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WEAK DISCRETE MAXIMUM PRINCIPLES

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Abstract. We introduce weak discrete maximum principles for matrix equations associated with some elliptic problems. We also give an example on discrete maximum principles.

1. Introduction

The maximum principles play a basic role in the theory and applications of a wide class of real linear second order elliptic partial differential equations.

In this paper we consider the weak discrete maximum principle for matrix equations associated with some elliptic problems and give an example to illustrate the usefulness of the discrete maximum principles.

Let Ω be a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$. The second order elliptic partial differential operator L at $x = (x_1, x_2, \dots, x_n)$ takes the form

$$L[u(x)] = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i\, \partial x_j} + \sum_{i=1}^{n} a_i(x) \frac{\partial u}{\partial x_i} + a(x)u(x)\,,$$

where $u(x)$ is a twice continuously differentiable function defined in Ω, $a_{ij}(x)$, $a_i(x)$, $(i, j = 1, 2, \dots, n)$, $a(x)$ are continuous functions in $\overline{\Omega} = \Omega \cup \partial\Omega$, $a(x) \geq 0$, $a_{ij} = a_{ji}$, $(i, j = 1, 2, \dots n)$, and there exist a positive constant μ such that

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_j \xi_i \geq \mu \sum_{i=1}^{n} \xi_1^2 \quad \text{in} \quad \Omega$$

for all n-tuples $(\xi_1, \xi_2, \dots, \xi_n)$ of real numbers.

Let $\frac{\partial}{\partial v} = \sum\limits_{i=1}^{n} v_i \frac{\partial}{\partial x_i}$ denote the outward directional derivative on $\partial\Omega$ in the direction v such that $v \cdot n > 0$, where n is the unit outward normal on $\partial\Omega$.

The strong and weak maximum principles are given in the following Theorems [7].

Theorem 1.1 (Strong maximum principle). *Suppose that $u(x)$ satisfies $L[u] \leq 0$ in Ω. If u attains a nonnegative maximum M at an interior point of Ω, then u attains its maximum in $\overline{\Omega}$.*

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Theorem 1.2 (Weak maximum principle)**.** *Suppose that $u(x)$ satisfies $L[u] \le 0$ in Ω, then* $\max_{x\in\overline{\Omega}} u(x) \le \max\{0, \max_{x\in\partial\Omega} u(x)\}$.

Theorem 1.3. *Let $u(x)$ satisfy $L[u] \le 0$ in Ω. Suppose that u attains a nonnegative maximum M at a boundary point P. If P lies on the boundary of a ball in Ω, then $\frac{\partial u}{\partial y} > 0$ at P, unless $u = M$ in $\overline{\Omega}$.*

Remark 1.1. The condition that P lies on the boundary of a ball in Ω in Theorem 1.3 is called the *interior sphere property* at P of $\partial\Omega$. If this property is not satisfied, the conclusion of Theorem 1.3 will in general not be valid.

Let us first present some matrix notations.

Given a matrix $A = (a_{ij})$, $i, j = 1, 2, \ldots, n$, we say that A is *reducible* if there exist a permutation matrix B such that

$$BAB^t = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where B^t denotes the transpose of B, A_{11} is an $r \times r$ submatrix, and A_{22} is an $(n-r) \times (n-r)$ submatrix with $r \in \{1, 2, \ldots, n-1\}$.

If no such permutation matrix exists, then A is irreducible. We say that A is *diagonally dominant* if

$$|a_{ii}| \ge \sum_{\substack{j=1\\ j\ne 1}} |a_{ij}| \qquad i = 1, 2, \ldots, n\,.$$

Furthermore, A is irreducibly diagonally dominant if it is irreducible, diagonally dominant, and

$$|a_{i_0 i_0}| > \sum_{\substack{j=1\\ j\ne i_0}} |a_{i_0 j}|$$

holds for some $i_0 \in \{1, 2, \ldots, n\}$.

