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# A NOTE ON RAPID CONVERGENCE OF APPROXIMATE SOLUTIONS FOR SECOND ORDER PERIODIC BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we develop a generalized quasilinearization technique for a nonlinear second order periodic boundary value problem and obtain a sequence of approximate solutions converging uniformly and quadratically to a solution of the problem. Then we improve the convergence of the sequence of approximate solutions by establishing the convergence of order $k$ ($k \geq 2$).

## 1. Introduction

The technique of generalized quasilinearization developed by Lakshmikantham [1,2] has been found to be extremely useful to solve the nonlinear boundary value problems. A good number of examples can be seen in the text by Lakshmikantham and Vatsala [3] and in the references [4,5]. Recently, Mohapatra, Vajravelu and Yin [6] considered the periodic boundary value problem

$$-u''(x) = f(x, u(x)), \quad u(0) = u(\pi), \quad u'(0) = u'(\pi), \qquad x \in [0, \pi],$$

with the assumption that $\frac{\partial f}{\partial u} < 0$ and $\frac{\partial^2 f}{\partial u^2} \leq 0$ (condition (iii) of Theorem 3.3 [6]). In this paper, we replace the convexity (concavity) condition by a condition of the form $f \in C^2([0,\pi] \times R^2)$ and obtain a sequence of approximate solutions converging monotonically and quadratically to a solution of the problem. Then we discuss the convergence of order $k$ ($k \geq 2$).

## 2. Preliminary results

We know that the homogeneous periodic boundary value problem

$$-u''(x) - \lambda u(x) = 0, \qquad x \in [0, \pi], \qquad u(0) = u(\pi), \qquad u'(0) = u'(\pi), \tag{2.1}$$

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has only the trivial solution if and only if $\lambda \neq 4n^2$ for all $n \in \{0, 1, 2, \dots\}$. Consequently, for these values of $\lambda$ and for any $\sigma(x) \in C([0,\pi])$, the non homogenous problem

$$\begin{aligned} -u''(x) - \lambda u(x) &= \sigma(x)\,, \quad x \in [0,\pi]\,, \\ u(0) = u(\pi)\,, &\qquad u'(0) = u'(\pi)\,, \end{aligned} \tag{2.2}$$

has a unique solution

$$u(x) = \int_0^\pi G_\lambda(x,y)\sigma(y)dy\,,$$

where $G_\lambda(x,y)$ is the Green's function given by

$$G_\lambda(x,y) = \frac{-1}{2\sqrt{\lambda}\sin\sqrt{\lambda}\frac{\pi}{2}} \begin{cases} \cos\sqrt{\lambda}(\frac{\pi}{2} - (y-x))\,, & 0 \le x \le y \le \pi\,, \\ \cos\sqrt{\lambda}(\frac{\pi}{2} - (x-y))\,, & 0 \le y \le x \le \pi\,, \end{cases}$$

for $\lambda > 0$ and

$$G_\lambda(x,y) = \frac{1}{2\sqrt{-\lambda}\sinh\frac{\sqrt{-\lambda}\pi}{2}} \begin{cases} \cosh\sqrt{-\lambda}(\frac{\pi}{2} - (y-x))\,, & 0 \le x \le y \le \pi\,, \\ \cosh\sqrt{-\lambda}(\frac{\pi}{2} - (x-y))\,, & 0 \le y \le x \le \pi\,, \end{cases}$$

for $\lambda < 0$. Here, we note that $G_\lambda(x,y) \ge 0$ for $\lambda < 0$ and $G_\lambda(x,y) < 0$ for $\lambda > 0$.

Now, consider the following nonlinear periodic boundary value problem

$$\begin{aligned} -u''(x) &= f(x,u(x))\,, \quad x \in [0,\pi]\,, \\ u(0) = u(\pi)\,, &\qquad u'(0) = u'(\pi)\,, \end{aligned} \tag{2.3}$$

where $f \in [0,\pi] \times R \to R$ is continuous.

We say that $\alpha \in C^2([0,\pi])$ is a lower solution of (2.3) if

$$\begin{aligned} -\alpha''(x) &\le f(x,\alpha(x))\,, \qquad x \in [0,\pi]\,, \\ \alpha(0) = \alpha(\pi)\,, &\qquad \alpha'(0) \ge \alpha'(\pi)\,. \end{aligned} \tag{2.4}$$

Similarly, $\beta \in C^2([0,\pi])$ is an upper solution of (2.3) if

$$\begin{aligned} -\beta''(x) &\ge f(x,\beta(x))\,, \qquad x \in [0,\pi]\,, \\ \beta(0) = \beta(\pi)\,, &\qquad \beta'(0) \le \beta'(\pi)\,. \end{aligned} \tag{2.5}$$

Now, we state some theorems without proof which are useful in the sequel (for the proof, see reference [3]).

