ARCHIVUM MATHEMATICUM (BRNO)
Tomus 41 (2005), 123 – 128

SINGULAR SOLUTIONS FOR THE DIFFERENTIAL EQUATION WITH p-LAPLACIAN

MIROSLAV BARTUŠEK

Abstract. In the paper a sufficient condition for all solutions of the differential equation with p-Laplacian to be proper. Examples of super-half-linear and sub-half-linear equations $(|y'|^{p-1}y')' + r(t)|y|^{\lambda} \operatorname{sgn} y = 0$, $r > 0$ are given for which singular solutions exist (for any $p > 0$, $\lambda > 0$, $p \neq \lambda$).

Consider the differential equation with p-Laplacian

$$\big(a(t)|y'|^{p-1}y'\big)' + r(t)f(y) = 0 \tag{1}$$

where $p > 0$, $a \in C^0(R_+)$, $r \in C^0(R_+)$, $f \in C^0(R)$, $R_+ = [0,\infty)$, $R = (-\infty,\infty)$ and

$$a > 0,\ r \geq 0 \quad \text{on} \quad R_+, f(x)x \geq 0 \quad \text{on} \quad R\,. \tag{2}$$

A solution y of (1) is called proper if it is defined on R_+ and $\sup_{t\in[\tau,\infty)} |y(t)| > 0$ for every $\tau \in (0,\infty)$. It is called singular of the first kind if it is defined on R_+, there exists $\tau \in (0,\infty)$ such that $y \equiv 0$ on $[\tau,\infty)$ and $\sup_{T\leq t<\tau} |y(t)| > 0$ for every $T \in [0,\tau)$. It is called singular of the second kind if it is defined on $[0,\tau)$, $\tau < \infty$ and $\sup_{0\leq t<\tau} |y'(t)| = \infty$. A singular solution y is called oscillatory if there exists a sequence of its zeros $\{t_k\}_1^\infty$, $t_k \in [0,\tau)$ tending to τ.

Eq. (1) and its special case

$$\big(a(t)|y'|^{p-1}y'\big)' + r(t)|y|^{\lambda} \operatorname{sgn} y = 0 \tag{3}$$

where $\lambda > 0$ is studied by many authors now, see e.g. [5, 6, 8] and the references therein.

One important problem is the existence of proper and singular solutions, respectively. It is known that all solutions of (3) are defined on R_+ if $\lambda \leq p$ and there exists no singular solution of the first kind if $\lambda \geq p$ (see Theorem 1 bellow); hence in case of half-linear equations, $\lambda = p$, all solutions are proper. But the set of Eqs. (3) with solutions to be proper is larger, Mirzov [8] proved that all solutions of (3) are proper if the functions a and $r > 0$ are locally absolute continuous on R_+. In the present paper we generalize these results to (1). Other results for

2000 *Mathematics Subject Classification*: 34C10, 34C15, 34D05.
Key words and phrases: singular solutions, noncontinuable solutions, second order equations.
Received June 10, 2003.

the nonexistence of singular solutions of the second order differential equations (1) with $a \equiv 1$ and $p = 1$ see e.g. in [2], [4] and [9].

Our second goal is to generalize results of [3] and [7] concerning to the second order equation ($p \equiv 1$, $a \equiv 1$). We prove that for $\lambda \neq p$, $a \equiv 1$ there exist equations of the form (3) with singular solutions.

The following theorem is a special case of Theorems 1.1 and 1.2 in [8]; the equivalent expression of results is also given in [5].

Theorem 1. *Let $M \in (0, \infty)$ and $M_1 \in (0, \infty)$.*

(i) *If $|f(x)| \leq M_1 |x|^p$ for $|x| \leq M$, then there exists no singular solution of the 1-st kind of (1).*

(ii) *If $|f(x)| \leq M_1 |x|^p$ for $|x| \geq M$, then there exists no singular solution of the 2-nd kind of (1).*

