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ON A SUBCLASS OF α-UNIFORM CONVEX FUNCTIONS

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Abstract. In this paper we define a subclass of α-uniform convex functions by using the S'al'agean differential operator and we obtain some properties of this class.

1. Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U, $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$, $\mathcal{H}_u(U) = \{f \in \mathcal{H}(U) : f \text{ is univalent in } U\}$ and $S = \{f \in A : f \text{ is univalent in } U\}$.

Let consider the integral operator $L_a : A \to A$ defined as:

$$f(z) = L_a F(z) = \frac{1+a}{z^a} \int_0^z F(t) \cdot t^{a-1}\, dt\,, \quad a \in \mathbb{C}\,, \Re\, a \geq 0\,. \tag{1}$$

In the case $a = 1, 2, 3, \ldots$ this operator was introduced by S.D. Bernardi and it was studied by many authors in different various cases. In the form (1) it was used first time by N. N. Pascu.

Let D^n be the S'al'agean differential operator (see [10]) defined as:

$$D^n : A \to A\,, \quad n \in \mathbb{N} \quad \text{and} \quad D^0 f(z) = f(z)\,,$$

$$D^1 f(z) = Df(z) = zf'(z)\,, \quad D^n f(z) = D\left(D^{n-1} f(z)\right)\,.$$

2. Preliminary results

Definition 2.1 ([4])**.** Let $f \in A$. We say that f is n-uniform starlike function of order γ and of type β if

$$\Re\left(\frac{D^{n+1} f(z)}{D^n f(z)}\right) \geq \beta \cdot \left|\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right| + \gamma\,, \quad z \in U$$

where $\beta \geq 0$, $\gamma \in [-1, 1)$, $\beta + \gamma \geq 0$, $n \in \mathbb{N}$. We denote this class with $US_n(\beta, \gamma)$.

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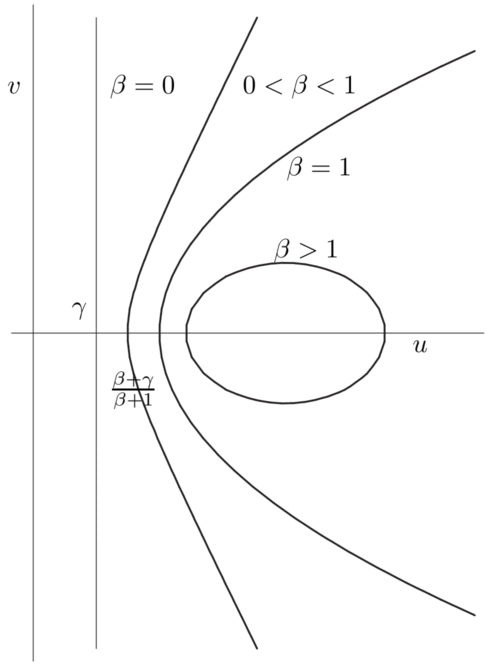
Remark 2.1. Geometric interpretation: $f \in US_n(\beta, \gamma)$ if and only if $\frac{D^{n+1}f(z)}{D^n f(z)}$ takes all values in the convex domain $D_{\beta,\gamma}$ which is included in right half plane. More, $D_{\beta,\gamma}$ is an elliptic region for $\beta > 1$, a parabolic region for $\beta = 1$, a hyperbolic region for $0 < \beta < 1$, the half plane $u > \gamma$ for $\beta = 0$.



Remark 2.2. If we take $n = 0$ and $\beta = 1$ in Definition 2.1 we obtain $US_0(1, \gamma) = SP\left(\frac{1-\gamma}{2}, \frac{1+\gamma}{2}\right)$, where the class $SP(\alpha, \beta)$ was introduced by F. Ronning in [9]. Also we have $US_n(\beta, \gamma) \subset S^*$, where S^* is the well know class of starlike functions (see [2]).

Definition 2.2 ([4])**.** Let $f \in A$. We say that f is α-uniform convex function, $\alpha \in [0, 1]$ if

$$\Re\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} \geq \left|(1-\alpha)\left(\frac{zf'(z)}{f(z)} - 1\right) + \alpha\frac{zf''(z)}{f'(z)}\right|, \quad z \in U\,.$$

We denote this class with UM_α.

Remark 2.3. Geometric interpretation: $f \in UM_\alpha$ if and only if

$$J(\alpha, f; z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)$$

takes all values in the parabolic region $\Omega = \{w : |w-1| \leq \Re w\} = \{w = u+iv; v^2 \leq 2u - 1\}$. Also, we have $UM_\alpha \subset M_\alpha$, where M_α is the well know class of α-convex functions introduced by P. T. Mocanu in [8].

