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SOLUTIONS OF A MULTI-POINT BOUNDARY VALUE PROBLEM FOR HIGHER-ORDER DIFFERENTIAL EQUATIONS AT RESONANCE (II)

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Abstract. In this paper, we are concerned with the existence of solutions of the following multi-point boundary value problem consisting of the higher-order differential equation

$$x^{(n)}(t) = f(t, x(t), x'(t), \dots, x^{(n-1)}(t)) + e(t)\,, \quad 0 < t < 1\,, \tag{$*$}$$

and the following multi-point boundary value conditions

$$\begin{gathered} x^{(i)}(0) = 0 \quad \text{for} \quad i = 0, 1, \dots, n-3\,, \\ x^{(n-1)}(0) = \alpha x^{(n-1)}(\xi)\,, \quad x^{(n-2)}(1) = \sum_{i=1}^{m} \beta_i x^{(n-2)}(\eta_i)\,. \end{gathered} \tag{$**$}$$

Sufficient conditions for the existence of at least one solution of the BVP $(*)$ and $(**)$ at resonance are established. The results obtained generalize and complement those in [13, 14]. This paper is directly motivated by Liu and Yu [J. Pure Appl. Math. 33 (4)(2002), 475–494 and Appl. Math. Comput. 136 (2003), 353–377].

1. Introduction

Recently, there has been considerable interest in the solvability of multi-point boundary value problems for second order differential equations, which can arise in many applications, we refer the reader to the monographs [1–3] and the references [6–11, 19–21].

In [14], Liu and Yu studied the existence of solutions of the following multi-point boundary value problem

$$\begin{cases} x''(t) = f(t, x(t), x'(t)) + e(t)\,, & 0 < t < 1\,, \\ x'(0) = \alpha x'(\xi)\,, & x(1) = \beta x(\eta)\,, \end{cases} \tag{1}$$

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where f is continuous, $\alpha \geq 0$ and $\beta \geq 0$, $e \in L^1[0,1]$. They proved that, under some assumptions, BVP (1) has at least one solution in the following cases:

Case 1. $\alpha = 1$, $\beta = 0$ (see [14, Theorem 2.2]);
Case 2. $\alpha = 1$, $\beta = 1/\eta$ (see [14, Theorem 2.4]);
Case 3. $\alpha = 1$, $\beta = 1$ (see [14, Theorem 2.6]);
Case 4. $\alpha = 0$, $\beta = 1$ (see [14, Theorem 2.8]).

In [14], Liu studied the solvability of the following multi-point boundary value problem

$$(2) \qquad \begin{cases} x''(t) = f(t, x(t), x'(t)) + e(t)\,, & 0 < t < 1\,, \\ x'(0) = \alpha x'(\xi)\,, & x(1) = \sum_{i=1}^m \beta_i x(\eta_i)\,, \end{cases}$$

where $0 < \eta_1 < \cdots < \eta_m < 1$, $\beta_i \in R$, $0 < \xi < 1$, $\alpha \geq 0$ and f is continuous. He established the existence results for the following cases:

Case 1′. $\alpha = 1$, $\sum_{i=1}^m \beta_i = 0$, $\sum_{i=1}^m \beta_i \eta_i \neq 1$ (see [15, Theorem 3.1]);
Case 2′. $\alpha = 1$, $\sum_{i=1}^m \beta_i \eta_i = 1$, $\sum_{i=1}^m \beta_i \eta_i^2 \neq 1$ (see [15, Theorem 3.2]);
Case 3′. $\alpha = 1$, $1 - \sum_{i=1}^m \beta_i = \sum_{i=1}^m \beta_i \eta_i - 1 \neq 1$ (see [15, Theorem 3.3]);
Case 4′. $\alpha = 0$, $\sum_{i=1}^m \beta_i = 1$, $\sum_{i=1}^m \beta_i \eta_i - 1 = 1$ and $\sum_{i=1}^m \beta_i \eta_i^2 \neq 1$ (see [15, Theorem 3.4]).

We note that if

$$\begin{vmatrix} 1 - \alpha & 0 \\ 1 - \sum_{i=1}^m \beta_i \eta_i & 1 - \sum_{i=1}^m \beta_i \end{vmatrix} = 0\,,$$

then the linear operator $Lx(t) = x''(t)$ defined in a suitable Banach space is not invertible, i.e. the problem

$$\begin{aligned} &x''(t) = 0\,, && 0 < t < 1\,, \\ &x'(0) = \alpha x'(\xi)\,, && x(1) = \textstyle\sum_{i=1}^m \beta_i x(\eta_i) \end{aligned}$$

has non-trivial solutions, which is called resonance case, i.e. $\dim \operatorname{Ker} L \geq 1$. In Cases $1' - 4'$ and 1-4 mentioned above, we find $\dim \operatorname{Ker} L = 1$. It is easy to check that if

$$\alpha = 1\,, \quad \sum_{i=1}^m \beta_i \eta_i = 1 \quad \text{and} \quad \sum_{i=1}^m \beta_i = 1\,,$$

then $\dim \operatorname{Ker} L = 2$. However, this case was not discussed in [14, 15] by Liu and Yu.

Furthermore, to the best of our knowledge, there has been no paper concerned with the existence of solutions of the multi-point boundary value problems for higher-order differential equations at resonance, although there were considerable papers concerned with the existence of positive solutions or solutions of higher-order differential equations at non-resonance cases, we refer the reader to [1–3] and the papers [4, 5, 16].