Theorem 2. *Let the function $a^{\frac{1}{p}} r$ be locally absolute continuous on R_+ and $\frac{1}{r} \in L_{\mathrm{loc}}(R_+)$. Then every nontrivial solution y of (1) is proper. Moreover, if $a^{\frac{1}{p}}(t) r(t) = r_0(t) - r_1(t)$, $t \in R_+$ and*

$$\rho(t) = a^{\frac{p+1}{p}}(t) |y'(t)|^{p+1} + \frac{p+1}{p}\, a^{\frac{1}{p}}(t) r(t) \int_0^{y(t)} f(s)\, ds \tag{4}$$

where r_0 and r_1 are nonnegative,nondecreasing and continuous functions, then for $0 \leq s < t < \infty$

$$\rho(s) \exp\Big\{ -\int_s^t \frac{r_1'(\sigma)}{a^{\frac{1}{p}}(\sigma) r(\sigma)}\, d\sigma \Big\} \leq \rho(t) \leq \rho(s) \exp\Big\{ \int_s^t \frac{r_0'(\sigma)}{a^{\frac{1}{p}}(\sigma) r(\sigma)}\, d\sigma \Big\}. \tag{5}$$

Proof. As $a^{\frac{1}{p}} r$ has locally bounded variation, the continuous nondecreasing functions r_0 and r_1 exist such that $a^{\frac{1}{p}} r = r_0 - r_1$ and they can be chosen to be nonnegative on R_+. Moreover, $r_0 \in L_{\mathrm{loc}}(R_+)$, $r_1 \in L_{\mathrm{loc}}(R_+)$. Let y be a solution of (1) defined on $[s, t]$. Then

$$\rho'(\tau) = \frac{p+1}{p} [a^{\frac{1}{p}}(t) r(t)]'_{|t=\tau} \int_0^{y(\tau)} f(\sigma)\, d\sigma\,, \quad \tau \in [s, t] \quad \text{a.e.}$$

Let $\varepsilon > 0$ be arbitrary. Then

$$\frac{\rho'(\tau)}{\rho(\tau) + \varepsilon} = \frac{p+1}{p} \frac{a^{\frac{1}{p}}(\tau) r(\tau)}{\rho(\tau) + \varepsilon} \int_0^{y(\tau)} f(\sigma)\, d\sigma \frac{r_0'(\tau) - r_1'(\tau)}{a^{\frac{1}{p}}(\tau) r(\tau)}\,,$$

$$-\frac{r_1'(\tau)}{a^{\frac{1}{p}}(\tau) r(\tau)} \leq \frac{\rho'(\tau)}{\rho(\tau) + \varepsilon} \leq \frac{r_0'(\tau)}{a^{\frac{1}{p}}(\tau) r(\tau)}\,, \quad \text{a.e. on} \quad [s, t]$$

and the integration and (4) yield

$$\exp\Big\{ -\int_s^t \frac{r_1'(\sigma)\, d\sigma}{a^{\frac{1}{p}}(\sigma) r(\sigma)} \Big\} \leq \frac{\rho(t) + \varepsilon}{\rho(s) + \varepsilon} \leq \exp\Big\{ \int_s^t \frac{r_0'(\sigma)}{a^{\frac{1}{p}}(\sigma) r(\sigma)}\, d\sigma \Big\}.$$

As $\varepsilon > 0$ is arbitrary, (5) holds and due to $r^{-1} \in L_{\mathrm{loc}}(R_+)$, y is proper. $\square$

Remark 1. The assumption $\frac{1}{r} \in L_{\mathrm{loc}}(R_+)$ holds e.g. if $r > 0$ on R_+.

Theorem 3. *Let the assumption of Theorem 2 be valid with $r > 0$ on R_+ and let*

$$\rho_1(t) = \frac{a(t)}{r(t)}|y'(t)|^{p+1} + \frac{p+1}{p}\int_0^{y(t)} f(s)\,ds\,.$$

Then for $0 \le s < t < \infty$ we have

$$\rho_1(s)\exp\Big\{-\int_s^t \frac{r_0'(\sigma)\,d\sigma}{a^{\frac{1}{p}}(\sigma)r(\sigma)}\Big\} \le \rho_1(t) \le \rho_1(s)\exp\Big\{\int_s^t \frac{r_1'(\sigma)}{a^{\frac{1}{p}}(\sigma)r(\sigma)}\,d\sigma\Big\}\,.$$

