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ON LEFT (θ, ϕ)-DERIVATIONS OF PRIME RINGS

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Abstract. Let R be a 2-torsion free prime ring. Suppose that θ, ϕ are automorphisms of R. In the present paper it is established that if R admits a nonzero Jordan left (θ, θ)-derivation, then R is commutative. Further, as an application of this resul it is shown that every Jordan left (θ, θ)-derivation on R is a left (θ, θ)-derivation on R. Finally, in case of an arbitrary prime ring it is proved that if R admits a left (θ, ϕ)-derivation which acts also as a homomorphism (resp. anti-homomorphism) on a nonzero ideal of R, then $d = 0$ on R.

1. Introduction

Throughout the present paper R will denote an associative ring with centre $Z(R)$. Recall that R is prime if $aRb = \{0\}$ implies that $a = 0$ or $b = 0$. As usual $[x, y]$ will denote the commutator $xy - yx$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$ for all $u \in U$, $r \in R$. Suppose that θ, ϕ are endomorphisms of R. An additive mapping $d : R \longrightarrow R$ is called a (θ, ϕ)-derivation (resp. Jordan (θ, ϕ)-derivation) if $d(xy) = d(x)\phi(y) + \theta(x)d(y)$, (resp. $d(x^2) = d(x)\phi(x) + \theta(x)d(x)$) holds for all $x, y \in R$. Of course, every $(1, 1)$-derivation (resp. Jordan $(1, 1)$- derivation), where 1 is the identity mapping on R is a derivation (resp. Jordan derivation) on R. An additive mapping $d : R \to R$ is called a left (θ, ϕ)-derivation (resp. Jordan left (θ, ϕ)-derivation) if $d(xy) = \theta(x)d(y) + \phi(y)d(x)$ (resp. $d(x^2) = \theta(x)d(x) + \phi(x)d(x)$) holds for all $x, y \in R$. Clearly, every left $(1, 1)$-derivation (resp. Jordan left $(1, 1)$-derivation) is a left derivation (resp. Jordan left derivation) on R. Obviously, every left derivation is a Jordan left derivation but the converse need not be true in general. Recently the author together with Nadeem [1] proved that the converse statement is true in the case when the underlying ring is prime and 2-torsion free. In the present paper we shall show that if a 2-torsion free prime ring R admits an additive mapping satisfying $d(u^2) = 2\theta(u)d(u)$ for all $u \in U$, then either $d(U) = \{0\}$ or $U \subseteq Z(R)$ where U is a Lie ideal of R with $u^2 \in U$ for all $u \in U$ and θ is an automorphism

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of R. In fact this result generalizes the main theorem proved in [4]. Further, some more related results are also obtained. Final section of the present paper deals with the study of left (θ,ϕ)-derivation which acts also as a homomorphism of the ring.

2. Preliminaries

We shall make use of the following results, all but one of which are known.

Lemma 2.1 ([9, Lemma 2]). *If $U \not\subseteq Z(R)$ is a Lie ideal of a 2-torsion free prime ring R and $a,b \in R$ such that $aUb = \{0\}$, then $a = 0$ or $b = 0$.*

Lemma 2.2 ([11, Lemma 4]). *Let G and H be additive groups and let R be a 2-torsion free ring. Let $f : G\times G \to H$ and $g : G\times G \to R$ be biadditive mappings. Suppose that for each pair $a,b \in G$ either $f(a,b) = 0$ or $g(a,b)^2 = 0$. In this case either $f = 0$ or $g(a,b)^2 = 0$ for all $a,b \in G$ respectively.*

Lemma 2.3 ([13, Theorem 4]). *Let R be a 2-torsion free prime ring and U a Lie ideal of R. If R admits a derivation d such that $d(u)^n = 0$ for all $u \in U$, where $n \geq 1$ is a fixed integer, then $d(u) = 0$ for all $u \in U$.*

Lemma 2.4 ([16, Lemma 1.3]). *Let R be a 2-torsion free semiprime ring. If U is a commutative Lie ideal of R, then $U \subseteq Z(R)$.*

Now we shall prove the following

Lemma 2.5. *Let R be a 2-torsion free ring and let U be a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. Suppose that θ is an endomorphism of R. If $d : R \to R$ is an additive mapping satisfying $d(u^2) = 2\theta(u)d(u)$ for all $u,v \in U$ then*

