u^k u^l u^\mu + \Gamma^j_{kl}C(u)^i_{ij}u^k u^l + \Gamma^j_{kl}\Gamma^{m+i}_{m+im+j}u^k u^l] \\
&= -2\cdot\Gamma^i_{ij}u^j + [2\cdot\Gamma^i_{ij} + A(u)^i_{ji}]\cdot u^j \\
&= A(u)^i_{ji}\cdot u^j,
\end{aligned}$$

so that $(\mathrm{div}_G\xi)_{(x,u)}$ is the trace of the endomorphism of M_x given by $X \to A(u;u,X)$. But

$$\begin{aligned}
A(u;u,X) &= \tfrac{1}{\alpha}\{-\tfrac{\alpha_1\alpha_2}{2}R(X,u)u + \tfrac{\alpha_2(\beta_1+\beta_3)}{2}g_x(u,u)X + \tfrac{1}{\phi}\{\phi\tfrac{\alpha_2(\beta_1+\beta_3)}{2} \\
&\quad + \phi_2\alpha(\alpha_1+\alpha_3)' + \alpha\phi_2(\beta_1+\beta_3)'\cdot r^2 + (\beta_1+\beta_3)\cdot r^2[\alpha_2(\phi_2\beta_2 \\
&\quad - \phi_1(\beta_1+\beta_3)) + (\alpha_1+\alpha_3)(\phi_1\beta_2 - \phi_2\beta_1)]\}g_x(X,u)u\}\,,
\end{aligned}$$

so that

$$\begin{aligned}
A(u)^i_{ij} &= \tfrac{1}{\alpha}\{-\tfrac{\alpha_1\alpha_2}{2}R^i_{jik}u^j u^k + m\cdot\tfrac{\alpha_2(\beta_1+\beta_3)}{2}\cdot r^2 + \tfrac{1}{\phi}\{\phi\tfrac{\alpha_2(\beta_1+\beta_3)}{2} \\
&\quad + \phi_2\alpha(\alpha_1+\alpha_3)' + \alpha\phi_2(\beta_1+\beta_3)'\cdot r^2 + (\beta_1+\beta_3)\cdot r^2[\alpha_2(\phi_2\beta_2 \\
&\quad - \phi_1(\beta_1+\beta_3)) + (\alpha_1+\alpha_3)(\phi_1\beta_2-\phi_2\beta_1)]\}r^2\} \\
&= \tfrac{1}{\alpha}\{-\tfrac{\alpha_1\alpha_2}{2}R_{jk}u^j u^k + (m-1)\cdot\tfrac{\alpha_2(\beta_1+\beta_3)}{2}\cdot r^2 + \tfrac{r^2}{\phi}\{\phi_2\alpha[(\alpha_1+\alpha_3)' \\
&\quad + (\beta_1+\beta_3)'\cdot r^2] + \alpha_2(\beta_1+\beta_3)[\phi + \phi_2\beta_2\cdot r^2 - \phi_1(\beta_1+\beta_3)\cdot r^2 \\
&\quad - (\alpha_1+\alpha_3)\beta_1\cdot r^2] + \alpha_1(\alpha_1+\alpha_3)\beta_2(\beta_1+\beta_3)\cdot r^2\}\} \\
&= \tfrac{1}{\alpha}\{-\tfrac{\alpha_1\alpha_2}{2}R_{jk}u^j u^k + (m-1)\cdot\tfrac{\alpha_2(\beta_1+\beta_3)}{2}\cdot r^2 + \tfrac{r^2}{\phi}\{\phi_2\alpha[(\alpha_1+\alpha_3)' \\
&\quad + (\beta_1+\beta_3)'\cdot r^2] + \alpha_2(\beta_1+\beta_3)[\alpha_1(\alpha_1+\alpha_3) - \alpha_2\phi_2] \\
&\quad + \alpha_1(\alpha_1+\alpha_3)\beta_2(\beta_1+\beta_3)\cdot r^2\}\} \\
&= \tfrac{1}{\alpha}\{-\tfrac{\alpha_1\alpha_2}{2}R_{jk}u^j u^k + (m-1)\cdot\tfrac{\alpha_2(\beta_1+\beta_3)}{2}\cdot r^2 + \tfrac{r^2}{\phi}\{\phi_2\alpha[(\alpha_1+\alpha_3)' \\
&\quad + (\beta_1+\beta_3)'\cdot r^2] + \alpha_1(\alpha_1+\alpha_3)(\beta_1+\beta_3)\phi_2 - \alpha_2^2\phi_2(\beta_1+\beta_3)\}\} \\
&= -\tfrac{\alpha_1\alpha_2}{2\alpha}R_{jk}u^j u^k + \{(m-1)\cdot\tfrac{\alpha_2(\beta_1+\beta_3)}{2\alpha} + \tfrac{\phi_2(\phi_1+\phi_3)'}{\phi}\}\cdot r^2\,,
\end{aligned}$$

