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FIXED POINTS THEOREMS OF NON-EXPANDING FUZZY MULTIFUNCTIONS

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Abstract. We prove the existence of a fixed point of non-expanding fuzzy multifunctions in α-fuzzy preordered sets. Furthermore, we establish the existence of least and minimal fixed points of non-expanding fuzzy multifunctions in α-fuzzy ordered sets.

1. Introduction

In [19], Zadeh introduced the notion of fuzzy order and similarity, which was investigated by several authors (see [1, 3, 7, 13]). During the last few decades many authors have established the existence of a lots of fixed point theorems in fuzzy setting: Beg [2, 4], Bose and Sahani [6], Fang [8], Hadzic [9], Heilpern [10], Kaleva [11] and the present author [13, 14, 15, 16]. The aim of this paper is to study the existence of fixed points of non-expanding fuzzy multifunctions in α-fuzzy setting.

Let X be a nonempty crisp set, with generic element of X denoted by x. A fuzzy subset A of X is characterized by its membership function $\mu_A : X \to [0,1]$ and $\mu_A(x)$ is interpreted as the degree of membership of element x in fuzzy subset A for each $x \in X$. Let A and B be two fuzzy subsets of X. We say that A is included in B and we write $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$, for all $x \in X$. In particular, if $x \in X$ and $\mu_A(x) = 1$, then $\{x\} \subseteq A$.

Let X be a nonempty crisp set and $\alpha \in]0,1]$. An α-fuzzy preorder relation on X is a fuzzy subset r_α of $X \times X$ satisfying the following two properties:

(i) for all $x \in X$, $r_\alpha(x,x) = \alpha$,

(ii) for all $x, y \in X$, $r_\alpha(x,y) + r_\alpha(y,x) > \alpha$ implies $x = y$.

A nonempty set X with an α-fuzzy preorder r_α defined on it, is called an α-fuzzy preorder and we denote it by (X, r_α).

An α-fuzzy preordered set (X, r_α) is called an α-fuzzy ordered set (see [14]) if

(iii) for all $x, z \in X, r_\alpha(x,z) \geq \sup\limits_{y\in X} [\inf\{r_\alpha(x,y), r_\alpha(y,z)\}]$.

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Let (X, r_α) be a nonempty α-fuzzy preordered set. A fuzzy multifunction $T : X \to [0,1]^X \setminus \{\emptyset\}$ is called non-expanding if for every $x \in X$ there exists $y \in X$ such that $\{y\} \subseteq T(x)$ and $r_\alpha(y, x) > \frac{\alpha}{2}$.

In the third section of this paper, we first prove the following result (Theorem 3.1): if (X, r_α) is a nonempty α-fuzzy preordered complete set and $T : X \to [0,1]^X \setminus \{\emptyset\}$ is a non-expanding fuzzy multifunction, then T has a fixed point.

Secondly, we establish the existence of least and minimal fixed points of non--expanding fuzzy multifunctions in α-fuzzy ordered sets (Theorems 3.3 and 3.5). As consequences we obtain some fixed point theorems for non-expanding maps.

2. Preliminaries

In order to establish our main results, we give some concepts and results.

Definition 2.1. Let (X, r_α) be an α-fuzzy preordered set. Then

(a) The α-fuzzy preorder r_α is said to be total if for all $x \neq y$ we have either $r_\alpha(x, y) > \frac{\alpha}{2}$ or $r_\alpha(y, x) > \frac{\alpha}{2}$. An α-fuzzy ordered set on which fuzzy order is total is called r_α-fuzzy chain.

(b) Let A be a subset of X. An element $l \in X$ is a r_α-lower bound of A if $r_\alpha(l, y) > \frac{\alpha}{2}$ for all $y \in A$. If l is a r_α-lower bound of A and $l \in A$, then l is called a least element of A. Similarly, we can define r_α-upper bounds and greatest elements of A.

(c) An element m of A is called a minimal element of A if $r_\alpha(y, m) > \frac{\alpha}{2}$ for some $y \in A$, then $y = m$. Maximal elements are defined analogously.

Let A be a nonempty subset of X. Then,

$$\sup_{r_\alpha}(A) = \text{ the least element of } r_\alpha\text{-upper bounds of } A \text{ (if it exists)},$$

and

$$\inf_{r_\alpha}(A) = \text{ the greatest element of } r_\alpha\text{-lower bounds of } A \text{ (if it exists)}.$$

Definition 2.2. Let (X, r_α) be a nonempty α-fuzzy preordered set. A map $f : X \to X$ is called non-expanding if for every $x \in X$, $r_\alpha(f(x), x) > \frac{\alpha}{2}$.