(i) *$d(uvu) = \theta(u^2)d(v) + 3\theta(u)\theta(v)d(u) - \theta(v)\theta(u)d(u)$ for all $u,v \in U$.*
(ii) *$[\theta(u),\theta(v)]\theta(u)d(u) = \theta(u)[\theta(u),\theta(v)]d(u)$ for all $u,v \in U$.*
(iii) *$[\theta(u),\theta(v)]d([u,v]) = 0$ for all $u,v \in U$.*
(iv) *$d(vu^2) = \theta(u^2)d(v) + (3\theta(v)\theta(u) - \theta(u)\theta(v))d(u) - \theta(u)d([u,v])$ for all $u,v \in U$.*

Proof. (i) Since $uv + vu = (u+v)^2 - u^2 - v^2$, we find that $uv + vu \in U$ for all $u,v \in U$. Hence by linearizing $d(u^2) = 2\theta(u)d(u)$ on u, we get

$$d(uv + vu) = 2\theta(u)d(v) + 2\theta(v)d(u) \quad \text{for all} \quad u,v \in U\,. \tag{2.1}$$

Further, replacing v by $uv + vu$ in (2.1), we get

$$d(u(uv+vu) + (uv+vu)u) = 4\theta(u^2)d(v) + 6\theta(u)\theta(v)d(u) + 2\theta(v)\theta(u)d(u)\,. \tag{2.2}$$

On the other hand,

$$\begin{aligned} d(u(uv+vu) + (uv+vu)u) &= d(u^2v + vu^2) + 2d(uvu) \\ &= 2\theta(u^2)d(v) + 4\theta(v)\theta(u)d(u) + 2d(uvu)\,. \end{aligned}$$

Combining the above equation with (2.2), we get (i).

(ii) By linearizing (i) on u, we get

$$\begin{aligned} d((u+w)v(u+w)) &= \theta(u^2)d(v)+\theta(w^2)d(v)+\{\theta(u)\theta(w)+\theta(w)\theta(u)\}d(v)\\ &\quad+3\theta(u)\theta(v)d(w)+3\theta(u)\theta(v)d(u)+3\theta(w)\theta(v)d(w)\\ &\quad+3\theta(w)\theta(v)d(u)-\theta(v)\theta(u)d(u)-\theta(v)\theta(u)d(w)\\ &\quad-\theta(v)\theta(w)d(u)-\theta(v)\theta(w)d(w)\,. \end{aligned} \tag{2.3}$$

On the other hand,

$$\begin{aligned} d((u+w)v(u+w)) &= d(uvu)+d(wvw)+d(uvw+wvu)\\ &= \theta(u^2)d(v)+3\theta(u)\theta(v)d(u)-\theta(v)\theta(u)d(u)+\theta(w^2)d(v)\\ &\quad+3\theta(w)\theta(v)d(w)-\theta(v)\theta(w)d(w)+d(uvw+wvu)\,. \end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4), we arrive at

$$\begin{aligned} d(uvw+wvu) =& \{\theta(u)\theta(w)+\theta(w)\theta(u)\}d(v)+3\theta(u)\theta(v)d(w)+3\theta(w)\theta(v)d(u)\\ &-\theta(v)\theta(u)d(w)-\theta(v)\theta(w)d(u)\quad\text{for all}\quad u,v\in U\,. \end{aligned} \tag{2.5}$$

Since $uv+vu$ and $uv-vu$ both belong to U we find that $2uv\in U$ for all $u,v\in U$. Hence, by our hypothesis we find that $d((2uv)^2)=2\theta(2uv)d((2uv))$ i.e., $4d(uv)^2=8\theta(uv)d(uv)$. Since char$R\neq 2$, we have $d(uv)^2=2\theta(u)\theta(v)d(uv)$. Replace w by $2uv$ in (2.5), and use the fact that char $R\neq 2$, to get

$$\begin{aligned} d(uv(uv)+(uv)vu) &= \{\theta(u^2)\theta(v)+\theta(u)\theta(v)\theta(u)\}d(v)+3\theta(u)\theta(v)d(uv)\\ &\quad+3\theta(u)\theta(v^2)d(u)-\theta(v)\theta(u)d(uv)\\ &\quad-\theta(v)\theta(u)\theta(v)d(u)\,. \end{aligned} \tag{2.6}$$

