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# ON NATURALITY OF THE HELMHOLTZ OPERATOR

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Abstract. We deduce that all natural operators of the type of the Helmholtz map from the variational calculus in fibered manifolds are the constant multiples of the Helmholtz operator.

## 0 Introduction

Given two fibered manifolds $Z_1 \to M$ and $Z_2 \to M$ over the same base $M$, we denote by $\mathcal{C}^\infty_M(Z_1, Z_2)$ the space of all base preserving fibered manifold morphisms of $Z_1$ into $Z_2$. In [2], Kolář and Vitolo studied the $s$-th order Helmholtz map of the variational calculus on a fibered manifold $p : Y \to M$, $\dim M = m$, as a morphism operator

$$H : \mathcal{C}^\infty_Y(J^sY, V^*Y \otimes \bigwedge^m T^*M) \to \mathcal{C}^\infty_{J^sY}(J^{2s}Y, V^*J^sY \otimes V^*Y \otimes \bigwedge^m T^*M)\,.$$

They also deduced that for $s = 1, 2$ all $\mathcal{FM}_{m,n}$-natural operators of this type (in the sense of [1]) are of the form $cH$, $c \in \mathbf{R}$. In the present paper we deduce that the same result holds for arbitrary $s$. In other words we prove the following theorem.

**Theorem 1.** *Let $m$, $n$, $s$ be natural numbers with $n \geq 2$. Then any $\pi^{2s}_s$-local and $\mathcal{FM}_{m,n}$-natural (regular) operator*

$$D : \mathcal{C}^\infty_Y(J^sY, V^*Y \otimes \bigwedge^m T^*M) \to \mathcal{C}^\infty_{J^sY}(J^{2s}Y, V^*J^sY \otimes V^*Y \otimes \bigwedge^m T^*M)$$

*is of the form $D = cH$, $c \in \mathbf{R}$, where $\pi^{2s}_s : J^{2s}Y \to J^sY$ is the jet projection.*

From now on $\mathbf{R}^{m,n}$ is the trivial bundle $\mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$ and $x^1, \dots, x^m$, $y^1, \dots, y^n$ are the usual coordinates on $\mathbf{R}^{m,n}$.

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## 1 Proof of Theorem 1

Let $D$ be an operator in question.

Since an $\mathcal{FM}_{m,n}$-map $\big(x, y-\sigma(x)\big)$ sends $j^{2s}_0(\sigma)$ into $\Theta = j^{2s}_0(0) \in J^{2s}_0(\mathbf{R}^m, \mathbf{R}^n) = J^{2s}_0(\mathbf{R}^{m,n})$, $J^{2s}(\mathbf{R}^{m,n})$ is the $\mathcal{FM}_{m,n}$-orbit of $\Theta$. Then $D$ is uniquely determined by the evaluations

$$\langle D(E)_\Theta, w \otimes v\rangle \in \bigwedge^m T^*_0\mathbf{R}^m$$

for all $E \in \mathcal{C}^\infty_{\mathbf{R}^{m,n}}\big(J^s(\mathbf{R}^{m,n}), V^*\mathbf{R}^{m,n} \otimes \bigwedge^m T^*\mathbf{R}^m\big)$, $w \in V_{\pi^{2s}_s(\Theta)}J^s(\mathbf{R}^{m,n})$ and $v \in T_0\mathbf{R}^n = V_{(0,0)}\mathbf{R}^{m,n}$.

Using the invariance of $D$ with respect to $\mathcal{FM}_{m,n}$-morphisms of the form $id_{\mathbf{R}^m} \times \psi$ for linear $\psi$ (since $n \geq 2$) we get that $D$ is uniquely determined by the evaluations

$$\Big\langle D(E)_\Theta, \frac{d}{dt}{}_0\big(tj^s_0(f(x), 0, \dots, 0)\big) \otimes \frac{\partial}{\partial y^2}{}_0\Big\rangle \in \bigwedge^m T^*_0\mathbf{R}^m$$

for all $E \in \mathcal{C}^\infty_{\mathbf{R}^{m,n}}\big(J^s(\mathbf{R}^{m,n}), V^*\mathbf{R}^{m,n} \otimes \bigwedge^m T^*\mathbf{R}^m\big)$ and all $f : \mathbf{R}^m \to \mathbf{R}$.