where R_{jk} denote the components of the Ricci tensor field on M. We deduce then that

$$\begin{aligned}
(5.2)\quad & (\mathrm{div}_G\xi)_{(x,u)} \\
&= -\frac{\alpha_1\alpha_2}{2\alpha}\mathrm{Ric}_x(u,u) + \{\frac{\phi_2(\phi_1+\phi_3)'}{\phi} + \frac{(m-1)\alpha_2(\beta_1+\beta_3)}{2\alpha}\}g_x(u,u)\,.
\end{aligned}$$

Now, $(\mathrm{div}_G \xi) \equiv 0$ if and only if

$$\begin{cases} \frac{\beta_2 \cdot r^2}{\phi} \cdot (\phi_1 + \phi_3)'(r^2) = 0 & \text{if} \quad \alpha_2(r^2) = 0\,, \\ \mathrm{Ric}_x(u,u) = \theta(r^2) g_x(u,u) & \text{if} \quad \alpha_2(r^2) \neq 0\,, \end{cases}$$

or equivalently,

$$\begin{cases} (\phi_1 + \phi_3)'(r^2) = 0 & \text{if} \quad \alpha_2(r^2) = 0\,, \quad \beta_2(r^2) \neq 0 \quad \text{and} \quad r \neq 0\,, \\ \mathrm{Ric}_x(u,u) = \theta(r^2) g_x(u,u) & \text{if} \quad \alpha_2(r^2) \neq 0\,. \end{cases}$$

Using the continuity of $(\phi_1 + \phi_3)'$ at 0, the last system is equivalent to

$$\begin{cases} (\phi_1 + \phi_3)' = 0 & \text{whenever} \quad \alpha_2 = 0 \quad \text{and} \quad \beta_2 \neq 0, \\ \mathrm{Ric}_x(u,u) = \theta(r^2) g_x(u,u) & \text{if} \quad \alpha_2(r^2) \neq 0\,. \end{cases}$$

□

Corollary 5.3. *Let (M,g) be a Riemannian manifold and G be a g-natural metric on the tangent bundle TM, with respect to which horizontal and vertical distributions are orthogonal. Then the geodesic flow of (M,g) is incompressible with respect to G.*

Proof. According to (2.2), the orthogonality of the horizontal and vertical distributions is equivalent to the vanishing of the functions α_2 and β_2 identically. Since the conditions (i) and (ii) of Proposition 5.2 deal with values where β_2 and α_2 don't vanish respectively, then they are automatically satisfied. This completes the proof. □

Corollary 5.4. *Let (M,g) be a Riemannian manifold and G be a g-natural metric on the tangent bundle TM, such that $I_0 = \mathbb{R}^+ \setminus [(\alpha_2)^{-1}(0)]$ is dense in $\mathbb{R}^+$. Then the geodesic flow of (M,g) is incompressible with respect to G if and only if the function θ of Theorem 5.2 is a constant θ_0 on I_0 and (M,g) is an Einstein manifold with* $\mathrm{Ric} = \theta_0 g$.