Motivated and inspired by Liu [14, 15], we are concerned with the following higher-order differential equation

$$x^{(n)}(t) = f(t, x(t), x'(t), \dots, x^{(n-1)}(t)) + e(t)\,, \quad 0 < t < 1\,, \tag{3}$$

subjected to the following multi-point boundary value conditions

$$\begin{cases} x^{(i)}(0) = 0 & \text{for} \quad i = 0, 1, \dots, n-3\,, \\ x^{(n-1)}(0) = \alpha x^{(n-1)}(\xi)\,, & x^{(n-2)}(1) = \sum_{i=1}^{m} \beta_i x^{(n-2)}(\eta_i)\,, \end{cases} \tag{4}$$

where $0 < \xi < 1$, $0 < \eta_1 < \cdots < \eta_m < 1$, $\alpha \in R$, $\beta_i \in R (i = 1, \dots, m)$ are fixed and f is continuous, $e \in L^1[0, 1]$. The purpose of this paper is to generalize and complement the results in [14, 15]. By the way, we, in [17, 18], investigated the solvability of the following boundary value problems for higher-order differential equations

$$\begin{cases} x^{(n)}(t) = f\big(t, x(t), x'(t), \dots, x^{(n-1)}(t)\big) + e(t)\,, \quad 0 < t < 1\,, \\ x^{(i)}(0) = 0 \quad \text{for} \quad i = 0, 1, \dots, n-3\,, \\ x^{(n-1)}(0) = \alpha x^{(n-1)}(\xi)\,, \quad x^{(n-1)}(1) = \sum_{i=1}^{m} \beta_i x^{(n-1)}(\eta_i)\,, \end{cases}$$

and

$$\begin{cases} x^{(n)}(t) = f\big(t, x(t), x'(t), \dots, x^{(n-1)}(t)\big) + e(t)\,, \quad 0 < t < 1 \quad , \\ x^{(i)}(0) = 0 \quad \text{for} \quad i = 0, 1, \dots, n-3\,, \\ x^{(n-2)}(0) = \alpha x^{(n-1)}(\xi)\,, \quad x^{(n-1)}(1) = \beta x^{(n-2)}(\eta)\,, \end{cases}$$

respectively. Using the similar method in this paper, we can study the solvability of the following boundary value problem similar to BVP (3) and (4)

$$\begin{cases} x^{(n)}(t) = f\big(t, x(t), x'(t), \dots, x^{(n-1)}(t)\big) + e(t)\,, \quad 0 < t < 1\,, \\ x^{(i)}(0) = 0 \quad \text{for} \quad i = 0, 1, \dots, n-3\,, \\ x^{(n-1)}(1) = \alpha x^{(n-1)}(\xi)\,, \quad x^{(n-1)}(0) = \sum_{i=1}^{m} \beta_i x^{(n-1)}(\eta_i)\,. \end{cases}$$

We omit the details.

To obtain the main results, we need the following notations and an abstract existence theorem by Gaines and Mawhin [22, 23].

Let X and Y be Banach spaces, $L:\ D(L) \subset X \to Y$ be a Fredholm operator of index zero, $P:\ X \to X$, $Q:\ Y \to Y$ be projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L\,,\ \operatorname{Ker} Q = \operatorname{Im} L\,,\ X = \operatorname{Ker} L \oplus \operatorname{Ker} P\,,\ Y = \operatorname{Im} L \oplus \operatorname{Im} Q\,.$$

It follows that

$$L|_{D(L) \cap \operatorname{Ker} P} : D(L) \cap \operatorname{Ker} P \to \operatorname{Im} L$$

is invertible, we denote the inverse of that map by K_p.

If Ω is an open bounded subset of X, $D(L) \cap \overline{\Omega} \neq \emptyset$, map $N:\ X \to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N:\ \overline{\Omega} \to X$ is compact.

Theorem GM ([22, 23])**.** *Let L be a Fredholm operator of index zero and let N be L-compact on Ω. Assume that the following conditions are satisfied:*

(i) $Lx \neq \lambda Nx$ *for every* $(x,\lambda) \in [(D(L)/\operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$*;*
(ii) $Nx \notin \operatorname{Im} L$ *for every* $x \in \operatorname{Ker} L \cap \partial\Omega$*;*
(iii) $\deg(\Lambda QN|_{\operatorname{Ker} L}, \ \Omega \cap \operatorname{Ker} L, 0) \neq 0$*, where* $\Lambda : \ Y/\operatorname{Im} L \to \operatorname{Ker} L$ *is the isomorphism.*

Then the equation $Lx = Nx$ *has at least one solution in* $D(L) \cap \overline{\Omega}$*.*

We use the classical Banach space $C^k[0,1]$, let $X = C^{n-1}[0,1]$ and $Y = L^1[0,1]$. $C^0[0,1]$ is endowed with the norm $\|y\|_\infty = \max_{t\in[0,1]} |y(t)|$, X is endowed with the norm $\|x\| = \max\big\{\|x\|_\infty, \|x'\|_\infty, \ldots, \|x^{(n-1)}\|_\infty\big\}$. $L^1[0,1]$ is endowed the norm $\|x\|_1$ for $x \in L^1[0,1]$. We also use the Sobolev space $W^{n,1}(0,1)$ defined by

$$W^{n,1}(0,1) = \big\{x : [0,1] \to R \ \text{ such that } x, x' \ldots, x^{(n-1)} \text{ are absolutely continuous on } [0,1] \text{ with } x^{(n)} \in L^1[0,1]\big\}.$$