The next theorem is a result of the so called "admissible functions method" introduced by P. T. Mocanu and S. S. Miller (see [5], [6], [7]).

Theorem 2.1. *Let h be convex in $\mathcal{U}$ and $\Re[\beta h(z)+\delta]>0,\ z\in\mathcal{U}$. If $p\in\mathcal{H}(\mathcal{U})$ with $p(0)=h(0)$ and p satisfied the Briot-Bouquet differential subordination $p(z)+\dfrac{zp'(z)}{\beta p(z)+\delta}\prec h(z)$, then $p(z)\prec h(z)$, where by "$\prec$" we denote the subordination relation.*

3. Main results

Definition 3.1. Let $\alpha\in[0,1]$ and $n\in\mathbb{N}$. We say that $f\in A$ is in the class $UD_{n,\alpha}(\beta,\gamma)$, $\beta\geq 0$, $\gamma\in[-1,1)$, $\beta+\gamma\geq 0$, if

$$\Re\left[(1-\alpha)\frac{D^{n+1}f(z)}{D^nf(z)}+\alpha\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right]$$
$$\geq\beta\left|(1-\alpha)\frac{D^{n+1}f(z)}{D^nf(z)}+\alpha\frac{D^{n+2}f(z)}{D^{n+1}f(z)}-1\right|+\gamma\,.$$

Remark 3.1. We have $UD_{n,0}(\beta,\gamma)=US_n(\beta,\gamma)\subset S^*$,$UD_{0,\alpha}(1,0)=UM_\alpha$ and $UD_{0,1}(\beta,\gamma)=US^c(\beta,\gamma)\subset S^c\left(\dfrac{\beta+\gamma}{\beta+1}\right)$, where $US^c(\beta,\gamma)$ is the class of the uniform convex functions of type β and order γ introduced by I. Magdaş in [4] and $S^c(\delta)$ is the well know class of convex functions of order δ (see [2]).

Remark 3.2. Geometric interpretation: $f\in UD_{n,\alpha}(\beta,\gamma)$ if and only if

$$J_n(\alpha,f;z)=(1-\alpha)\frac{D^{n+1}f(z)}{D^nf(z)}+\alpha\frac{D^{n+2}f(z)}{D^{n+1}f(z)}$$

takes all values in the convex domain $D_{\beta,\gamma}$, where $D_{\beta,\gamma}$ is defined in Remark 2.1.

Theorem 3.1. *For all $\alpha,\alpha'\in[0,1]$ with $\alpha<\alpha'$, we have*

$$UD_{n,\alpha'}(\beta,\gamma)\subset UD_{n,\alpha}(\beta,\gamma)\,.$$

Proof. From $f\in UD_{n,\alpha'}(\beta,\gamma)$ we have

$$\Re\left[(1-\alpha')\frac{D^{n+1}f(z)}{D^nf(z)}+\alpha'\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right]$$
$$\geq\beta\left|(1-\alpha')\frac{D^{n+1}f(z)}{D^nf(z)}+\alpha'\frac{D^{n+2}f(z)}{D^{n+1}f(z)}-1\right|+\gamma\,.$$

With the notations $\dfrac{D^{n+1}f(z)}{D^n f(z)} = p(z)$, where $p(z) = 1 + p_1 z + \dots$, we have

$$zp'(z) = z\frac{\left(D^{n+1}f(z)\right)' \cdot D^n f(z) - D^{n+1}f(z) \cdot \left(D^n f(z)\right)'}{\left(D^n f(z)\right)^2}$$
$$= \frac{D^{n+2}f(z)}{D^n f(z)} - \left(\frac{D^{n+1}f(z)}{D^n f(z)}\right)^2,$$

$$\frac{zp'(z)}{p(z)} = \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - \frac{D^{n+1}f(z)}{D^n f(z)},$$

and thus we obtain

$$J_n(\alpha', f; z) = p(z) + \alpha' \cdot \frac{zp'(z)}{p(z)}.$$

Now we have that $p(z) + \alpha' \cdot \dfrac{zp'(z)}{p(z)}$ takes all values in the convex domain $D_{\beta,\gamma}$ which is included in right half plane.

If we consider $h \in \mathcal{H}_u(U)$, with $h(0) = 1$, which maps the unit disc U into the convex domain $D_{\beta,\gamma}$, we have $\Re\, h(z) > 0$ and from hypothesis $\alpha' > 0$. From here follows that $\Re\dfrac{1}{\alpha'} \cdot h(z) > 0$. In this conditions from Theorem 2.1 , whit $\delta = 0$ we obtain $p(z) \prec h(z)$, or $p(z)$ take all values in $D_{\beta,\gamma}$.