Proof. It is similar to one of Theorem 2 as

$$\rho_1'(t) = \Big[\frac{(a(t)|y'(t)|^p)^{\frac{p+1}{p}}}{a^{\frac{1}{p}}(t)r(t)} + \frac{p+1}{p}\int_0^{y(t)} f(s)\,ds\Big]' = -\frac{[a^{\frac{1}{p}}(t)r(t)]'}{a^{\frac{1}{p}}(t)r(t)}\frac{a(t)|y'(t)|^{p+1}}{r(t)}\,.$$

□

Remark 2. For $p = 1$, $a \equiv 1$ and $r > 0$ on R_+ Theorems 1 and 2 are proved in [9], Th. 17.1 and Cor. 17.2; for Eq. (3), if a and $r > 0$ are locally absolutely continuous they are proved in [8], Th. 9.4.

In [1] there is an example of Eq. (3) with $a \equiv 1$, $0 < \lambda < 1$ and $p = 1$ for which there exists a solution y with infinitely many accumulation points of zeros. The following corollary gives a sufficient condition under which every solution of (1) has no accumulation points of zeros in R_+.

Corollary 1. *If the assumptions of Th. 1 are fulfilled, there every nontrivial solution of* (1) *has only finite number of zeros on a finite interval and it has no double zeros.*

Proof. Let $\tau \in R_+$ be an accumulation point of zeros or a double zero of a solution y of (1). As y is proper, $y(\tau) = y'(\tau) = 0$ and (1) has a solution $\bar{y}$ such that $\bar{y} = y$ for $t \le \tau$ and $\bar{y} \equiv 0$ on (τ, ∞). Hence $\bar{y}$ is singular of the first kind that contradicts Th. 1. □

The following theorem shows that singular solutions exist. It enlarges the same results for the second order differential equation, obtained in [3] and [7], to (3).

Lemma 1. *For an arbitrary integer k there exists $q_k \in C[0,1]$ such that*

$$q_k(0) = q_k(1) = 0\,, \tag{6}$$

$$\lim_{k\to\infty} q_k(t) = 0 \quad \textit{uniformly on} \quad [0,1] \tag{7}$$

and the equation

$$\big(|u'|^{p-1}u'\big)' + (C + q_k(t))\,|u|^\lambda \operatorname{sgn} u = 0 \tag{8}$$

has a solution u_k fulfilling

$$u_k(0) = 1\,, \quad u_k(1) = \Big(\frac{k+1}{k}\Big)^{\frac{2(p+1)}{\lambda - p}}\,, \quad u_k'(0) = u_k'(1) = 0 \tag{9}$$

where C is a suitable positive constant. Moreover, $C + q_k(t) > 0$ on $[0,1]$.

Proof. Consider a solution w of the problem

$$\left(|\dot{w}|^{p-1}\dot{w}\right)^{\cdot} + |w|^{\lambda}\operatorname{sgn} w = 0\,, \quad w(0) = 1\,, \quad w'(0) = 0\,, \quad \frac{d}{dx} = {}^{\cdot}\,.$$

Then $|\dot{w}(x)|^{p+1} + \frac{p+1}{p(\lambda+1)}|w(x)|^{\lambda+1} \equiv \frac{p+1}{p(\lambda+1)}$ on the definition interval and it is clear that w is a periodic function with period $T > 0$ with the local maximum at $x = T$. Transformation $x = tT$ yields the existence of a solution Z of the problem

$$\left(|Z'|^{p-1}Z'\right)' + C|Z|^{\lambda}\operatorname{sgn} Z = 0\,, \quad Z(0) = Z(1) = 1\,, \quad Z'(0) = Z'(1) = 0 \tag{10}$$

where $C = T^{p+1} > 0$. Note that $Z' > 0$ in a left neighbourhood of $t = 1$.