An element x of X is called a fixed point of a map $f : X \to X$ if $f(x) = x$. We denote by $\operatorname{Fix}(f)$ the set of all fixed points of f.

Definition 2.3. Let (X, r_α) be a nonempty α-fuzzy preordered set and let (x_β) be a family of X. We say that (x_β) is an α-fuzzy decreasing family if $r_\alpha(x_{\beta+1}, x_\beta) > \frac{\alpha}{2}$.

Definition 2.4. A nonempty α-fuzzy preordered set (X, r_α) is said to be an α-fuzzy ordered complete set if r_α is total and for every decreasing family (x_β) of X, $\inf_{r_\alpha}(x_\beta)$ exists in X.

Let X be a nonempty crisp set. A fuzzy multifunction is any map $T : X \to [0,1]^X \setminus \{\emptyset\}$ such that for every $x \in X$, $T(x)$ is a nonempty fuzzy subset of X.

An element x of X is called a fixed point of a fuzzy multifunction $T : X \to [0,1]^X \setminus \{\emptyset\}$ if $\{x\} \subseteq T(x)$. We denote by $\mathcal{F}_T$ the set of all fixed points of T.

Definition 2.5 ([13]). Let (X, r_α) be an α-fuzzy ordered set. The inverse fuzzy relation s_α of r_α is defined by $s_\alpha(x, y) = r_\alpha(y, x)$, for all $x, y \in X$.

In [13], we established the following results.

Lemma 2.6 ([13, Lem. 3.6]). *Let (X, r_α) be a nonempty α-fuzzy ordered set. If every nonempty r_α-fuzzy chain has a r_α-upper bound, then X has a maximal element.*

Lemma 2.7 ([13, Prop. 3.6]). *Let (X, r_α) be a nonempty α-fuzzy ordered set and let s_α be the inverse α-fuzzy relation of r_α. Then,*

(i) *The α-fuzzy relation s_α is an α-fuzzy order on X.*

(ii) *If every nonempty r_α-fuzzy chain has a r_α-infimum, then every nonempty s_α-fuzzy chain has a r_α-supremum.*

3. Main results

We begin this section by proving the existence of fixed point of non-expanding fuzzy multifunctions. More precisely, we shall show the following:

Theorem 3.1. *Let (X, r_α) be a nonempty α-fuzzy preordered complete set and let $T : X \to [0, 1]^X \setminus \{\emptyset\}$ be a non-expanding fuzzy multifunction. Then, T has a fixed point.*

Proof. Let (X, r_α) be a nonempty α-fuzzy preordered complete set and let $T : X \to [0, 1]^X \setminus \{\emptyset\}$ be an expanding fuzzy multifunction. Assume that T has no fixed point and let x_0 be a given element of X.

Next, we shall produce an α-fuzzy decreasing family (x_β) of X where β is an ordinal as follows:

(i) First case: if $\beta = 0$, then the element x_0 is given by our hypothesis.

(ii) Second case: β is a nonzero non limit ordinal. Since T is an expanding fuzzy multifunction and r_α is total, then for $x_{\beta-1}$ there is $x_\beta \in X$ such that

$$\begin{cases} \{x_\beta\} \subseteq T(x_{\beta-1}) \\ \alpha > r_\alpha(x_\beta, x_{\beta-1}) > \frac{\alpha}{2} \,. \end{cases}$$

(iii) Third case: β is a limit ordinal. As (X, r_α) is an α-fuzzy ordered complete set, hence we have

$$x_\beta = \inf_{r_\alpha}\{x_\gamma : \gamma < \beta\} \,.$$

It follows that if β and γ are two ordinals with $\beta \neq \gamma$, then we have $x_\beta \neq x_\gamma$.