Consider the discrete approximation to the boundary value problem

$$L\big(u(x)\big) = f(x) \quad \text{in } \Omega\,, \tag{1.1}$$

$$\beta\big(u(x)\big) = g(x) \quad \text{on } \partial\Omega\,. \tag{1.2}$$

Here β is the boundary operator, for example

$$\beta\big(u(x)\big) = \alpha_1(x)\frac{\partial u}{\partial n} + \alpha_2(x)\,u\,.$$

By P_i, $i = 1, 2, \ldots, n$ (or P_i, $i = n+1, \ldots, n+m$) we denote the nodal points which belong to Ω (or $\partial\Omega$). The finite difference method or the finite element method with mesh h size leads to the following equations which approximate (1.1), (1.2).

$$\sum_{j=1}^{n} a_{ij}w_j + \sum_{j=n+1}^{n+m} a_{ij}w_j = f_i\,, \qquad i = 1, 2, \ldots, n\,, \tag{1.3}$$

$$\sum_{j=1}^{n} a_{ij}w_j + \sum_{j=n+1}^{n+m} a_{ij}w_j = g_i\,, \qquad i = n+1, \ldots, n+m\,. \tag{1.4}$$

Here $(w_1, w_2, \ldots, w_{n+m})$, $(f_1, f_2, \ldots, f_n)$, $(g_{n+1}, g_{n+2}, \ldots, g_{n+m})$, are approximate values of $\big(u(P_1), u(P_2), \ldots, u(P_{n+m})\big)$, $\big(f(P_1), f(P_2), \ldots, f(P_n)\big)$, $\big(g(P_{n+1}), u(P_{n+2}), \ldots, g(P_{n+m})\big)$, respectively.

The matrices (a_{ij}), $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n+m$, and (a_{ij}), $i = n+1, \ldots, n+m$, $j = 1, 2, \ldots, n+m$, are the approximations to the operators L and β, respectively.

Now define sets as follows:

$$\begin{aligned} D_n &= \{1, 2, \ldots, n\}\,, \\ F_{i,m} &= \{j, j = n+1, \ldots, n+m, a_{ij} \neq 0\}\,, \quad i = 1, 2, \ldots, n\,. \\ G_m &= \bigcup_{i=1}^{n} F_{i,m}\,, \\ H_m &= \{n+1, n+2, \ldots, n+m\} \backslash G_m\,. \end{aligned}$$

A strong discrete maximum principle for (1.3), (1.4) is given in the following Theorem [6].

Theorem 1.4. *Assume that* $a_{ii} > 0$, $a_{ij} \leq 0$, $i \neq j$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n+m$.

$$\sum_{j=1}^{n+m} a_{ij} > 0\,, \qquad i = 1, 2, \ldots, n\,,$$

and that $A = (a_{ij})$, $i, j, = 1, 2, \ldots, n$, *is irreducibly diagonally dominant. Let* $(w_1, w_2, \ldots, w_{n+m})$ *satisfy*

$$\sum_{j=1}^{n+m} a_{ij}w_j \leq 0\,, \qquad i = 1, 2, \ldots, n\,.$$

If there exist some $r \in \{1, 2, \ldots, n\}$ *such that*

$$\max_{j=1,2,\ldots,n+m} w_j = w_r \geq 0\,,$$

then

$$\begin{aligned} w_j &= w_r\,, \qquad j \in D_n \cup G_m\,, \\ w_j &\leq w_r\,, \qquad j \in H_m\,. \end{aligned}$$

2. The weak discrete maximum principle

In this section we introduce the weak discrete maximum principle for (1.3), (1.4).

Theorem 2.1 (Weak discrete maximum principle [1])**.** *Assume that* $a_{ii} > 0$*,* $a_{ij} \leq 0$*,* $i \neq j$*,* $i = 1, 2, \ldots, n$*,* $j = 1, 2, \ldots, n+m$*.*

$$\sum_{j=1}^{n+m} a_{ij} > 0\,, \qquad i = 1, 2, \ldots, n\,,$$

and that $A = (a_{ij})$*,* $i, j = 1, 2, \ldots, n$*, is irreducibly diagonally dominant. Let* $(w_1, w_2, \ldots, w_{n+m})$ *satisfy*

$$\sum_{j=1}^{n+m} a_{ij} w_j \leq 0\,, \qquad i = 1, 2, \ldots, n\,.$$

Then,

$$\max_{1 \leq i \leq n+m} w_i \leq \max\left\{0, \max_{n+1 \leq j \leq n+m}\right\}.$$