If we consider the function $g : [0, \alpha'] \to \mathbb{C}$, $g(u) = p(z) + u \cdot \dfrac{zp'(z)}{p(z)}$, with $g(0) = p(z) \in D_{\beta,\gamma}$ and $g(\alpha') \in D_{\beta,\gamma}$. Since the geometric image of $g(\alpha)$ is on the segment obtained by the union of the geometric image of $g(0)$ and $g(\alpha')$, we have $g(\alpha) \in D_{\beta,\gamma}$, or

$$p(z) + \alpha \cdot \frac{zp'(z)}{p(z)} \in D_{\beta,\gamma}.$$

Thus $J_n(\alpha, f; z)$ takes all values in $D_{\beta,\gamma}$, or $f \in UD_{n,\alpha}(\beta, \gamma)$. □

Remark 3.3. From Theorem 3.1 we have $UD_{n,\alpha}(\beta,\gamma) \subset UD_{n,0}(\beta,\gamma)$ for all $\alpha \in [0, 1]$, and from Remark 3.1 we obtain that the functions from the class $UD_{n,\alpha}(\beta, \gamma)$ are univalent.

Theorem 3.2. *If $F(z) \in UD_{n,\alpha}(\beta,\gamma)$ then $f(z) = L_a(F)(z) \in US_n(\beta,\gamma)$, where L_a is the integral operator defined by* (1)*.*

Proof. From (1) we have

$$(1 + a)F(z) = af(z) + zf'(z).$$

By means of the application of the linear operator D^{n+1} we obtain

$$(1 + a)D^{n+1}F(z) = aD^{n+1}f(z) + D^{n+1}(zf'(z))$$

or

$$(1 + a)D^{n+1}F(z) = aD^{n+1}f(z) + D^{n+2}f(z).$$

Thus:

$$\frac{D^{n+1}F(z)}{D^nF(z)} = \frac{D^{n+2}f(z) + aD^{n+1}f(z)}{D^{n+1}f(z) + aD^nf(z)}$$

$$= \frac{\dfrac{D^{n+2}f(z)}{D^{n+1}f(z)} \cdot \dfrac{D^{n+1}f(z)}{D^nf(z)} + a \cdot \dfrac{D^{n+1}f(z)}{D^nf(z)}}{\dfrac{D^{n+1}f(z)}{D^nf(z)} + a}.$$

With the notation $\dfrac{D^{n+1}f(z)}{D^nf(z)} = p(z)$ where $p(z) = 1 + p_1z + \dots$, we have:

$$zp'(z) = z \cdot \left(\frac{D^{n+1}f(z)}{D^nf(z)}\right)' = \frac{z\left(D^{n+1}f(z)\right)' \cdot D^nf(z) - D^{n+1}f(z) \cdot z\left(D^nf(z)\right)'}{\left(D^nf(z)\right)^2}$$

$$= \frac{D^{n+2}f(z) \cdot D^nf(z) - \left(D^{n+1}f(z)\right)^2}{\left(D^nf(z)\right)^2}$$

and

$$\frac{1}{p(z)} \cdot zp'(z) = \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - \frac{D^{n+1}f(z)}{D^nf(z)} = \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - p(z).$$

It follows:

$$\frac{D^{n+2}f(z)}{D^{n+1}f(z)} = p(z) + \frac{1}{p(z)} \cdot zp'(z).$$

Thus we obtain:

$$\frac{D^{n+1}F(z)}{D^nF(z)} = \frac{p(z) \cdot \left(zp'(z) \cdot \frac{1}{p(z)} + p(z)\right) + a \cdot p(z)}{p(z) + a}$$

$$= p(z) + \frac{1}{p(z) + a} \cdot zp'(z).$$

If we denote $\dfrac{D^{n+1}F(z)}{D^nF(z)} = q(z)$, with $q(0) = 1$, and we consider $h \in \mathcal{H}_u(U)$, with $h(0) = 1$, which maps the unit disc U into the convex domain $D_{\beta,\gamma}$, we have from $F(z) \in UD_{n,\alpha}(\beta,\gamma)$ (see Remark 3.2):

$$q(z) + \alpha \cdot \frac{zq'(z)}{q(z)} \prec h(z).$$

From Theorem 2.1, with $\delta = 0$ we obtain $q(z) \prec h(z)$, or

$$p(z) + \frac{1}{p(z) + a} \cdot zp'(z) \prec h(z).$$

Using the hypothesis and the construction of the function $h(z)$ we obtain from Theorem 2.1 $p(z) \prec h(z)$ or $f(z) \in US_n(\beta,\gamma)$ (see Remark 2.1). $\square$

Remark 3.4. From Theorem 3.2 with $\alpha = 0$ we obtain the Theorem 3.1 from [1] which assert that the integral operator L_a, defined by (1), preserve the class $US_n(\beta, \gamma)$.

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