Now, we shall defining an ordinal valued function G by assign to every $x \in X$, an ordinal $G(x)$ as follows:

$$G(x) = \begin{cases} \beta & \text{if } x = x_\beta \\ 0 & \text{otherwise} \,. \end{cases}$$

Therefore, the range of G is the set of all ordinals. From ZF Axioms of substitution [12, page 261], we conclude that the range of G is a set. That is a contradiction. Therefore, T has a fixed point. ∎

As an application of Theorem 3.1, we obtain the following:

Corollary 3.2. *Let (X, r_α) be a nonempty α-fuzzy preordered complete set and let $f : X \to X$ be a non-expanding map. Then, f has a fixed point.*

For the existence of the least fixed point of non-expanding fuzzy multifunctions, we shall show the following:

Theorem 3.3. *Let (X, r_α) be a nonempty α-fuzzy ordered set with a least element l and let $T : X \to [0,1]^X \setminus \{\emptyset\}$ be a non-expanding fuzzy multifunction. Then the set $\mathcal{F}_T$ of all fixed points of T is nonempty and l is the least element of $\mathcal{F}_T$.*

Proof. Let (X, r_α) be a nonempty α-fuzzy ordered set with a least element l and let $T : X \to [0,1]^X \setminus \{\emptyset\}$ be a non-expanding fuzzy multifunction. Since T is an non-expanding fuzzy multifunction, there exists an element x of X such that $\{x\} \subseteq T(l)$ and $r_\alpha(x, l) > \frac{\alpha}{2}$. As $l = \inf_{r_\alpha}(X)$, then $r_\alpha(l, x) > \frac{\alpha}{2}$. Hence, $r_\alpha(x, l) + r_\alpha(l, x) > \alpha$. Therefore, $x = l$. So l is fixed point of T. On the other hand, l is the least element of X. Therefore, we deduce that l is the least fixed point of T. $\square$

As a consequence of Theorem 3.3, we have:

Corollary 3.4. *Let (X, r_α) be a nonempty α-fuzzy ordered set with a least element l and $f : X \to X$ be a non-expanding map. Then, the set $\mathrm{Fix}\,(f)$ of all fixed points of f is nonempty and l is the least element of $\mathrm{Fix}\,(f)$.*

Next, we shall establish the existence of a minimal fixed point of non-expanding fuzzy multifunctions.

Theorem 3.5. *Let (X, r_α) be a nonempty α-fuzzy ordered set with the property that every nonempty r_α-fuzzy chain has a r_α-infimum. Let $T : X \to [0,1]^X \setminus \{\emptyset\}$ be a non-expanding r_α-fuzzy multifunction. Then, the set $\mathcal{F}_T$ of all fixed points of T is nonempty and has a minimal element.*

To prove Theorem 3.5, we shall need the following lemma.

Lemma 3.6. *Let (X, r_α) be a nonempty α-fuzzy ordered set with the property that every nonempty r_α-fuzzy chain has a r_α-infimum. Then, X has a minimal element.*

Proof. Let (X, r_α) be a nonempty α-fuzzy ordered set with the property that every nonempty r_α-fuzzy chain has a r_α-infimum. Let s_α be the α-fuzzy inverse order relation of r_α. From Lemma 2.7, every nonempty r_α-fuzzy chain has a s_α-supremum. Then, by Lemma 2.6, X has a maximal element m (say) in (X, s_α). Let x be an element of X such that $r_\alpha(x, m) > \frac{\alpha}{2}$. Then, $s_\alpha(m, x) > \frac{\alpha}{2}$. Since m is a maximal element in (X, s_α), hence $x = m$. Therefore, m is a minimal element in (X, r_α). $\square$

Now we are ready to give the proof of Theorem 3.5.

Proof of Theorem 3.5. Let (X, r_α) be a nonempty α-fuzzy ordered set with the property that every nonempty r_α-fuzzy chain has a r_α-infimum and let

$T : X \to [0,1]^X \setminus \{\emptyset\}$ be a non-expanding fuzzy multifunction. By using Lemma 3.6, we deduce that X has a minimal element m (say). As T is a non-expanding fuzzy multifunction, so there is an element x of X such that $\{x\} \subseteq T(m)$ and $r_\alpha(x,m) > \frac{\alpha}{2}$. Since m is a minimal element of X, then $x = m$. Thus, m is a fixed point of T. Using the fact that m is a minimal element of X, we conclude that m is a minimal fixed point of T. $\square$

By using Theorem 3.5, we get:

Corollary 3.7. *Let (X, r_α) be a nonempty α-fuzzy ordered set with the property that every nonempty r_α-fuzzy chain has a r_α-infimum and let $f : X \to X$ be a non-expanding map. Then, the set of all fixed points of f is nonempty and has a minimal element.*

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