Proof. Fix $x \in M$, $t_0 \in I_0 \setminus \{0\}$ and $u_0 \in M_x$ such that $g_x(u_0,u_0) = t_0$. Suppose that ξ is incompressible with respect to G. Then from Theorem 5.2,

$$\mathrm{Ric}_x(\lambda u_0, \lambda u_0) = \theta(\lambda^2 t_0) g_x(\lambda u_0, \lambda u_0)\,,$$

for all λ such that $\lambda^2 t_0 \in I_0$. By virtue of bilinearity of Ric_x and g_x, we see that $\theta(t) = \theta(t_0) =: \theta_0$, for all $t \in I_0$. Furthermore,

$$\mathrm{Ric}_x(tu_0, tu_0) = \theta_0 g_x(tu_0, tu_0)\,, \tag{5.3}$$

for all $t \in I_0$. Since I_0 is dense in $\mathbb{R}^+$, then by continuity, (5.3) is valid for all $t \in \mathbb{R}$. Now, x and u_0 being arbitrary and since θ depends only on the norms of vectors under consideration, $\mathrm{Ric}_x(u,u) = \theta_0 g_x(u,u)$, for all $(x,u) \in TM$. Using once again the bilinearity and the symmetry of Ric_x and g_x, we have $\mathrm{Ric} = \theta_0 g$.

Conversely, if θ is a constant θ_0 on I_0 and $\mathrm{Ric} = \theta_0 g$, then we have, by (5.2), $(\mathrm{div}_G)_{(x,u)} = 0$, for all (x,u) such that $g_x(u,u) \in I_0$. Fix t_0 and u_0 as before. Then

$(\mathrm{div}_G)_{(x,\lambda u_0)} = 0$, for all λ such that $\lambda^2 t_0 \in I_0$. Since $\{(x, \lambda u_0)/\lambda^2 t_0 \in I_0\}$ is dense in $\{(x, \lambda u_0)/\lambda \in \mathbb{R}\}$, the continuity of div_G implies that $(\mathrm{div}_G)_{(x,\lambda u_0)} = 0$, for all $\lambda \in \mathbb{R}$. u_0 and x being arbitrary, we have $(\mathrm{div}_G)_{(x,u)} = 0$, for all $(x, u) \in TM$. □

Remark 5.5. If $I_0 = \mathbb{R}^+ \setminus [(\alpha_2)^{-1}(0)]$ is dense in $\mathbb{R}^+$ and θ is constant on I_0, then Corollary 5.4 shows that the metric G presents a certain kind of rigidity. For example, if we consider G Riemannian such that

(a) α_1, α_3 and $\beta_1 + \beta_3$ are constant on $\mathbb{R}^+$;
(b) α_2 doesn't vanish on $\mathbb{R}^+$, except may be at isolated points of $\mathbb{R}^+$,

then $\theta = 0$ on I_0. It follows from Corollary 5.4 that the geodesic flow of (M, g) is incompressible with respect to G if and only if (M, g) is an Einstein manifold with vanishing scalar curvature.

As examples of such Riemannian g-natural metrics G, we mention:

- the elements $G = a \cdot g^s + b \cdot g^h + c \cdot g^v$ of $\mathcal{C}$, such that $b \neq 0$;
- $G = a \cdot g^s + \alpha_2 \cdot g^h + c \cdot g^v$, where $\alpha_2 = \sqrt{a(a + c)} \cdot \sin \circ \rho$ on $\mathbb{R}^+$, ρ being any function on $\mathbb{R}^+$ and $a, a + c > 0$.

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M. T. H. K. Abbassi
Département des Mathématiques, Faculté des sciences Dhar El Mahraz
Université Sidi Mohamed Ben Abdallah
B. P. 1796, Fes-Atlas, Fes, Morocco
E-mail: mtk_abbassi@Yahoo.fr

M. Sarih
Département des Mathématiques et Informatique
Faculté des sciences et techniques de Settat
Université Hassan 1^{er}
B. P. 577, 26000 Morocco