Define the linear operator L and the nonlinear operator N by

$$L: \ X \cap \operatorname{dom} L \to Y, \quad Lx(t) = x^{(n)}(t) \ \text{ for } x \in X \cap \operatorname{dom} L,$$

$$N: \ X \to Y, \qquad Nx(t) = f\big(t, x(t), x'(t), \ldots, x^{(n-1)}(t)\big) + e(t) \text{ for } x \in X,$$

respectively. This paper can be placed in the existence theory of boundary value problems for ordinary differential equations, The foundation and the most vital impact on this theory are closely related to mathematicians: Agarwal, O'Regan and Wong, whose scientific output is represented in monographs [1–3]. It is observed that this particular branch of differential equations has been constantly developed and gained prominence since the early 1980s.

2. Existence of solutions of BVP (3) and (4)

In this section, we establish the existence results for BVP (3) and (4) in the following cases:

Case (i) $\alpha = 1$, $\sum_{i=1}^m \beta_i = 1$, $\sum_{i=1}^m \beta_i\eta_i = 1$;
Case (ii) $\alpha = 0$, $\sum_{i=1}^m \beta_i = 1$;
Case (iii) $\alpha = 1$, $\sum_{i=1}^m \beta_i = 1$, $\sum_{i=1}^m \beta_i\eta_i \neq 1$;
Case (iv) $\alpha = 1$, $\sum_{i=1}^m \beta_i \neq 1$, $\sum_{i=1}^m \beta_i\eta_i = 1$.

We first consider Case (i). Let

$$\operatorname{dom} L = \Big\{x \in W^{n,1}(0,1), \ x^{(i)}(0) = 0 \text{ for } \ i = 0, 1, \ldots, n-3, \\ x^{(n-1)}(1) = x^{(n-1)}(\xi), x^{(n-2)}(1) = \sum_{i=1}^m \beta_i x^{(n-2)}(\eta_i)\Big\}.$$

Lemma 2.1. *The following results hold.*

(i) $\operatorname{Ker} L = \{at^{n-1} + bt^{n-2}, \ t \in [0,1], \ a, b \in R\}$*;*
(ii) $\operatorname{Im} L = \{y \in Y, \int_0^\xi y(s)\,ds = 0, \ \sum_{i=1}^m \beta_i \int_0^{\eta_i} (\eta_i - s)y(s)\,ds = \int_0^1 (1 - s)y(s)\,ds\}$*;*
(iii) *L is a Fredholm operator of index zero;*

(iv) *There is* $k \in \{0, 1, \ldots, m\}$ *such that*

$$\sum_{i=1}^{m} \beta_i \eta_i^{k+1} = 1 \quad \textit{and} \quad \sum_{i=1}^{m} \beta_i \eta^{k+2} \neq 1\,;$$

(v) *There are projectors* $P: X \to X$ *and* $Q: Y \to Y$ *such that* $\operatorname{Ker} L = \operatorname{Im} P$, $\operatorname{Ker} Q = \operatorname{Im} L$. *Furthermore, let* $\Omega \subset X$ *be an open bounded subset with* $\overline{\Omega} \cap \operatorname{dom} L \neq$, *then* N *is* L*-compact on* $\overline{\Omega}$*;*

(vi) $x(t)$ *is a solution of* BVP (3)–(4) *if and only if* x *is a solution of the operator equation* $Lx = Nx$ *in* $\operatorname{dom} L$.

Proof. (i) Let $x \in \operatorname{Ker} L$, then $x^{(n)}(t) = 0$ and $x^{(i)}(0) = 0$ for $i = 0, 1, \ldots, n-3$ and $x^{(n-1)}(0) = x^{(n-1)}(\xi)$ and $x^{(n-2)}(1) = \sum_{i=1}^{m} \beta_i x^{(n-2)}(\eta_i)$. It is easy to get $x(t) = at^{n-1} + bt^{n-2}$, thus $x \in \{at^{n-1} + bt^{n-2} : \ t \in [0,1], \ c \in R\ \}$. On the other hand, if $x(t) = at^{n-1} + bt^{n-2}$, then we find that $x \in \operatorname{Ker} L$. This completes the proof of (i).

(ii) For $y \in ImL$, then there is $x \in \operatorname{dom} L$ such that $x^{(n)}(t) = y(t)$ and $x^{(i)}(0) = 0$ for $i = 0, 1, \ldots, n-3$ and $x^{(n-1)}(0) = x^{(n-2)}(\xi)$ and $x^{(n-1)}(1) = \sum_{i=1}^{m} \beta_i x^{(n-2)}(\eta_i)$. Thus

$$x(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)\, ds + at^{n-1} + bt^{n-2}\,.$$

It follows from the boundary value conditions that

$$\int_0^{\xi} y(s)\, ds = 0\,, \ \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s) y(s)\, ds = \int_0^1 (1-s) y(s)\, ds\,. \tag{5}$$

On the other hand, if (5) holds, let

$$x(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)\, ds + at^{n-1} + bt^{n-2}\,, \quad t \in [0,1]\,.$$

Then $x \in \operatorname{dom} L \cap X$ and $Lx = y$. Thus the proof of (ii) is completed.