**Theorem 1.** *Suppose that $\alpha, \beta \in C^2([0,\pi], R)$ are lower and upper solutions of (2.3) respectively. If $f(x,u)$ is strictly decreasing in $u$, then $\alpha(x) \le \beta(x)$ on $[0,\pi]$.*

**Theorem 2.** *Suppose that $\alpha, \beta \in C^2([0,\pi], R)$ are lower and upper solutions of (2.3) respectively such that*

$$\alpha(x) \le \beta(x)\,, \qquad \forall\, x \in [0,\pi]\,.$$

*Then there exists at least one solution $u(x)$ of (2.3) such that $\alpha(x) \le u(x) \le \beta(x)$ on $[0,\pi]$.*

Now, we are in a position to present the main result.

## 3. Main result

**Theorem 3.** *Assume that*

($A_1$) $\alpha, \beta \in C^2([0,\pi], R)$ *are lower and upper solutions of* (2.3) *such that* $\alpha(x) \leq \beta(x)$ *on* $[0,\pi]$.

($A_2$) $f \in C^2([0,\pi] \times R^2)$ *and* $\frac{\partial f}{\partial u}(x,u) < 0$ *for every* $(x,u) \in S$*, where*

$$S = \{(x,u) \in R^2 : x \in [0,\pi] \quad \text{and} \quad u \in [\alpha(x), \beta(x)]\}\,.$$

*Then there exists a monotone sequence* $\{q_n\}$ *which converges uniformly and quadratically to a unique solution of* (2.3).

**Proof.** In view of the assumption $(A_2)$ and the mean value theorem, we have

$$f(x,u) \geq f(x,v) + \Big[\frac{\partial}{\partial u} f(x,v) + 2mv\Big](u-v) - m(u^2 - v^2)\,, \qquad m > 0\,,$$

for every $x \in [0,\pi]$ and $u, v \in R$ such that $\alpha(x) \leq v \leq u \leq \beta(x)$ on $[0,\pi]$. In passing, we remark that we have used $\frac{\partial^2 f}{\partial u^2}(x,u) \geq -2m$, $(x,u) \in S$ here, which follows from $(A_2)$. We define the function $g(x,u,v)$ as

$$g(x,u,v) = f(x,v) + \Big[\frac{\partial}{\partial u} f(x,v) + 2mv\Big](u-v) - m\left(u^2 - v^2\right)\,.$$

Observe that

$$g(x,u,v) \leq f(x,u)\,, \qquad g(x,u,u) = f(x,u)\,. \tag{3.1}$$

It follows from $(A_2)$ and (3.1) that $g(x,u,v)$ is strictly decreasing in $u$ for each fixed $(x,v) \in [0,\pi] \times R$ and satisfies one sided Lipschitz condition

$$g(x,u_1,v) - g(x,u_2,v) \leq L(u_1 - u_2)\,, \quad L > 0\,. \tag{3.2}$$

Now, set $\alpha = q_0$ and consider the periodic boundary value problem

$$\begin{gathered} -u''(x) = g(x, u(x), q_0(x))\,, \quad x \in [0,\pi]\,, \\ u(0) = u(\pi)\,, \quad u'(0) = u'(\pi)\,. \end{gathered} \tag{3.3}$$

In view of ($A_1$) and (3.3), we have

$$\begin{gathered} -q_0''(x) \leq f(x, q_0(x)) = g(x, q_0(x), q_0(x))\,, \quad x \in [0,\pi]\,, \\ q_0(0) = q_0(\pi)\,, \quad q_0'(0) \geq q_0'(\pi)\,, \end{gathered}$$

and

$$\begin{gathered} -\beta''(x) \geq f(x, \beta(x)) \geq g(x, \beta(x), q_0(x))\,, \quad x \in [0,\pi]\,, \\ \beta(0) = \beta(\pi)\,, \quad \beta'(0) \leq \beta'(\pi)\,, \end{gathered}$$

which imply that $q_0(x)$ and $\beta(x)$ are lower and upper solutions of (3.3) respectively. Hence, by Theorem 2 and (3.2), there exists a unique solution $q_1(x)$ of (3.3) such that

$$q_0(x) \leq q_1(x) \leq \beta(x) \quad \text{on} \quad [0,\pi]\,.$$

Next, consider the periodic boundary value problem

$$-u''(x) = g(x, u(x), q_1(x)), \quad x \in [0, \pi], \qquad u(0) = u(\pi), \quad u'(0) = u'(\pi). \tag{3.4}$$