Using the invariance of $D$ with respect to $\mathcal{FM}_{m,n}$-maps $(x^1, \dots, x^m, y^1 + f(x)y^1, y^2, \dots, y^n)$ preserving $\Theta$ we get that $D$ is uniquely determined by the evaluations

$$\Big\langle D(E)_\Theta, \frac{d}{dt}{}_0\big(tj^s_0(1, 0, \dots, 0)\big) \otimes \frac{\partial}{\partial y^2}{}_0\Big\rangle \in \bigwedge^m T^*_0\mathbf{R}^m$$

for all $E \in \mathcal{C}^\infty_{\mathbf{R}^{m,n}}\big(J^s(\mathbf{R}^{m,n}), V^*\mathbf{R}^{m,n} \otimes \bigwedge^m T^*\mathbf{R}^m\big)$.

Let $E \in \mathcal{C}^\infty_{\mathbf{R}^{m,n}}\big(J^s(\mathbf{R}^{m,n}), V^*\mathbf{R}^{m,n} \otimes \bigwedge^m T^*\mathbf{R}^m\big)$. Using the invariance of $D$ with respect to $\mathcal{FM}_{m,n}$-maps $\psi_\tau = (x^1, \dots, x^m, \frac{1}{\tau^1}y^1, \dots, \frac{1}{\tau^n}y^n)$ for $\tau^j \neq 0$ we get the homogeneity condition

$$\begin{aligned}
&\Big\langle D\big((\psi_\tau)_*E\big)_\Theta, \frac{d}{dt}{}_0\big(tj^s_0(1, 0, \dots, 0)\big) \otimes \frac{\partial}{\partial y^2}{}_0\Big\rangle \\
&= \tau^1\tau^2\Big\langle D(E)_\Theta, \frac{d}{dt}{}_0\big(tj^s_0(1, 0, \dots, 0)\big) \otimes \frac{\partial}{\partial y^2}{}_0\Big\rangle
\end{aligned}$$

for $\tau = (\tau^1, \dots, \tau^n)$. By Corollary 19.8 in [1] of the non-linear Peetre theorem we can assume that $E$ is a polynomial (with arbitrary degree). It is easily seen that coordinates of polynomial $(\psi_\tau)_*E$ are the multiplication by monomials in $\tau$ of respective coordinates of polynomial $E$. The regularity of $D$ implies that $\big\langle D(E)_\Theta, \frac{d}{dt}{}_0\big(tj^s_0(1, 0, \dots, 0)\big) \otimes \frac{\partial}{\partial y^2}{}_0\big\rangle$ is smooth with respect to the coordinates of $E$. Then by the homogeneous function theorem (and the above type of homogeneity) we deduce that $\big\langle D(E)_\Theta, \frac{d}{dt}{}_0\big(tj^s_0(1, 0, \dots, 0)\big) \otimes \frac{\partial}{\partial y^2}{}_0\big\rangle$ depends linearly on the coordinates of $E$ on all $x^\beta y^1_\alpha dy^2 \otimes dx^\mu$ and $x^\beta y^2 dy^1 \otimes dx^\mu$, it depends bilinearly on the coordinates of $E$ on all $x^\rho dy^1 \otimes dx^\mu$ and $x^\beta dy^2 \otimes dx^\mu$, and it is independent of the other coordinates of $E$, where $(x^i, y^j_\alpha)$ is the induced coordinate system on

$J^s(\mathbf{R}^{m,n})$ and $dx^\mu = dx^1 \wedge \dots \wedge dx^m$. (Here and from now on $\alpha$, $\rho$ and $\beta$ are arbitrary $m$-tuples with $|\alpha| \le s$).

In other words (and more precisely) $\left\langle D(E)_\Theta, \frac{d}{dt}_0\big(tj^s_0(1,0,\dots,0)\big) \otimes \frac{\partial}{\partial y^2}_0 \right\rangle$ is determined by the values

$$\left\langle D(x^\beta y^2_\alpha dy^1 \otimes dx^\mu)_\Theta, \frac{d}{dt}_0\big(tj^s_0(1,0,\dots,0)\big) \otimes \frac{\partial}{\partial y^2}_0 \right\rangle,$$
$$\left\langle D(x^\beta y^1_\alpha dy^2 \otimes dx^\mu)_\Theta, \frac{d}{dt}_0\big(tj^s_0(1,0,\dots,0)\big) \otimes \frac{\partial}{\partial y^2}_0 \right\rangle,$$
$$\left\langle D(x^\rho dy^1 \otimes dx^\mu + x^\beta dy^2 \otimes dx^\mu)_\Theta, \frac{d}{dt}_0\big(tj^s_0(1,0,\dots,0)\big) \otimes \frac{\partial}{\partial y^2}_0 \right\rangle.$$