(iii) and **(iv)** For an integer k, let

$$\Delta_k = \begin{vmatrix} \int_0^{\xi} s^k\, ds & \int_0^{\xi} s^{k-1}\, ds \\ A_3 & A_2 \end{vmatrix}\,,$$

where

$$A_1 = \int_0^1 (1-s) y(s)\, ds - \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s) y(s)\, ds\,,$$

$$A_2 = \int_0^1 (1-s) s^{k-1}\, ds - \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s) s^{k-1}\, ds\,,$$

$$A_3 = \int_0^1 (1-s) s^k\, ds - \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s) s^k\, ds\,.$$

From (i), $\dim \operatorname{Ker} L = 2$. On the other hand, we claim that there is $k \in \{0, 1, \ldots, m\}$ such that
$$\sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s)s^k \, ds \neq \int_0^1 (1-s)s^k \, ds \,.$$
In fact, if for all $k \in \{0, 1, \ldots, m\}$, we have
$$\sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s)s^k \, ds = \int_0^1 (1-s)s^k \, ds \,.$$
Consider the equations
$$x_0 \int_0^1 (1-s)s^k \, ds + \sum_{i=1}^{m} x_i \int_0^{\eta_i} (\eta_i - s)s^k \, ds = 0 \,, \quad i = 0, 1, \ldots, m \,.$$
Since the determinant of coefficients of above equations is
$$D = \begin{vmatrix} \int_0^1 (1-s) \, ds & \int_0^{\eta_1} (\eta_1 - s) \, ds & \ldots & \int_0^{\eta_m} (\eta_m - s) \, ds \\ \vdots & \vdots & & \vdots \\ \int_0^1 (1-s)s^m ds & \int_0^{\eta_1} (\eta_1 - s)s^m ds & \ldots & \int_0^{\eta_m} (\eta_m - s)s^m \, ds \end{vmatrix} ,$$
it is easy to check that $D \neq 0$ since $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$. We get $x_0 = x_1 = \cdots = x_m = 0$, this contradicts $\sum_{i=1}^m \beta_i = 1$. Together eith $\sum_{i=1}^m \beta_i \eta_i = 1$, the proof of (iv) is complete.

If $y \in Y$, let k be defined in (iv), suppose
$$y - (At^k + Bt^{k-1}) \in \operatorname{Im} L \,.$$
It follows that
$$\int_0^{\xi} y(s) \, ds = A \int_0^{\xi} s^k ds + B \int_0^{\xi} s^{k-1} \, ds \,,$$
$$\begin{aligned} \int_0^1 (1-s)y(s)ds \quad & - \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s)y(s)ds \\ & = A\Big(\int_0^1 (1-s)s^k ds - \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s)s^k ds \Big) \\ & \quad + B\Big(\int_0^1 (1-s)s^{k-1} ds - \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s)s^{k-1} ds \Big) \,. \end{aligned}$$
It is easy to see $\Delta_k \neq 0$ from (iv). Then we get
$$A = \frac{1}{\Delta_k} \begin{vmatrix} \int_0^{\xi} y(s)ds & \int_0^{\xi} s^{k-1} ds \\ A_1 & A_2 \end{vmatrix} , \quad B = \frac{1}{\Delta_k} \begin{vmatrix} \int_0^{\xi} s^k ds & \int_0^{\xi} y(s)ds \\ A_3 & A_1 \end{vmatrix} \,.$$
For $y \in Y$, let
$$y_0 = y - \frac{t^k}{\Delta_k} \begin{vmatrix} \int_0^{\xi} y(s) \, ds & \int_0^{\xi} s^{k-1} \, ds \\ A_1 & A_2 \end{vmatrix} + \frac{t^{k-1}}{\Delta_k} \begin{vmatrix} \int_0^{\xi} s^k \, ds & \int_0^{\xi} y(s) \, ds \\ A_3 & A_1 \end{vmatrix} \,.$$

It is easy to check that $y_0 \in \operatorname{Im} L$. Let

$$\overline{R} = \{at^k + bt^{k-1} : \ t \in [0,1], \ c \in R\}.$$

We get $Y = \overline{R} + \operatorname{Im} L$. Again, $\overline{R} \cap \operatorname{Im} L = \{0\}$, so $Y = \overline{R} \oplus \operatorname{Im} L$. Hence $\dim Y/\operatorname{Im} L = 2$. On the other hand, $\operatorname{Im} L$ is closed. So L is a Fredholm operator of index zero. This completes the proof of (iii).

(v) Define the projectors $Q: \ Y \to Y$ and $P: \ X \to X$ by

$$Qy(t) = \frac{t^k}{\Delta_k} \begin{vmatrix} \int_0^\xi y(s)\,ds & \int_0^\xi s^{k-1}\,ds \\ A_1 & A_2 \end{vmatrix} + \frac{t^{k-1}}{\Delta_k} \begin{vmatrix} \int_0^\xi s^k\,ds & \int_0^\xi y(s)\,ds \\ A_3 & A_1 \end{vmatrix} \quad \text{for } \ y \in Y,$$

and

$$Px(t) = \frac{x^{(n-1)}(0)}{(n-1)!} t^{n-1} + \frac{x^{(n-2)}(0)}{(n-2)!} t^{n-2} \quad \text{for} \quad x \in X,$$

respectively. It is easy to prove that $\operatorname{Ker} L = \operatorname{Im} P$ and $\operatorname{Im} L = \operatorname{Ker} Q$. Then the inverse $K_p : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ of the map $L : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ can be written by

$$K_p y(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)\,ds \quad \text{for} \quad y \in \operatorname{Im} L.$$