Using $(A_1)$ and employing the fact that $q_1(x)$ is a solution of (3.3), we obtain

$$-q_1''(x) = g\big(x, q_1(x), q_0(x)\big) \le g\big(x, q_1(x), q_1(x)\big), \quad x \in [0, \pi], \qquad q_1(0) = q_1(\pi), \quad q_1'(0) \ge q_1'(\pi), \tag{3.5}$$

and

$$-\beta''(x) \ge f(x, \beta) \ge g(x, \beta(x), q_1(x)), \quad x \in [0, \pi], \qquad \beta(0) = \beta(\pi), \quad \beta'(0) \le \beta'(\pi). \tag{3.6}$$

From (3.5) and (3.6), we find that $q_1(x)$ and $\beta(x)$ are lower and upper solutions of (3.4) respectively. Again, by Theorem 2 and (3.2), there exists a unique solution $q_2(x)$ of (3.4) such that

$$q_1(x) \le q_2(x) \le \beta(x) \quad \text{on} \quad [0, \pi].$$

This process can be continued successively to obtain a monotone sequence $\{q_n(x)\}$ satisfying

$$q_0(x) \le q_1(x) \le q_2(x) \le \cdots \le q_{n-1}(x) \le q_n(x) \le \beta(x) \quad \text{on} \quad [0, \pi],$$

where the element $q_n(x)$ of the sequence $\{q_n(x)\}$ is a solution of the problem

$$-u''(x) = g(x, u(x), q_{n-1}(x)), \quad x \in [0, \pi], \qquad u(0) = u(\pi), \quad u'(0) = u'(\pi).$$

Since the sequence $\{q_n\}$ is monotone, it follows that it has a pointwise limit $q(x)$. To show that $q(x)$ is in fact a solution of (2.3), we note that $q_n(x)$ is a solution of the following problem

$$-u''(x) - \lambda u(x) = \Psi_n(x), \quad x \in [0, \pi], \qquad u(0) = u(\pi), \quad u'(0) = u'(\pi), \tag{3.7}$$

where $\Psi_n(x) = g\big(x, q_n(x), q_{n-1}(x)\big) - \lambda q_n(x)$ for every $x \in [0, \pi]$. Since $g(x, u, v)$ is continuous on $S$ and $\alpha(x) \le q_n(x) \le \beta(x)$ on $[0, \pi]$, it follows that $\{\Psi_n(x)\}$ is bounded in $C[0, \pi]$. Thus, $q_n(x)$, the solution of (3.7) can be written as

$$q_n(x) = \int_0^\pi G_\lambda(x, y) \Psi_n(y)\, dy. \tag{3.8}$$

This implies that $\{q_n(x)\}$ is bounded in $C^2([0, \pi])$ and hence $\{q_n(x)\} \nearrow q(x)$ uniformly on $[0, \pi]$. Consequently, taking limit $n \to \infty$ of (3.8) yields

$$q(x) = \int_0^\pi G_\lambda(x, y) \big[f\big(y, q(y)\big) - \lambda q(y)\big]\, dy, \quad x \in [0, \pi].$$

Thus, we have shown that $q(x)$ is a solution of (2.3).
Now, we prove that the convergence of the sequence is quadratic. For that, we define

$$F(x,u)=f(x,u)+mu^2\,. \tag{3.9}$$

In view of $(A_2)$ we can find a constant $C$ such that

$$0\le \frac{\partial^2}{\partial u^2}F(x,u)\le C\,. \tag{3.10}$$

Letting $e_n(x)=q(x)-q_n(x)$, $n=1,2,3,\dots$, we have

$$\begin{aligned}
-e_n''(x)&=q_n''(x)-q''(x)\\
&=F\big(x,q(x)\big)-F\big(x,q_{n-1}(x)\big)-\big(q_n(x)-q_{n-1}(x)\big)\frac{\partial}{\partial u}F\big(x,q_{n-1}(x)\big)\\
&\quad-m\big(q^2(x)-q_{n-1}^2(x)\big)\,,\\
e_n(0)&=e_n(\pi)\,,\quad e_n'(0)=e_n'(\pi)\,.
\end{aligned}$$