Let $t_0 \in (0,1)$ be such that

$$Z(t) > 0 \quad \text{and} \quad Z'(t) > 0 \quad \text{for} \quad t_0 \le t < 1 \tag{11}$$

and put

$$u_k(t) = \begin{cases} Z(t) & \text{for} \quad t \in [0, t_0]\,, \\ \left(\frac{k+1}{k}\right)^{\frac{2(p+1)}{\lambda-p}} - 1 + Z(t) & \\ \quad + \int_t^1 Z'(s)\left[\alpha_k(s-t_0)^3 + \beta_k(s-T_0)^2\right] ds & \text{for} \quad t \in (t_0, 1] \end{cases} \tag{12}$$

where α_k and β_k fulfil the system

$$\alpha_k \int_{t_0}^1 Z'(s)(s-t_0)^3\, ds + \beta_k \int_{t_0}^1 Z'(s)(s-t_0)^2\, ds = 1 - \left(\frac{k+1}{k}\right)^{\frac{2(p+1)}{\lambda-p}}\,, \tag{13}$$

$$\alpha_k(1-t_0)^3 + \beta_k(1-t_0)^2 = 1 - \left(\frac{k+1}{k}\right)^{\frac{2(p+1)\lambda}{(\lambda-p)p}}\,. \tag{14}$$

Note that the determinant of the system is negative, as due to $Z' > 0$ we have

$$(1-t_0)^2 \int_{t_0}^1 Z'(s)(s-t_0)^3\, ds - (1-t_0)^3 \int_{t_0}^1 Z'(s)(s-t_0)^2\, ds < 0$$

and it is clear that

$$\lim_{k\to\infty} \alpha_k = 0\,, \qquad \lim_{k\to\infty} \beta_k = 0\,. \tag{15}$$

As

$$u_k'(t) = Z'(t)[1 - \alpha_k(t-t_0)^3 - \beta_k(t-t_0)^2]\,, \qquad t \in (t_0, 1]\,, \tag{16}$$

(13) yields $u_k \in C^1[0,1]$ and according to (15) there exists k_0 such that

$$u_k(t) > 0, u_k'(t) \ge 0 \quad \text{on} \quad [t_0, 1] \quad \text{for} \quad k \ge k_0\,. \tag{17}$$

Further, from this

$$\begin{aligned}\left(|u_k'(t)|^{p-1}u_k'(t)\right)' &= \left(Z'(t)^p\left[1-\alpha_k(t-t_0)^3-\beta_k(t-t_0)^2\right]^p\right)' \\ &= -CZ^\lambda(t)(1-\alpha_k(t-t_0)^3-\beta_k(t-t_0)^2)^p \\ &\quad - pZ'(t)^p(1-\alpha_k(t-t_0)^3-\beta_k(t-t_0)^2)^{p-1} \\ &\quad \times [3\alpha_k(t-t_0)^2+2\beta_k(t-t_0)]\,.\end{aligned}$$

Hence, (11) and (12) yield

$$|u_k'(t)|^{p-1}u_k'(t) \in C^1[0,1]\,, \quad \left(|u_k'(t)|^{p-1}u_k'(t)\right)' < 0 \tag{18}$$

on $[t_0, 1]$ for large k, say, $k \geq k_1 \geq k_0$.

Define q_k by

$$q_k(t) = \begin{cases} 0 & \text{for} \quad t \in [0, t_0] \\ C + [u_k(t)]^{-\lambda}(|u_k'(t)|^{p-1}u_k'(t))' & \text{for} \quad t \in (t_0, 1]\,. \end{cases} \tag{19}$$

Then $q \in C[0,1]$ and (14) yields (6) be valid; it is clear that u_k is a solution of (8) on $[0,1]$ and according to (10), (12), (16), the relation (9) holds. As according to (12) and (15) $\lim_{k\to\infty} \frac{u_k(t)}{Z(t)} = 1$ uniformly on $[t_0, 1]$, then (15) and (19) yield (7). Note that according to (17), (18) and (19) $C + q_k(t) > 0$ on $[0,1]$ for $k \geq k_1$. □

Theorem 4. *Let $a \equiv 1$ and $p > \lambda$ $(p < \lambda)$. Then there exists a positive continuous function r such that Eq.* (3) *has a singular solution of the first* (*second*) *kind.*

Proof. Consider the sequence $\{t_k\}_{k=1}^\infty$ such that $t_1 = 0$, $t_k = \sum_{i=1}^{k-1} \frac{1}{i^2}$, $k = 2, 3, \ldots$ Then $\lim_{k\to\infty} t_k = \frac{\pi^2}{6}$. Let r and y be functions defined by

$$r(t) = \left(C + q_k(k^2(t-t_k))\right)\,, \quad y(t) = k^{\frac{2(p+1)}{\lambda-p}} u_k(k^2(t-t_k) \qquad \text{for} \quad t \in [t_k, t_{k+1})\,, \quad k = 1, 2, \ldots \tag{20}$$

where q_k and u_k are given by Lemma 1.