In fact, for $y \in \operatorname{Im} L$, we have $(LK_p)y(t) = y(t)$. On the other hand, for $x \in \operatorname{Ker} P \cap \operatorname{dom} L$, it follows that

$$\begin{aligned} (K_p L)x(t) = K_p(x^{(n)}(t)) &= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} x^{(n)}(s)\,ds \\ &= -\frac{x^{(n-1)}(0)}{(n-1)!} t^{n-1} - \frac{x^{(n-2)}(0)}{(n-2)!} t^{n-2} + x(t) = x(t). \end{aligned}$$

Furthermore, let $\wedge : \operatorname{Ker} L \to \overline{R}$ be the isomophism with $\wedge(at^{n-1} + bt^{n-2}) = at^k + bt^{k-1}$. One has

$$\begin{aligned} QNx(t) &= Q\big(f(t, x(t), x'(t), \ldots, x^{(n-1)}(t)) + e(t)\big) \\ &= \frac{t^k}{\Delta_k} \begin{vmatrix} \int_0^\xi \big(f(t, x(t), x'(t), \ldots, x^{(n-1)}(t)) + e(t)\big)\,ds & \int_0^\xi s^{k-1}\,ds \\ A_1 & A_2 \end{vmatrix} \\ &\quad + \frac{t^{k-1}}{\Delta_k} \begin{vmatrix} \int_0^\xi s^k\,ds & \int_0^\xi \big(f(t, x(t), x'(t), \ldots, x^{(n-1)}(t)) + e(t)\big)\,ds \\ A_3 & A_1 \end{vmatrix}. \end{aligned}$$

$$\begin{aligned}
K_p(I-Q)Nx(t) &= K_p\Big[\big(f(t,x(t),x'(t),\dots,x^{(n-1)}(t))+e(t)\big)\\
&\quad -\Big(\frac{t^k}{\Delta_k}\begin{vmatrix}\int_0^\xi \big(f(t,x(t),x'(t),\dots,x^{(n-1)}(t))+e(t)\big)\,ds & \int_0^\xi s^{k-1}\,ds\\ A_1 & A_2\end{vmatrix}\\
&\quad +\frac{t^{k-1}}{\Delta_k}\begin{vmatrix}\int_0^\xi s^k\,ds & \int_0^\xi \big(f(t,x(t),x'(t),\dots,x^{(n-1)}(t))+e(t)\big)\,ds\\ A_3 & A_1\end{vmatrix}\Big)\Big]\\
&= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\big(f(s,x(s),x'(s),\dots,x^{(n-1)}(s))+e(s)\big)\,ds\\
&\quad -\Big(\frac{1}{\Delta_k}\begin{vmatrix}\int_0^\xi \big(f(t,x(t),x'(t),\dots,x^{(n-1)}(t))+e(t)\big)\,ds & \int_0^\xi s^{k-1}\,ds\\ A_1 & A_2\end{vmatrix}\\
&\quad \times\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}s^k\,ds\\
&\quad +\frac{1}{\Delta_k}\begin{vmatrix}\int_0^\xi s^k\,ds & \int_0^\xi \big(f(t,x(t),x'(t),\dots,x^{(n-1)}(t))+e(t)\big)\,ds\\ A_3 & A_1\end{vmatrix}\\
&\quad \times\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}s^{k-1}\,ds\Big)\,.
\end{aligned}$$

Since f is continuous, using the Ascoli-Arzela theorem, we can prove that $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N:\ \overline{\Omega}\to X$ is compact, thus N is L-compact on $\overline{\Omega}$.

(vi) The proof is simple and is omitted. □

Theorem 2.1. *For Case* (i)*, assume the following conditions hold.*

(A$_1$) *There exist functions $a_i(i=0,1,\dots,n-1)$, b and $r\in L^1[0,1]$ and a constant $\theta\in[0,1)$ such that for all $x_i\in R(i=0,1,\dots,n-1)$, the following inequality hold:*

$$|f(t,x_0,x_1,x_2,\dots,x_{n-1})|\le\sum_{i-0}^{n-1}a_i(t)|x_i|+b(t)|x_{n-1}|^\theta+r(t)\,;$$

(A$_2$) *There is $M>0$ such that for any $x\in\operatorname{dom}L/\operatorname{Ker}L$, if $|x^{(n-1)}(t)|>M$ for all $t\in[0,1]$, then either*

$$\int_0^\xi \big(f(s,x(s),x'(s),\dots,x^{(n-1)}(s))+e(s)\big)\,ds\ne 0\,,$$

or

$$\sum_{i=1}^m\beta_i\int_0^{\beta_i}(\beta_i-s)\big(f(s,x(s),x'(s),\dots,x^{(n-1)}(s))+e(s)\big)\,ds$$
$$-\int_0^1(1-s)\big(f(s,x(s),x'(s),\dots,x^{(n-1)}(s))+e(s)\big)\,ds\ne 0\,;$$