Moreover
$$\left\langle D(E)_\Theta, \frac{d}{dt}_0\big(tj^s_0(1,0,\dots,0)\big) \otimes \frac{\partial}{\partial y^2}_0 \right\rangle$$
is linear in $E$ for $E$ from the vector subspace (over $\mathbf{R}$) spaned by all $x^\beta y^1_\alpha dy^2 \otimes dx^\mu$ and $x^\beta y^2_\alpha dy^1 \otimes dx^\mu$,

$$\begin{aligned}&\left\langle D(dy^1 \otimes dx^\mu + E)_\Theta, \frac{d}{dt}_0\big(tj^s_0(1,0,\dots,0)\big) \otimes \frac{\partial}{\partial y^2}_0 \right\rangle \\ &= \left\langle D(E)_\Theta, \frac{d}{dt}_0\big(tj^s_0(1,0,\dots,0)\big) \otimes \frac{\partial}{\partial y^2}_0 \right\rangle \end{aligned} \tag{1}$$

for $E$ from the vector subspace (over $\mathbf{R}$) spaned by all $x^\beta y^1_\alpha dy^2 \otimes dx^\mu$ and $x^\beta y^2_\alpha dy^1 \otimes dx^\mu$, and

$$\begin{aligned}&\left\langle D(ax^\rho dy^1 \otimes dx^\mu + bx^\beta dy^2 \otimes dx^\mu)_\Theta, \frac{d}{dt}_0\big(tj^s_0(1,0,\dots,0)\big) \otimes \frac{\partial}{\partial y^2}_0 \right\rangle \\ &= ab\left\langle D(x^\rho dy^1 \otimes dx^\mu + x^\beta dy^2 \otimes dx^\mu)_\Theta, \frac{d}{dt}_0\big(tj^s_0(1,0,\dots,0)\big) \otimes \frac{\partial}{\partial y^2}_0 \right\rangle \end{aligned} \tag{2}$$

for all real numbers $a$ and $b$.

Then by the invariance of $D$ with respect to $(\tau^1 x^1, \dots, \tau^m x^m, y^1, \dots, y^n)$ for $\tau^i \neq 0$ we get

$$\begin{aligned}&\left\langle D(x^\beta y^2_\alpha dy^1 \otimes dx^\mu)_\Theta, \frac{d}{dt}_0\big(tj^s_0(1,0,\dots,0)\big) \otimes \frac{\partial}{\partial y^2}_0 \right\rangle \\ &= \left\langle D(x^\beta y^1_\alpha dy^2 \otimes dx^\mu)_\Theta, \frac{d}{dt}_0\big(tj^s_0(1,0,\dots,0)\big) \otimes \frac{\partial}{\partial y^2}_0 \right\rangle = 0 \end{aligned} \tag{3}$$

if only $\beta \neq \alpha$, and

$$\left\langle D(x^\rho dy^1 \otimes dx^\mu + x^\beta dy^2 \otimes dx^\mu)_\Theta, \frac{d}{dt}_0\big(tj^s_0(1,0,\dots,0)\big) \otimes \frac{\partial}{\partial y^2}_0 \right\rangle = 0$$

for all $\rho$ and $\beta$.

Suppose $\alpha=(\alpha_1,\dots,\alpha_m)$ be an $m$-tuple with $|\alpha|\le s$ and $\alpha_i\ne 0$ for some $i$. Then using the invariance of $D$ with respect to locally defined $\mathcal{FM}_{m,n}$-map $\psi=(x^1,\dots,x^m,y^1,y^2+x^iy^2\dots,y^n)^{-1}$ preserving $x^1,\dots,x^m$, $y^1$, $\Theta$, $j^s_0(1,0,\dots,0)$ and $\frac{\partial}{\partial y^2}{}_0$ and sending $y^2_\alpha$ into $y^2_\alpha+x^iy^2_\alpha+y^2_{\alpha-1_i}$ $\big($as $y^2_\alpha\circ J^s\psi^{-1}(j^s_{x_0}\sigma)=\partial_\alpha(\sigma^2+x^i\sigma^2)(x_0)=\partial_\alpha\sigma^2(x_0)+x^i_0\partial_\alpha\sigma^2(x_0)+\partial_{\alpha-1_i}\sigma^2(x_0)=(y^2_\alpha+x^iy^2_\alpha+y^2_{\alpha-1_i})(j^s_{x_0}\sigma)$ for $j^s_{x_0}\sigma\in J^s\mathbf{R}^{m,n}$, where $\partial\alpha$ is the iterated partial derivative with erspect to the index $\alpha$ multiplied by $\frac{1}{\alpha!}\big)$ from

$$\Big\langle D(x^{\alpha-1_i}y^2_\alpha dy^1\otimes dx^\mu)_\Theta,\frac{d}{dt}{}_0\big(tj^s_0(1,0,\dots,0)\big)\otimes\frac{\partial}{\partial y^2}{}_0\Big\rangle=0$$