On the other hand,

$$\begin{aligned} d((uv)^2+uv^2u) &= 2\theta(u)\theta(v)d(uv)+2\theta(u^2)\theta(v)d(v)\\ &\quad+3\theta(u)\theta(v^2)d(u)-\theta(v^2)\theta(u)d(u)\,. \end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7), we get

$$[\theta(u),\theta(v)]d(uv)=\theta(u)[\theta(u),\theta(v)]d(v)+\theta(v)[\theta(u),\theta(v)]d(u) \tag{2.8}$$

Replacing $u+v$ for v in (2.8), we have

$$\begin{aligned} 2[\theta(u),\theta(v)]\theta(u)d(u)+[\theta(u),\theta(v)]d(uv) &= 2\theta(u)[\theta(u),\theta(v)]d(u)\\ &\quad+\theta(u)[\theta(u),\theta(v)]d(v)+\theta(v)[\theta(u),\theta(v)]d(u)\,. \end{aligned}$$

Now application of (2.8) yields (ii).

(iii) Linearize (ii) on u, to get

$$\begin{aligned}[\theta(u),\theta(v)]\theta(u)d(u)+[\theta(u),\theta(v)]\theta(v)d(v)+[\theta(u),\theta(v)]\theta(u)d(v)\\ +[\theta(u),\theta(v)]\theta(v)d(u)= \theta(u)[\theta(u),\theta(v)]d(u)\\ +\theta(u)[\theta(u),\theta(v)]d(v)+\theta(v)[\theta(u),\theta(v)]d(u)\\ +\theta(v)[\theta(u),\theta(v)]d(v) \quad \text{for all} \quad u,v\in U\,.\end{aligned}$$

Now application of (2.8) and (ii) yields that

$$[\theta(u),\theta(v)]\theta(u)d(v)+[\theta(u),\theta(v)]\theta(v)d(u)=[\theta(u),\theta(v)]d(uv)$$

and hence

$$[\theta(u),\theta(v)]\{d(uv)-\theta(u)d(v)-\theta(v)d(u)\}=0 \quad \text{for all} \quad u,v\in U\,. \tag{2.9}$$

Combining (2.1) and (2.9) we find that,

$$[\theta(u),\theta(v)]\{d(vu)-\theta(u)d(v)-\theta(v)d(u)\}=0 \quad \text{for all} \quad u,v\in U\,. \tag{2.10}$$

Further, combining of (2.9) and (2.10) yields the required result.

(iv) Replace v by $2vu$ in (2.1), and use the fact that char$R\neq 2$, to get

$$d(uvu+vu^2)=2(\theta(u)d(uv)+\theta(v)\theta(u)d(u)) \quad \text{for all} \quad u,v\in U\,. \tag{2.11}$$

Again, replacing v by $2uv$ in (2.1), we get

$$d(u^2v+uvu)=2(\theta(u)d(uv)+\theta(u)\theta(v)d(u)) \quad \text{for all} \quad u,v\in U\,. \tag{2.12}$$

Now, combining (2.11) and (2.12), we get

$$d(u^2v-vu^2)=2(\theta(u)d([u,v])+[\theta(u),\theta(v)]d(u)) \quad \text{for all} \quad u,v\in U\,. \tag{2.13}$$

Replacing u by u^2 in (2.1), we have

$$d(u^2v+vu^2)=2(\theta(u^2)d(v)+2\theta(v)\theta(u)d(u)) \quad \text{for all} \quad u,v\in U\,. \tag{2.14}$$

Hence, subtracting (2.13) from (2.14) and using the fact that characteristic of R is different from two we find that

$$d(vu^2)=\theta(u^2)d(v)+\{3\theta(v)\theta(u)-\theta(u)\theta(v)\}d(u)-\theta(u)d([u,v]) \quad \text{for all} \quad u,v\in U\,.$$

3. Left derivation and commutativity of prime ring

A mapping $f:R\to R$ is said to be commuting on R if $f(x)x=xf(x)$ holds for all $x\in R$. Comparing Jordan left derivation with commuting mapping on a ring R, it turns out that notion of Jordan left derivation is in a close connection with the commuting mapping on R. There has been considerable interest for commuting mappings on prime rings. The fundamental result in this direction is due to Posner [18] who proved that if a prime ring R admits a non-zero derivation that is commuting on R, then R is commutative. Using rather weak hypotheses Bresar and Vukman [12] obtained a result which shows that the existence of a non-zero Jordan left derivation on a 2-torsion free and 3-torsion free prime ring R forces R to be commutative. It was also remarked by Bresar and Vukman that the assumption "R is 3-torsion free" in the hypotheses of the above result may be avoided. In this direction we have obtained the following theorem which also includes the main result of [4].