(A$_3$) *There is $M^* > 0$ such that for $x(t) = at^{n-1} + bt^{n-2}$ either the equations*

$$\begin{cases} -\lambda a = (1-\lambda)\frac{1}{\Delta_k}\begin{vmatrix} \int_0^\xi \big(f(s,x(s),x'(s),\dots,x^{(n-1)}(s)) + e(s)\big)\,ds & \int_0^\xi s^{k-1}\,ds \\ A_1 & A_2 \end{vmatrix} \\ -\lambda b = (1-\lambda)\frac{1}{\Delta_k}\begin{vmatrix} \int_0^\xi s^k\,ds & \int_0^\xi \big(f(s,x(s),x'(s),\dots,x^{(n-1)}(s)) + e(s)\big)\,ds \\ A_3 & A_1 \end{vmatrix} \end{cases}$$

or

$$\begin{cases} \lambda a = (1-\lambda)\frac{1}{\Delta_k}\begin{vmatrix} \int_0^\xi \big(f(s,x(s),x'(s),\dots,x^{(n-1)}(s)) + e(s)\big)\,ds & \int_0^\xi s^{k-1}\,ds \\ A_1 & A_2 \end{vmatrix} \\ \lambda b = (1-\lambda)\frac{1}{\Delta_k}\begin{vmatrix} \int_0^\xi s^k\,ds & \int_0^\xi \big(f(s,x(s),x'(s),\dots,x^{(n-1)}(s)) + e(s)\big)\,ds \\ A_3 & A_1 \end{vmatrix} \end{cases}$$

has no solution (a, b) satisfying $|a| > M^*$ or $|b| > M^*$;

(A$_4$) *There exist $\alpha > 0$, $\beta \geq 0$ and $L_1 \geq 0$ such that*

$$|f(t, x_0, x_1, \dots, x_{n-1})| \geq \alpha|x_{n-2}| - \beta|x_{n-1}| - L_1$$

for all $t \in [0,1]$ and $x_i \in R$ for $i = 0, 1, \dots, n-1$;

(A$_5$) $\left(\frac{\beta}{\alpha} + 3\right)\left(\sum_{i=0}^{n-1} \|a_i\|_1\right) < 1$.

Then BVP (3) *and* (4) *has at least one solution.*

Proof. To apply Theorem GM, we should define an open bounded subset Ω of X so that (i), (ii) and (iii) of Theorem GM hold. To obtain Ω, we base it upon three steps. The proof of this theorem is divide the proof into four steps.

Step 1. Let

$$\Omega_1 = \{x \in \operatorname{dom} L / \operatorname{Ker} L,\ Lx = \lambda Nx \text{ for some } \lambda \in (0,1)\}.$$

For $x \in \Omega_1$, $x \notin= \operatorname{Ker} L$, $\lambda \neq 0$ and $Nx \in \operatorname{Im} L$, thus $QNx = 0$. Then

$$\int_0^\xi \big(f(s,x(s),x'(s),\dots,x^{(n-1)}(s)) + e(s)\big)\,ds = 0\,,$$

$$\begin{aligned} \text{(6)} \qquad & \int_0^1 (1-s)\big(f(s,x(s),x'(s),\dots,x^{(n-1)}(s)) + e(s)\big)\,ds \\ & \quad - \sum_{i=1}^m \beta_i \int_0^{\beta_i} (\xi_i - s)\big(f(s,x(s),x'(s),\dots,x^{(n-1)}(s)) + e(s)\big)\,ds = 0\,. \end{aligned}$$

Hence by (A_2), we know that there is $t_0 \in [0,1]$ such that $|x^{(n-1)}(t_0)| \le M$. Thus

$$\begin{aligned}|x^{(n-1)}(0)| &\le |x^{(n-1)}(t_0)| + \Big|\int_0^{t_0} x^{(n)}(s)\,ds\Big| \\ &\le M + \int_0^1 |x^{(n)}(s)|\,ds \le M + \|Nx\|_1 .\end{aligned}$$

Similarly, we have $|x^{(n-1)}(t)| \le M + \|Nx\|_1$. From (6), there $t_1 \in [0,1]$ such that

$$f\big(t_1, x(t_1), x'(t_1), \ldots, x^{(n-1)}(t_1)\big) = \frac{1}{\xi}\int_0^{\xi} e(s)\,ds .$$

By (A_4), we get

$$\frac{1}{\xi}\|e\|_1 \ge \big|f\big(t_1, x(t_1), x'(t_1), \ldots, x^{(n-1)}(t_1)\big)\big| \ge \alpha\big|x^{(n-2)}(t_1)\big| - \beta\big|x^{(n-1)}(t_1)\big| - L_1 .$$

This implies that

$$\begin{aligned}\big|x^{(n-2)}(t_1)\big| &\le \frac{\beta}{\alpha}\big|x^{(n-1)}(t_1)\big| + \frac{L_1}{\alpha} + \frac{\|e\|_1}{\alpha\xi} \\ &\le \frac{\beta}{\alpha}(M + \|Nx\|_1) + \frac{L_1}{\alpha} + \frac{\|e\|_1}{\alpha\xi} .\end{aligned}$$

Hence

$$\begin{aligned}\big|x^{(n-2)}(0)\big| &= \Big|\int_0^{t_1} x^{(n-1)}(t)\,dt - x^{(n-2)}(t_1)\Big| \\ &\le \int_0^1 \big|x^{(n-1)}(t)\big|\,dt + \frac{\beta}{\alpha}(M + \|Nx\|_1) + \frac{L_1}{\alpha} + \frac{\|e\|_1}{\alpha\xi} \\ &\le M + \|Nx\|_1 + \frac{\beta}{\alpha}M + \frac{\beta}{\alpha}\|Nx\|_1 + \frac{L_1}{\alpha} + \frac{\|e\|_1}{\alpha\xi} \\ &= \Big(\frac{\beta}{\alpha} + 1\Big)\|Nx\|_1 + \Big(\frac{\beta}{\alpha} + 1\Big)M + \frac{L_1}{\alpha} + \frac{\|e\|_1}{\alpha\xi} .\end{aligned}$$