Theorem 2.2. *Assume that* $a_{ii} > 0$*,* $a_{ij} \leq 0$*,* $i \neq j$*,* $i = 1, 2, \ldots, n$*,* $j = 1, 2, \ldots, n+m$*,*

$$\sum_{j=1}^{n+m} a_{ij} \geq 0\,, \qquad i = 1, 2, \ldots, n+m\,,$$

and that $A = (a_{ij})$*,* $i, j = 1, 2, \ldots, n$ *is irreducibly diagonally dominant. Let* $(w_1, w_2, \ldots, w_{n+m})$ *satisfy*

$$\sum_{j=1}^{n+m} a_{ij} w_j \leq 0\,, \qquad i = 1, 2, \ldots, n\,.$$

If there exist some $r(n+1 \leq r \leq n+m)$ *and at least one* q $(1 \leq q \leq n)$ *such that*

$$\max\left\{0, \max_{1 \leq j \leq n+m} w_j\right\} = w_r \geq 0\,,$$

$$\sum_{j=1}^{n+m} a_{rj} \geq 0\,, \qquad j \neq r\,, \quad j = 1, 2, \ldots, n+m\,, \quad a_{rq} \leq 0\,,$$

then

$$\sum_{j=1}^{n+m} a_{rj} w_j > 0$$

unless

$$\begin{aligned} w_j &= w_r\,, \qquad && j \in D_n \cup G_m\,,\\ w_j &\neq w_r\,, \qquad && j \in H_m\,. \end{aligned}$$

Proof. Assume that

$$\sum_{j=1}^{n+m} a_{rj} w_j \leq 0 \,.$$

Then we have

$$\begin{aligned} 0 \geq \sum_{j=1}^{n+m} a_{rj} w_j &= a_{rr} w_r + \sum_{\substack{j=1 \\ j \neq r}}^{n+m} a_{rj} w_j \\ &\geq \Bigg[-\sum_{\substack{j=1 \\ j \neq r}}^{n+m} a_{rj} \Bigg] w_r + \sum_{\substack{j=1 \\ j \neq r}}^{n+m} a_{rj} w_j \\ &= -\sum_{\substack{j=1 \\ j \neq r}}^{n+m} a_{rj} (w_r - w_j) \\ &= -a_{rq}(w_r - w_q) - \sum_{\substack{j=1 \\ j \neq r \\ j \neq q}}^{n+m} a_{rj}(w_r - w_j) \geq 0 \,. \end{aligned}$$

From the fact $a_{rq} < 0$, it follows that

$$w_q = w_r + \max_{1 \leq j \leq n+m} w_j \geq 0 \,.$$

Since $1 \leq q \leq n$, an application of Theorem 1.4 leads to

$$w_j = w_q = w_r \qquad j \in D_n \cup G_m \,. \tag{2.1}$$

On the other hand, it is clear that

$$w_j \leq w_q = w_r \,, \qquad j \in H_m \,. \tag{2.2}$$

Hence, (2.1), (2.2) contradicts the hypotheses in Theorem 2.2. Thus, we have

$$\sum_{j=1}^{n+m} a_{rj} w_j \leq 0 \,.$$

This completes the proof. □

3. Example

In this section we give an example to illustrate the discrete maximum principles. Consider the domain $\Omega \subset R^2$ with boundary $\partial\Omega$, P_1, P_2 denote the interior points of Ω and $P_3 - -P_{12}$ denote the boundary points shown in Figure 1 with a uniform mesh ($h = 1, n = 2, m = 10$).

$$\begin{aligned} -\Delta u + u &= -5 \qquad \text{in} \quad \Omega \,, \\ u &= -9 \qquad \text{on} \quad \partial\Omega \,. \end{aligned}$$