Theorem 3.1. *Let R be a 2-torsion free prime ring and let U be a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. Suppose that θ is an automorphism of R. If $d : R \to R$ is an additive mapping satisfying $d(u^2) = 2\theta(u)d(u)$ for all $u \in U$, then either $d(U) = \{0\}$ or $U \subseteq Z(R)$.*

Proof. Suppose that $U \nsubseteq Z(R)$. By Lemma 2.5(ii) we have

$$\{\theta(u^2)\theta(v) - 2\theta(u)\theta(v)\theta(u) + \theta(v)\theta(u^2)\}d(u) = 0 \quad \text{for all} \quad u, v \in U\,. \tag{3.1}$$

Replacing $[u,w]$ for u in (3.1), we get

$$\begin{aligned}[\theta(u),\theta(w)]^2\theta(v)d([u,w]) - 2[\theta(u),\theta(w)]\theta(v)[\theta(u),\theta(w)]d([u,w])&\\ + \theta(v)[\theta(u),\theta(w)]^2 d([u,w]) = 0&\\ \text{for all} \quad u,v,w \in U\,.&\end{aligned}$$

Now, application of Lemma 2.5(iii) yields that $\theta^{-1}([\theta(u),\theta(w)]^2)U\theta^{-1}(d([u,w]) = \{0\}$. Hence by Lemma 2.1 we find that for each pair $u,w \in U$, either $[\theta(u),\theta(w)]^2 = 0$ or $d([u,w]) = 0$. This implies that either $[u,w]^2 = 0$ or $d([u,w]) = 0$. Note that the mappings $(u,w) \mapsto [u,w]$ and $(u,w) \mapsto d([u,w])$ satisfy the requirements of the Lemma 2.2. Hence, either $[u,w]^2 = 0$ for all $u,w \in U$ or $d([u,w]) = 0$ for all $u,w \in U$. If $[u,w]^2 = 0$ for all $u,w \in U$, then for each $u \in U, (I_u(w))^2 = 0$ for all $w \in U$, where I_u is the inner derivation such that $I_u(w) = [u,w]$. Thus by the application of Lemma 2.3 we find that U is a commutative Lie ideal of R, and hence by Lemma 2.4, $U \subseteq Z(R)$, a contradiction. Hence, we consider the remaining case that $d([u,w]) = 0$ for all $u,w \in U$, i.e., $d(uw) = d(wu)$ for all $u,w \in U$. Since $wu - uw$ and $wu + uw$ both belong to U, we find that $2wu \in U$ for all $u,w \in U$. This yields that $d((2wu)u) = d(u(2wu))$. Since (2.1) is valid in the present situation, we find that

$$\begin{aligned}4d((wu)u) &= d((2wu)u + u(2wu))\\ &= 4\theta(w)\theta(u)d(u) + 2\theta(u)d(2wu)\\ &= 4\theta(w)\theta(u)d(u) + 2\theta(u)d(wu+uw)\\ &= 4\{\theta(w)\theta(u)d(u) + \theta(u)\theta(w)d(u) + \theta(u^2)d(w)\}\,.\end{aligned}$$

Since R is a 2-torsion free, we obtain

$$d((wu)u) = \theta(u^2)d(w) + \theta(u)\theta(w)d(u) + \theta(w)\theta(u)d(u) \quad \text{for all} \quad u,w \in U \tag{3.2}$$

Since $d([u,w]) = 0$ for all $u,w \in U$, using Lemma 2.5(iv) and (3.2), we get $2[\theta(u),\theta(w)]d(u) = 0$. This implies that

$$[\theta(u),\theta(w)]d(u) = 0 \quad \text{for all} \quad u,w \in U\,. \tag{3.3}$$

Now, replacing w by $2wv$ in (3.3) and using the fact that $\operatorname{char} R \neq 2$ we get $[\theta(u),\theta(w)]\theta(v)d(u) = 0$ i.e., $\theta^{-1}([\theta(u),\theta(w)])U\theta^{-1}(d(u) = \{0\}$. Thus by Lemma 2.1, we find that for each $u \in U, \theta^{-1}([\theta(u),\theta(w)]) = 0$ or $\theta^{-1}(d(u)) = 0$. This implies that $[u,w] = 0$ or $d(u) = 0$. Now let $U_1 = \{u \in U \mid \ [u,w] = 0$ for all