Using the mean value theorem repeatedly, we obtain

$$\begin{aligned}
-e_n''(x)&=\Big[\frac{\partial}{\partial u}F(x,\xi)-\frac{\partial}{\partial u}F(x,q_{n-1})\Big]\big(q(x)-q_{n-1}(x)\big)\\
&\quad+\Big[\frac{\partial}{\partial u}F\big(x,q_{n-1}(x)\big)\Big]\big(q(x)-q_n(x)\big)-m\big(q^2(x)-q_{n-1}^2(x)\big)\\
&=\frac{\partial^2}{\partial u^2}F\big(x,\zeta(x)\big)e_{n-1}(x)\big(\xi-q_{n-1}(x)\big)\\
&\quad+\Big[\frac{\partial}{\partial u}F\big(x,q_{n-1}(x)\big)-m\big(q(x)+q_n(x)\big)\Big]e_n(x)\,,\\
e_n(0)&=e_n(\pi)\,,\quad e_n'(0)=e_n'(\pi)\,,
\end{aligned}\tag{3.11}$$

where $q_{n-1}(x)\le\zeta\le\xi\le q(x)$ on $[0,\pi]$ ($\zeta$ and $\xi$ also depend on $q_{n-1}(x)$ and $q(x)$). Substituting

$$\begin{aligned}
&\frac{\partial}{\partial u}F\big(x,q_{n-1}(x)\big)-m\big(q(x)+q_n(x)\big)=a_n(x)\,,\\
&\frac{\partial^2}{\partial u^2}F\big(x,\zeta(x)\big)e_{n-1}(x)\big(\xi-q_{n-1}(x)\big)=Ce_{n-1}^2(x)+b_n(x)\,,
\end{aligned}$$

in (3.11) gives $b_n(x)\le 0$ on $[0,\pi]$ and

$$\begin{aligned}
-e_n''(x)-e_n(x)a_n(x)&=Ce_{n-1}^2(x)+b_n(x)\,,\quad x\in[0,\pi]\,,\\
e_n(0)&=e_n(\pi)\,,\quad e_n'(0)=e_n'(\pi)\,.
\end{aligned}\tag{3.12}$$

Since $\lim_{n\to\infty}a_n(x)=\frac{\partial f}{\partial u}\big(x,q(x)\big)$ and $\frac{\partial f}{\partial u}\big(x,q(x)\big)<0$, therefore for $\lambda<0$, there exist $n_0\in N$ such that for $n\ge n_0$, we have $a_n(x)<\lambda<0$, $x\in[0,\pi]$. Therefore, the error function $e_n(x)$ satisfies the following problem

$$-e_n''(x)-\lambda e_n(x)=\big(a_n(x)-\lambda\big)e_n(x)+Ce_{n-1}^2(x)+b_n(x)\,,\quad x\in[0,\pi]\,,$$

whose solution is

$$e_n(x) = \int_0^\pi G_\lambda(x,y)\big[\big(a_n(y)-\lambda\big)e_n(y) + Ce_{n-1}^2(y) + b_n(y)\big]\,dy\,.$$

Since $a_n(y)-\lambda < 0$, $b_n(y) \le 0$, and $G_\lambda(x,y) \ge 0$ for $\lambda < 0$, therefore, it follows that

$$G_\lambda(x,y)[(a_n(y)-\lambda)e_n(y) + b_n(y) + Ce_{n-1}^2(y)] \le G_\lambda(x,y)Ce_{n-1}^2(y)\,.$$

Thus, we obtain

$$0 \le e_n(x) \le C\int_0^\pi G_\lambda(x,y)e_{n-1}^2(y)\,dy\,,$$

which can be expressed as

$$\|e_n\| \le C_1\|e_{n-1}\|^2\,,$$

where $C_1 = C\max\int_0^\pi G_\lambda(x,y)\,dy$ and $\|e_n\| = \max\big\{|e_n| : x\in[0,\pi]\big\}$ is the usual uniform norm.