So

$$\begin{aligned}\|Px\| &= \Big\|\frac{x^{(n-1)}(0)}{(n-1)!}t^{n-1} + \frac{x^{(n-2)}(0)}{(n-2)!}t^{n-2}\Big\| \\ &\le \big|x^{(n-2)}(0)\big| + \big|x^{(n-1)}(0)\big| \\ &\le \Big(\frac{\beta}{\alpha} + 2\Big)\|Nx\|_1 + \Big(\frac{\beta}{\alpha} + 2\Big)M + \frac{L_1}{\alpha} + \frac{\|e\|_1}{\alpha\xi} .\end{aligned}$$

On the other hand, for $x \in \Omega_1$, then $x \in \operatorname{dom} L/\operatorname{Ker} L$ and $(I-P)x \in \operatorname{dom} L \cap \operatorname{Ker} P$ and $LPx = 0$. By the definition of K_p, it is easy to prove that $\|K_p y\| \le \|y\|_1$. Hence

$$\|(I-P)x\| = \|K_p L(I-P)x\| \le \|L(I-P)x\|_1 = \|Lx\|_1 \le \|Nx\|_1 .$$

Thus one has

$$\begin{aligned}\|x\| &\le \|Px\| + \|(I-P)x\| \\ &\le \Big(\frac{\beta}{\alpha}+2\Big)\|Nx\|_1 + \Big(\frac{\beta}{\alpha}+2\Big)M + \frac{L_1}{\alpha} + \frac{\|e\|_1}{\alpha} + \|Nx\|_1 \\ &= \Big(\frac{\beta}{\alpha}+3\Big)\|Nx\|_1 + \Big(\frac{\beta}{\alpha}+2\Big)M + \frac{L_1}{\alpha} + \frac{\|e\|_1}{\alpha}\,.\end{aligned}$$

It is easy to see for $x \in X \cap \operatorname{dom} L$ that

$$\|x\| = \max\left\{\|x^{(n-2)}\|_\infty, \|x^{(n-1)}\|_\infty\right\}.$$

From (A_1), we get

$$\begin{aligned}\|x\| \le& \Big(\frac{\beta}{\alpha}+3\Big)\Big(\sum_{i=0}^{n-1}\|a_i\|_1\|x^{(i)}\|_\infty + \|b\|_1\|x^{(n-1)}\|_\infty^\theta + \|e\|_1 + \|r\|_1\Big) \\ &+ \Big(\frac{\beta}{\alpha}+2\Big)M + \frac{L_1}{\alpha} + \frac{\|e\|_1}{\alpha\xi} \\ \le& \Big(\frac{\beta}{\alpha}+3\Big)\Big(\sum_{i=0}^{n-1}\|a_i\|_1\|x\| + \|b\|_1\|x\|^\theta + \|e\|_1 + \|r\|_1\Big) \\ &+ \Big(\frac{\beta}{\alpha}+2\Big)M + \frac{L_1}{\alpha} + \frac{\|e\|_1}{\alpha\xi}\,.\end{aligned}$$

Since $\theta \in [0,1)$, from the above inequality, there is $M_1 > 0$ such that

$$\|x\| = \max\left\{\|x^{(n-2)}\|_\infty, \|x^{(n-1)}\|_\infty\right\} \le M_1\,.$$

It follows that Ω_1 is bounded.

Step 2. Let

$$\Omega_2 = \{x \in \operatorname{Ker} L :\ Nx \in \operatorname{Im} L\}.$$

For $x \in \Omega_2$, then $x(t) = at^{n-1} + bt^{n-2}$ for some $a, b \in R$. $Nx \in \operatorname{Im} L$ implies $QNx = 0$. Thus

$$\begin{aligned} & \textstyle\int_0^\xi (f(s,x(s),x'(s),\ldots,x^{(n-1)}(s)) + e(s))ds = 0, \\ (7) \qquad & \textstyle\int_0^1 (1-s)(f(s,x(s),x'(s),\ldots,x^{(n-1)}(s)) + e(s))ds \\ - & \textstyle\sum_{i=1}^m \beta_i \int_0^{\beta_i} (\eta_i - s)(f(s,x(s),x'(s),\ldots,x^{(n-1)}(s)) + e(s))ds = 0.\end{aligned}$$

From (A_2), we get that there is $t_1 \in [0,1]$ such that $|x^{(n-1)}(t_1)| \le M$, i.e. $a(n-1)!| \le M$. On the other hand, by (A_4), we get from (7) that there is

$t_2 \in [0,1]$ such that

$$\begin{aligned}\frac{1}{\xi}\|e\|_1 &\geq \big|f(t_1, x(t_1), x'(t_1), \ldots, x^{(n-1)}(t_1))\big| \\ &\geq \alpha\big|x^{(n-2)}(t_1)\big| - \beta\big|x^{(n-1)}(t_1)\big| - L_1 \\ &= \alpha|a(n-1)!t_1 + b(n-2)!| - \beta|a(n-1)!| - L_1 \\ &\geq \alpha|b(n-2)!| - \alpha|a(n-1)!t_1| - \beta|a(n-1)!| - L_1 \\ &\geq \alpha|b(n-2)!| - (\alpha+\beta)|a(n-1)!| - L_1\,.\end{aligned}$$

So $\alpha|b(n-2)!| \leq \|e\|_1 + (\alpha+\beta)M + L_1$. This shows Ω_2 is bounded.