$w \in U\}$ and $U_2 = \{u \in U \mid d(u) = 0\}$. Clearly, U_1 and U_2 are additive subgroups of U whose union is U. But a group can not be written as a union of two of its proper subgroups and hence by Brauer's trick either $U = U_1$ or $U = U_2$. If $U = U_1$, then $[u, w] = 0$ for all $u, w \in U$ and by using the similar arguments as above we get $U \subseteq Z(R)$, again a contradiction. Hence we have the remaining possibility that $d(u) = 0$ for all $u \in U$ i.e., $d(U) = \{0\}$. This completes the proof of the theorem. □

As an application of the above theorem we get the following result, which generalizes the main theorem of [1].

Theorem 3.2. *Let R be a 2-torsion free prime ring and U a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. Suppose that θ is an automorphism of R. If $d : R \to R$ is an additive mapping satisfying $d(u^2) = 2\theta(u)d(u)$ for all $u \in U$, then $d(uv) = \theta(u)d(v) + \theta(v)d(u)$ for all $u, v \in U$.*

Proof. Suppose that $d = 0$ on U. Since $2uv \in U$, $uv - vu$ and $uv + vu$ both belong to U, we find that $2d(uv) = d(2uv) = 0$. This implies that $d(uv) = 0$ for all $u, v \in U$. Hence, the result is obvious in the present case. Therefore now assume that $d(U) \neq \{0\}$. Then by the above theorem $U \subseteq Z(R)$. Thus R satisfies the property $d(u^2) = d(u)\theta(u) + \theta(u)d(u)$ for all $u \in U$ and hence by Theorem 3.2 of [3] we find that $d(uv) = d(u)\theta(v) + \theta(u)d(v)$ for all $u, v \in U$. Further since $\theta(U) \subseteq Z(R)$, we find that $d(uv) = \theta(u)d(v) + \theta(v)d(u)$ holds for all $u, v \in U$. □

Corollary 3.1. *Let R be a 2-torsion free prime ring. If $d : R \to R$ is a Jordan left derivation, then d is a left derivation.*

If the underlying ring is arbitrary, then we have the following

Theorem 3.3. *Let R be a 2-torsion free ring and U a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. Suppose that θ is an endomorphism of R and R has a commutator which is not a zero divisor. If $d : R \to R$ is an additive mapping satisfying $d(u^2) = 2\theta(u)d(u)$ for all $u \in U$, then $d(uv) = \theta(u)d(v) + \theta(v)d(u)$ for all $u, v \in U$.*

Proof. For any $u, v \in U$, define a map $f : U \times U \to R$ such that $f(u, v) = d(uv) - \theta(u)d(v) - \theta(v)d(u)$. Since θ and d both are additive, f is additive in both the arguments and is zero if d is a left (θ, θ)-derivation. Note that (2.9) is still valid in the present situation and hence we have

$$[\theta(u), \theta(v)]f(u, v) = 0 \quad \text{for all} \quad u, v \in U\,. \tag{3.4}$$

Let a, b be fixed elements of U such that $[\theta(a), \theta(b)]c = 0$ implies that $c = 0$. Application of (3.4) yields that

$$f(a, b) = 0\,. \tag{3.5}$$

Replacing u by $u + a$ in (3.4) and using (3.4), we find that

$$[\theta(u), \theta(v)]f(a, v) + [\theta(a), \theta(v)f(u, v) = 0 \quad \text{for all} \quad u, v \in U\,. \tag{3.6}$$

Replacing v by b in (3.6) and using (3.6), we have

$$f(u,b)=0 \quad \text{for all} \quad u\in U\,. \tag{3.7}$$

Further, substituting $v+b$ for v in (3.6) and using (3.5) and (3.7), we get

$$[\theta(u),\theta(b)]f(a,v)+[\theta(a),\theta(b)]f(u,v)=0 \quad \text{for all} \quad u,v\in U\,. \tag{3.8}$$

Now replacing u by a in (3.8) and using the fact that $\operatorname{char} R\neq 2$, we have

$$f(a,v)=0 \quad \text{for all} \quad v\in U\,. \tag{3.9}$$

Combining of (3.8) and (3.9) yields that $[\theta(a),\theta(b)]f(u,v)=0$. This implies that $f(u,v)=0$ for all $u,v\in U$ i.e., d is a left (θ,θ)-derivation. □