Let $k \in \{1, 2, \ldots\}$ be fixed. The transformation

$$t = t_k + \frac{x}{k^2}\,, \quad x \in [0,1]\,, \qquad y(t) = k^{\frac{2(p+1)}{\lambda-p}} u_k(x) \tag{21}$$

shows that y is a solution of (3) on $[t_k, t_{k+1}]$ and

$$y_+(t_k) = k^{\frac{2(p+1)}{\lambda-p}}\,, \quad y_-(t_{k+1}) = (k+1)^{\frac{2(p+1)}{\lambda-p}}\,, \quad y_+'(t_k) = y_-'(t_{k+1}) = 0\,, \tag{22}$$

$r_+(t_k) = r_-(t_{k+1}) = C$; here $h_+(\bar{t})$ $(h_-(\bar{t}))$ denote the right-hand side (left-hand side) limit of a function h. Hence function r is continuous on $[0, \frac{\pi^2}{6})$ and (7) yields $\lim_{t\to\frac{\pi^2}{6}} r(t) = C$. Similarly the function y, defined by (20) fulfils $y \in C^1[0, \frac{\pi^2}{6})$ and it is a solution of (3) on $[0, \frac{\pi^2}{6})$. Moreover, according to (12), (16), (21) and (22)

$$\lim_{t\to\frac{\pi^2}{6}} y(t) = \lim_{t\to\frac{\pi^2}{6}} y'(t) = 0 \qquad \text{if} \quad \lambda < p$$

and

$$\limsup_{t\to\frac{\pi^2}{6}} |y(t)| = \infty \qquad \text{if} \quad \lambda > p\,.$$

If we put $r(t) = C$ for $t \ge \frac{\pi^2}{6}$ then y is the singular solution of the second kind if $\lambda > p$ and

$$y(t) = \begin{cases} k^{\frac{2(p+1)}{\lambda-p}} u_k(k^2(t-t_k))\,, & t_k \le t < t_{k+1}\,,\ k = 1,2,\dots \\ 0\,, & t \ge \frac{\pi^2}{6} \end{cases}$$

is the singular solution of the first kind if $\lambda < p$. It is clear that $r > 0$ on R_+. $\square$

References

[1] Bartušek, M., *Asymptotic properties of oscillatory solutions of differential equations of n-th order*, Folia Fac. Sci. Natur. Univ. Masaryk. Brun. Math. 1992.

[2] Bartušek, M., Cecchi, M., Došlá, Z., Marini, M., *Global monotonicity and oscillation for second order differential equation*, Czechoslovak Math. J., to appear.

[3] Coffman, C. V., Ullrich, D. F., *On the continuation of solutions of a certain non-linear differential equation*, Monatsh. Math. **B 71** (1967), 385–392.

[4] Coffman, C. V., Wong, J. S. W., *Oscillation and nonoscillation theorems for second order differential equations*, Funkcial. Ekvac. **15** (1972), 119–130.

[5] Cecchi, M., Došlá, Z., Marini, M., *On nonoscillatory solutions of differential equations with p-Laplacian*, Adv. Math. Sci. Appl. **11** (2001), 419–436.

[6] Došlý, O., *Qualitative theory of half-linear second order differential equations*, Math. Bohem. **127** (2002), 181–195.

[7] Heidel, J. W., *Uniqueness, continuation and nonoscillation for a second order differential equation*, Pacific J. Math. **32** (1970), 715–721.

[8] Mirzov, D., *Asymptotic properties of solutions of systems of nonlinear nonautonomous ordinary differential equations*, Folia Fac. Sci. Natur. Univ. Masaryk. Brun. Math. **14** 2004.

[9] Kiguradze, I., Chanturia, T., *Asymptotic properties of solutions of nonautonomous ordinary differential equations*, Kluwer, Dordrecht 1993.

Faculty of Sciences, Masaryk University
Department of Mathematics
Janáčkovo nám. 2a, 662 95 Brno, Czech Republic
E-mail: bartusek@math.muni.cz