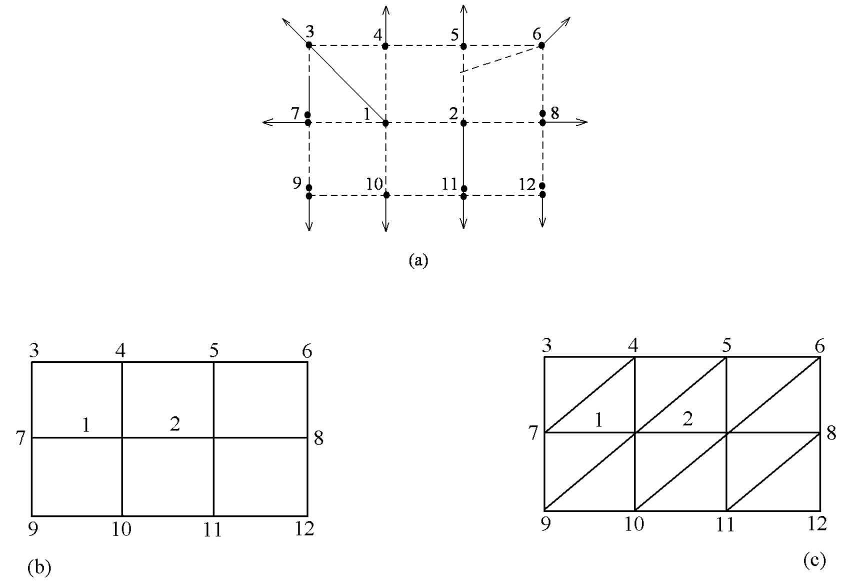

Fig. 1: Uniform mesh of the domain ($h = 1, n = 2, m = 10$):
(a) domain direction (b) finite difference (c) finite element.

The finite difference method with mesh points in Figure 1 (b) leads to

$$\begin{bmatrix} 5 & -1 & \dots & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 5 & \dots & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} -8 \\ -8 \\ \dots \\ -9 \\ -9 \\ -9 \\ -9 \\ -9 \\ -9 \\ -9 \\ -9 \\ -9 \\ -9 \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \end{bmatrix}$$

In this case $D_n = \{1, 2\}$, $G_m = \{4, 5, 7, 8, 10, 11\}$, $H_m = \{3, 6, 9, 12\}$. Since $\max_{1 \leq j \leq 12} w_j = w_1 = w_2 = -8$, the assumptions of Theorem 1.4 do not hold, but those of Theorem 2.1 hold.

4. Concluding remarks

If $A = (a_{ij})$, $i, j = 1, 2, \ldots, n$, is a matrix satisfying all conditions of Theorem 1.4, then $A^{-1} > 0$.

In view of applications to numerical analysis, the discrete maximum principle is useful in the resulting matrix equations, which approximate elliptic boundary value problems by employing the finite difference method or the finite element method. The weak discrete maximum principle is the discrete counterpart of Theorem 1.2 and well established in [2]. Moreover, this is applied not only to linear boundary value problems, but also to nonlinear boundary value problem [4], [5].

In the discrete matrix equations associated with the elliptic boundary value problems, it is desirable to reduce the number of the boundary points belonging to H_m as much as possible, in order to obtain good numerical solutions from the viewpoint of the maximum principle [2].

References

[1] Ciarlet, P. G., *Discrete maximum principles for finite difference operators*, Aequationes Math. **4** (1970), 336–352.

[2] Ciarlet, P. G. and Raviart, P. A., *Maximum principle and uniform convergence for the finite element method*, Comput. Methods Appl. Mech. Engrg. **2** (1973), 17–31.

[3] Gilbarg, D. and Trudinger, N. S., *Elliptic partial differential equations of second order*, Springer, New York, 1983.

[4] Ishihara, K., *On finite element schemes of the Dirichlet problem for a system of nonlinear elliptic equations*, Numer. Funct. Anal. Optim. **3** (1981), 105–136.

[5] Ishihara, K., *Finite element approximations applied to the nonlinear boundary value problem* $\Delta u = bu^2$, Publ. Res. Inst. Math. Sci. **18** (1982), 17–34.

[6] Al-Mahameed, M. M., *A discrete maximum principle*, Far East J. Appl. Math. **5** (3) (2001), 309–315.

[7] Greenspan, D., *Introductory numerical analysis of elliptic boundary value problems*, Harper and Row, New York (1965).

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