In the end of this section it is tempting to conjecture as follows

Conjecture 3.1. *Let R be a 2-torsion free prime ring and U a Lie ideal of R such that $u^2\in U$ for all $u\in U$. Suppose that θ,ϕ are automorphisms of R. If $d:R\to R$ is an additive mapping satisfying $d(u^2)=\theta(u)d(u)+\phi(u)d(u)$ for all $u\in U$, then either $d(U)=\{0\}$ or $U\subseteq Z(R)$.*

4. Left derivation as a homomorphism or as an anti-homomorphism

Let S be a non-empty subset of R and $d:R\to R$ a derivation of R. If $d(xy)=d(x)d(y)$ (resp. $d(xy)=d(y)d(x)$) holds for all $x,y\in S$, then d is said to act as a homomorphism (resp. anti-homomorphism) on S. Recently, Bell and Kappe [8] proved that if K is a non-zero right ideal of a prime ring R and $d:R\to R$ a derivation of R such that d acts as a homomorphism on K, then $d=0$ on R. This result was further extended for (θ,ϕ)-derivation in [2] as follows:

Theorem 4.1 ([2, Theorem 3.2])**.** *Let R be a prime ring and K a nonzero ideal of R, and let θ,ϕ be automorphisms of R. Suppose that $d:R\to R$ is a (θ,ϕ)-derivation of R.*

(i) *If d acts as a homomorphism on K, then $d=0$ on R.*
(ii) *If d acts as an anti-homomorphism on K, then $d=0$ on R.*

In the present section our objective is to extend the above study to the left derivation of a prime ring R which acts either as a homomorphism or as an anti-homomorphism of R.

Theorem 4.2. *Let R be a prime ring and K a nonzero ideal of R, and let θ,ϕ be automorphisms of R. Suppose $d:R\to R$ is a left (θ,ϕ)-derivation of R.*

(i) *If d acts as an anti-homomorphism on K, then $d=0$ on R.*
(ii) *If d acts as a homomorphism on K, then $d=0$ on R.*

Proof. (i) Let d act as an anti-homomorphism on K. By our hypothesis, we have

$$d(xy)=\theta(x)d(y)+\phi(y)d(x) \quad \text{for all} \quad x,y\in K\,. \tag{4.1}$$

In (4.1) replacing y by xy, we get

$$d(xy)d(x) = d(x(xy)) = \theta(x)d(xy) + \phi(xy)d(x) \quad \text{for all} \quad x, y \in K\,. \tag{4.2}$$

Now multiplying (4.1) in the right by $d(x)$ and using the fact that d is an anti-homomorphism on K, we get

$$d(xy)d(x) = \theta(x)d(xy) + \phi(y)d(x)d(x) \quad \text{for all} \quad x, y \in K\,. \tag{4.3}$$

Combining (4.2) and (4.3), we get

$$\phi(x)\phi(y)d(x) = \phi(y)d(x)d(x)\,. \tag{4.4}$$

In (4.4) replace y by ry, to get

$$\phi(x)\phi(r)\phi(y)d(x) = \phi(r)\phi(y)d(x)d(x) \quad \text{for all} \quad x, y \in K \quad \text{and} \quad r \in R\,. \tag{4.5}$$

Multiplying (4.4) on left by $\phi(r)$ and combining with (4.5), we obtain

$$[\phi(r), \phi(x)]\phi(y)d(x) = 0\,. \tag{4.6}$$

In (4.6) replacing y by sy, we get

$$[\phi(r), \phi(x)]\phi(s)\phi(y)d(x) = 0 \quad \text{for all} \quad x, y \in K \quad \text{and} \quad r, s \in R\,,$$