## 4. Rapid convergence

**Theorem 4.** *Assume that*

- $(B_1)$ $\alpha, \beta \in C^2(\Omega)$ *are lower and upper solutions of* (2.3) *respectively such that* $\alpha(x) \le \beta(x)$ *on* $[0,\pi]$.
- $(B_2)$ $f \in C^k([0,\pi]\times R^2)$ *and* $\frac{\partial f}{\partial u}(x,u) < 0$ *for every* $(x,u) \in S$, *where*

$$S = \{(x,u) \in R^2 : x\in[0,\pi] \text{ and } u \in [\alpha(x),\beta(x)]\}\,.$$

*Then there exists a monotone sequence* $\{q_n(x)\}$ *of solutions converging uniformly to a solution of* (2.3) *with the order of convergence* $k$ $(k \ge 2)$.

**Proof.** In view of the assumption $(B_2)$ and generalized mean value theorem, we obtain

$$f(x,u) \ge \sum_{i=0}^{k-1}\frac{\partial^i f}{\partial u^i}(x,v)\frac{(u-v)^i}{i!} - m_k(u-v)^k\,,\quad m_k > 0\,, \tag{4.1}$$

for every $x\in[0,\pi]$ and $u,v\in R$ such that $\alpha(x) \le v \le u \le \beta(x)$. In (4.1), we have used $\frac{\partial^k f}{\partial u^k}(x,u) \ge -k!m_k$, which follows from $(B_2)$. We define

$$g_r(x,u,v) = \sum_{i=0}^{k-1}\frac{\partial^i f}{\partial u^i}(x,v)\frac{(u-v)^i}{i!} - m_k(u-v)^k\,. \tag{4.2}$$

Observe that

$$g_r(x,u,v) \le f(x,u)\,,\quad g_r(x,u,u) = f(x,u)\,. \tag{4.3}$$

In view of $(B_2)$ and (4.3), we note that $g_r(x,u,v)$ satisfies one sided Lipschitz condition

$$g_r(x,u_1,v) - g_r(x,u_2,v) \le L(u_1-u_2)\,,\quad L > 0\,. \tag{4.4}$$

Now, set $\alpha(x) = q_0(x)$ and consider the periodic boundary value problem

$$-u''(x) = g_r(x, u(x), q_0(x))\,, \quad x \in [0,\pi]\,, \qquad u(0) = u(\pi)\,, \quad u'(0) = u'(\pi)\,. \tag{4.5}$$

From the assumption $(B_1)$ and (4.3), we get

$$-q_0''(x) \le f(x, q_0(x)) = g_r(x, q_0(x), q_0(x))\,, \quad x \in [0,\pi]\,,$$
$$q_0(0) = q_0(\pi)\,, \quad q_0'(0) \ge q_0'(\pi)\,,$$

and

$$-\beta''(x) \ge f\big(x, \beta(x)\big) \ge g_r\big(x, \beta(x), q_0(x)\big)\,, \quad x \in [0,\pi]\,,$$
$$\beta(0) = \beta(\pi)\,, \quad \beta'(0) \le \beta'(\pi)\,,$$

which imply that $q_0(x)$ and $\beta(x)$ are lower and upper solutions of (4.5) respectively. Therefore, by Theorem 2 and (4.4), there exists a unique solution $q_1(x)$ of (4.5) such that

$$q_0(x) \le q_1(x) \le \beta(x) \quad \text{on} \quad [0,\pi]\,.$$

Similarly, we conclude that the problem

$$-u''(x) = g_r(x, u(x), q_1(x))\,, \quad x \in [0,\pi]\,,$$
$$u(0) = u(\pi)\,, \quad u'(0) = u'(\pi)\,,$$

has a unique solution $q_2(x)$ such that

$$q_1(x) \le q_2(x) \le \beta(x)\,, \quad x \in [0,\pi]\,.$$

Continuing this process successively, we obtain a monotone sequence $\{q_n(x)\}$ of solutions satisfying

$$q_0(x) \le q_1(x) \le q_2(x) \le \cdots \le q_{n-1}(x) \le q_n(x) \le \beta(x) \quad \text{on} \quad [0,\pi]\,,$$

where the element $q_n(x)$ of the sequence $\{q_n(x)\}$ is a solution of the problem

$$-u''(x) - \lambda u(x) = g_r(x, q_n(x), q_{n-1}(x)) - \lambda q_n(x) = \Psi_n(x)\,, \quad x \in [0,\pi]\,, \qquad u(0) = u(\pi)\,, \quad u'(0) = u'(\pi)\,. \tag{4.6}$$

Since the sequence is monotone, it follows that it has a pointwise limit $q(x)$. Employing the arguments used in section 3, we find that $\{q_n(x)\} \nearrow q(x)$, uniformly on $[0,\pi]$. On the other hand, the solution of (4.6) is given by

$$q_n(x) = \int_0^\pi G_\lambda(x,y)\Psi_n(y)\,dy\,, \quad x \in [0,\pi]\,, \tag{4.7}$$

which, on taking limit $n \to \infty$, becomes

$$q(x) = \int_0^\pi G_\lambda(x,y)\big[f\big(y, q(y)\big) - \lambda q(y)\big]\,dy\,, \quad x \in [0,\pi]\,.$$

Thus, $q(x)$ is a solution of (2.3).