(see (3)) we deduce that

$$\begin{aligned}&\Big\langle D(x^{\alpha}y^2_\alpha dy^1\otimes dx^\mu)_\Theta,\frac{d}{dt}{}_0\big(tj^s_0(1,0,\dots,0)\big)\otimes\frac{\partial}{\partial y^2}{}_0\Big\rangle\\&=-\Big\langle D(x^{\alpha-1_i}y^2_{\alpha-1_i}dy^1\otimes dx^\mu)_\Theta,\frac{d}{dt}{}_0\big(tj^s_0(1,0,\dots,0)\big)\otimes\frac{\partial}{\partial y^2}{}_0\Big\rangle.\end{aligned}$$

Then for any $m$-tuple $\alpha$ with $|\alpha|\le s$ we have

$$\begin{aligned}&\Big\langle D(x^{\alpha}y^2_\alpha dy^1\otimes dx^\mu)_\Theta,\frac{d}{dt}{}_0\big(tj^s_0(1,0,\dots,0)\big)\otimes\frac{\partial}{\partial y^2}{}_0\Big\rangle\\&=(-1)^{|\alpha|}\Big\langle D(y^2_{(0)}dy^1\otimes dx^\mu)_\Theta,\frac{d}{dt}{}_0\big(tj^s_0(1,0,\dots,0)\big)\otimes\frac{\partial}{\partial y^2}{}_0\Big\rangle.\end{aligned}$$

By the same arguments (since $\psi$ sends $dy_2$ into $dy^2+x^idy^2$) from

$$\Big\langle D(x^{\alpha-1_i}y^1_\alpha dy^2\otimes dx^\mu)_\Theta,\frac{d}{dt}{}_0\big(tj^s_0(1,0,\dots,0)\big)\otimes\frac{\partial}{\partial y^2}{}_0\Big\rangle=0$$

we obtain

$$\Big\langle D(x^{\alpha}y^1_\alpha dy^2\otimes dx^\mu)_\Theta,\frac{d}{dt}{}_0\big(tj^s_0(1,0,\dots,0)\big)\otimes\frac{\partial}{\partial y^2}{}_0\Big\rangle=0$$

if $\alpha\ne(0)$.

Using the invariance of $D$ with respect to (locally defined) $\mathcal{FM}_{m,n}$-map $(x^1,\dots,x^m,y^1+y^1y^2,\dots,y^n)^{-1}$ preserving $\Theta$, $j^s_0(1,0,\dots,0)$ and $\frac{\partial}{\partial y^2}{}_0$ from

$$\Big\langle D(dy^1\otimes dx^\mu)_\Theta,\frac{d}{dt}{}_0\big(tj^s_0(1,0,\dots,0)\big)\otimes\frac{\partial}{\partial y^2}{}_0\Big\rangle=0$$

(see (2)) and (1) we deduce that

$$\begin{aligned}&\Big\langle D(y^2_{(0)}dy^1\otimes dx^\mu)_\Theta,\frac{d}{dt}{}_0\big(tj^s_0(1,0,\dots,0)\big)\otimes\frac{\partial}{\partial y^2}{}_0\Big\rangle\\&=-\Big\langle D(y^1_{(0)}dy^2\otimes dx^\mu)_\Theta,\frac{d}{dt}{}_0\big(tj^s_0(1,0,\dots,0)\big)\otimes\frac{\partial}{\partial y^2}{}_0\Big\rangle.\end{aligned}$$

Then $D$ is uniquely determined by

$$\left\langle D(y^2_{(0)} dy^1 \otimes dx^\mu)_\Theta, \frac{d}{dt}_0 \left(tj^s_0(1,0,\dots,0)\right) \otimes \frac{\partial}{\partial y^2}_0 \right\rangle \in \bigwedge^m T^*_0\mathbf{R}^m = \mathbf{R}\,.$$

Then the vector space of all $D$ in question is of dimension less or equal to 1. That is why $D = cH$ for some $c \in \mathbf{R}$. $\square$

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## References


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