and hence, $[r, x]Ry\phi^{-1}(d(x)) = \{0\}$ for all $x, y \in K$ and $r \in R$. Thus for each $x \in K$, the primeness of R forces that either $[r, x] = 0$ or $\phi(y)d(x) = 0$. Let $K_1 = \{x \in K \mid \phi(y)d(x) = 0 \text{ for all } y \in K\}$ and $K_2 = \{x \in K \mid [r, x] = 0 \text{ for all } r \in R\}$. Then clearly K_1 and K_2 are additive subgroups of K whose union is K. By Braur's trick, we have $\phi(y)d(x) = 0$ for all $x, y \in K$ or $[r, x] = 0$ for all $x \in K$ and $r \in R$. If $[r, x] = 0$, replace x by sx, to get $[r, s]x = 0$ for all $x \in K$ and $r, s \in R$, this implies that $[r, s]Rx = \{0\}$. The primeness of R forces that either $x = 0$ or $[r, s] = 0$, but $K \neq \{0\}$, we have $[r, s] = 0$ for all $r, s \in R$, i.e., R is commutative. So, $d(xy) = d(x)\phi(y) + \theta(x)d(y)$ for all $x, y \in K$ i.e., d is a (θ, ϕ)-derivation which acts as an anti-homomorphism on K. Hence by Theorem 4.1(ii), we have $d = 0$ on R. Henceforth, we have remaining possibility that

$$\phi(y)d(x) = 0 \quad \text{for all} \quad x, y \in K\,. \tag{4.7}$$

Replace y by yr in (4.7), to get $\phi(y)\phi(r)d(x) = 0$ for all $x, y \in K$ and $r \in R$, and hence $yR\phi^{-1}(d(x)) = \{0\}$. This implies that $\phi^{-1}(d(x)) = 0$, that is

$$d(x) = 0 \quad \text{for all} \quad y \in K\,. \tag{4.8}$$

Replace x by sx in (4.8), to get

$$\phi(x)d(s) = 0 \quad \text{for all} \quad x \in K \quad \text{and} \quad s \in R\,. \tag{4.9}$$

Replacing x by xr in (4.9), we get $\phi(x)\phi(r)d(s) = 0$ for all $x \in K$ and $r, s \in R$, and hence $xR\phi^{-1}(d(s)) = \{0\}$. Since R is prime, and K a nonzero ideal of R, we find that $d = 0$ on R.

(ii) If d acts as a homomorphism on K, then we have

$$d(x)d(y) = d(xy) = \theta(x)d(y) + \phi(y)d(x) \quad \text{for all} \quad x, y \in K\,. \tag{4.10}$$

Replacing x by xy in (4.10), we get

$$d(xy)d(y) = \theta(x)\theta(y)d(y) + \phi(y)d(xy) \quad \text{for all} \quad x, y \in K\,.$$

Now, application of (4.10) yields that $\theta(x)d(y)d(y) = \theta(x)\theta(y)d(y)$. This implies that

$$\theta(x)(d(y) - \theta(y))d(y) = 0 \quad \text{for all} \quad x, y \in K\,. \tag{4.11}$$

Replace x by xr in (4.11), to get $\theta(x)\theta(r)(d(y) - \theta(y))d(y) = 0$ for all $x, y \in K$ and $r \in R$, and hence, $xR\theta^{-1}((d(y) - \theta(y))d(y)) = \{0\}$ for all $x, y \in K$. The primeness of R forces that either $x = 0$ or $\theta^{-1}((d(y) - \theta(y))d(y)) = 0$. Since K is a nonzero ideal of R, we have $\theta^{-1}((d(y) - \theta(y))d(y)) = 0$, this yields that $(d(y) - \theta(y))d(y) = 0$ that is $d(y^2) = \theta(y)d(y)$. Since d is a left (θ, ϕ)-derivation, we find that $\phi(y)d(y) = 0$. Linearizing the latter relation, we have

$$\phi(y)d(x) + \phi(x)d(y) = 0 \quad \text{for all} \quad x, y \in K\,. \tag{4.12}$$

Replace x by yx in (4.12), to get

$$\phi(y)\phi(x)d(y) = 0 \quad \text{for all} \quad x, y \in K\,. \tag{4.13}$$

Substituting sx for x in (4.13), we get $\phi(y)\phi(s)\phi(x)d(y) = 0$ for all $x, y \in K$ and $s \in R$, and hence $yRx\phi^{-1}(d(y)) = \{0\}$. Thus for each $y \in K$; the primeness of R forces that either $y = 0$ or $x\phi^{-1}(d(y)) = 0$. But $y = 0$ also implies that $x\phi^{-1}(d(y)) = 0$, that is

$$\phi(x)d(y) = 0 \quad \text{for all} \quad x, y \in K\,. \tag{4.14}$$

Now using similar techniques as used to get (i) from (4.7) we get the required result.

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