In order to prove the convergence of order $k$ ($k \ge 2$), we define $e_n(x) = q(x) - q_n(x)$ and $a_n(x) = q_{n+1}(x) - q_n(x)$. Clearly $a_n(x) \ge 0$ and $e_n(x) \ge 0$. Further, $a_n(x) \le$

$e_n(x)$, $x \in [0, \pi]$, which implies that $a_n^k(x) \le e_n^k(x)$. By the generalized mean value theorem, we have

$$\begin{aligned} -e''_{n+1}(x) &= q''_{n+1}(x) - q''(x) \\ &= \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial u^i}\big(x, q_n(x)\big) \frac{e_n^i(x) - a_n^i(x)}{i!} - \frac{\partial^k f}{\partial u^k}(x, \xi) \frac{e_n^k(x)}{k!} + m_k a_n^k(x) \\ &\le \big(e_n(x) - a_n(x)\big) P_n(x) + C e_n^k(x)\,, \end{aligned}$$

$$e_{n+1}(0) = e_{n+1}(\pi)\,, \quad e'_{n+1}(0) = e'_{n+1}(\pi)\,,$$

where $C = 2m_k$, $q_{n-1}(x) \le \xi \le q(x)$, and

$$P_n(x) = \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial u^i}\big(x, q_n(x)\big) \frac{1}{i!} \sum_{j=0}^{i-1} e_n^{i-1-j}(x) a_n^j(x)\,, \quad x \in [0, \pi]\,.$$

Thus, for some $\tilde{w}(x) \le 0$, the error function $e_{n+1}(x)$ satisfies the problem

$$\begin{aligned} -e''_{n+1}(x) - e_{n+1}(x) P_n(x) &= C e_n^k(x) + \tilde{w}(x)\,, \quad x \in [0, \pi]\,, \\ e_{n+1}(0) &= e_{n+1}(\pi)\,, \quad e'_{n+1}(0) = e'_{n+1}(\pi)\,. \end{aligned}$$

Since $\lim_{n\to\infty} P_n(x) = \frac{\partial f}{\partial u}\big(x, q(x)\big) < 0$, therefore, for $\lambda < 0$, there exists $n_0 \in N$ such that for $n \ge n_0$, we have $P_n(x) < \lambda < 0$, $x \in [0, \pi]$. Thus, we can write

$$\begin{aligned} -e''_{n+1}(x) - \lambda e_{n+1}(x) &= (P_n(x) - \lambda) e_{n+1}(x) + C e_n^k(x) + \tilde{w}(x)\,, \quad x \in [0, \pi]\,, \\ e_{n+1}(0) &= e_{n+1}(\pi)\,, \quad e'_{n+1}(0) = e'_{n+1}(\pi)\,, \end{aligned}$$

whose solution is given by

$$e_{n+1}(x) = \int_0^\pi G_\lambda(x, y) \big[(P_n(y) - \lambda) e_{n+1}(y) + C e_n^k(y) + \tilde{w}(y)\big]\, dy\,. \tag{4.8}$$

Since $P_n(y) - \lambda < 0$, $\tilde{w}(y) \le 0$ and $G_\lambda(x, y) \ge 0$ for $\lambda < 0$, therefore, it follows that

$$G_\lambda(x, y) \big[(P_n(y) - \lambda) e_{n+1}(y) + C e_n^k(y) + \tilde{w}(y)\big] \le G_\lambda(x, y) C e_n^k(y)\,. \tag{4.9}$$

Combining (4.8) and (4.9), we obtain

$$0 \le e_{n+1}(x) \le C \int_0^\pi G_\lambda(x, y) e_n^k(y)\, dy\,.$$

Thus,

$$\|e_n(x)\| \le C_1 \|e_{n-1}(x)\|^k\,,$$

where $C_1 = C \max \int_0^\pi G_\lambda(x, y)\, dy$.

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