Step 3. If (a_1) in (A_3) holds, let

$$\Omega_3 = \{x \in \operatorname{Ker} L:\ -\lambda \wedge x + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\}\,,$$

where $\wedge$ is the isomorphism given by $\wedge(at^{n-1}+bt^{n-2}) = at^k + bt^{k-1}$ for all $a, b \in R$. If (a_2) in (A_3) holds, Let

$$\Omega_3 = \{x \in \operatorname{Ker} L:\ \lambda \wedge x + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\}\,.$$

Now, we prove that Ω_3 is bounded in both cases.

In fact, if (a_1) holds, and $x = at^{n-1} + bt^{n-2} \in \Omega_3$, we have

$$\begin{aligned}&-\lambda(at^k + bt^{k-1}) \\ &= (1-\lambda)\left(\frac{t^k}{\Delta_k}\begin{vmatrix}\int_0^\xi \big(f(s,x(s),x'(s),\ldots,x^{(n-1)}(s)) + e(s)\big)\,ds & \int_0^\xi s^{k-1}\,ds \\ A_1 & A_2\end{vmatrix}\right. \\ &\quad \left. + \frac{t^{k-1}}{\Delta_k}\begin{vmatrix}\int_0^\xi s^k\,ds & \int_0^\xi \big(f(s,x(s),x'(s),\ldots,x^{(n-1)}(s)) + e(s)\big)\,ds \\ A_3 & A_1\end{vmatrix}\right).\end{aligned}$$

If $\lambda = 1$, then $a = b = 0$. Otherwise, we have

$$\begin{aligned}-\lambda a &= (1-\lambda)\frac{1}{\Delta_k}\begin{vmatrix}\int_0^\xi \big(f(s,x(s),x'(s),\ldots,x^{(n-1)}(s)) + e(s)\big)\,ds & \int_0^\xi s^{k-1}\,ds \\ A_1 & A_2\end{vmatrix} \\ -\lambda b &= (1-\lambda)\frac{1}{\Delta_k}\begin{vmatrix}\int_0^\xi s^k\,ds & \int_0^\xi \big(f(s,x(s),x'(s),\ldots,x^{(n-1)}(s)) + e(s)\big)\,ds \\ A_3 & A_1\end{vmatrix}\,.\end{aligned}$$

It follows from (A_3) that $|a| \leq M^*$ and $|b| \leq M^*$. This shows that Ω_3 is bounded. Similarly to above argument, we can prove that Ω_3 is bounded if (a_2) holds.

In the following, we shall show that all conditions of Theorem GM are satisfied. Set Ω be an open bounded subset of X such that $\Omega \supset \cup_{i=1}^3 \overline{\Omega_i}$. By Lemma 2.1, L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. By the definition of Ω, we have

- **(a)** $Lx \neq \lambda Nx$ for $x \in (\operatorname{dom} L/\operatorname{Ker} L) \cap \partial\Omega$ and $\lambda \in (0,1)$;
- **(b)** $Nx \notin \operatorname{Im} L$ for $x \in \operatorname{Ker} L \cap \partial\Omega$.

Step 4. We prove

(c) $\deg(QN|_{\operatorname{Ker} L},\ \Omega \cap \operatorname{Ker} L, 0) \neq 0$.

In fact, let $H(x,\lambda) = \pm\lambda \wedge x + (1-\lambda)QNx$. According the definition of Ω, we know $H(x,\lambda) \neq 0$ for $x \in \partial\Omega \cap \operatorname{Ker} L$, thus by homotopy property of degree,

$$\begin{aligned}\deg(QN \mid \operatorname{Ker} L, \Omega \cap \operatorname{Ker} L, 0) &= \deg(H(\cdot,0), \Omega \cap \operatorname{Ker} L, 0)\\ &= \deg(H(\cdot,1), \Omega \cap \operatorname{Ker} L, 0)\\ &= \deg(\wedge, \Omega \cap \operatorname{Ker} L, 0) \neq 0\,.\end{aligned}$$

Thus by Theorem GM, $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$, which is a solution of BVP (3)–(4). The proof is complete. □

Now, we consider BVP (3) and (4) in the Case (ii), let

$$\begin{aligned}\operatorname{dom} L = \{x \in W^{n,1}(0,1),\ &x^{(i)}(0) = 0 \text{ for } i = 0,1,\dots,n-3,\ \ x^{(n-1)}(1) = 0\\ &x^{(n-2)}(1) = \sum_{i=1}^{m} \beta_i x^{(n-2)}(\eta_i)\}\,.\end{aligned}$$

We have the following lemma and theorem, whose proofs are similar to those of Lemma 2.1 and Theorem 2.1, respectively, and are omitted.

Lemma 2.2. *The following results hold.*
(i) $\operatorname{Ker} L = \{ct^{n-2},\ t \in [0,1],\ c \in R\}$;
(ii) $\operatorname{Im} L = \big\{y \in Y,\ \sum_{i=1}^m \beta_i \int_0^{\eta_i} (\eta_i - s)y(s)\,ds = \int_0^1 (1-s)y(s)\,ds\big\}$;
(iii) *L is a Fredholm operator of index zero;*
(iv) *There is $k \in \{0,1,\dots,m\}$ such that*

$$M = -\sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s)s^k\,ds + \int_0^1 (1-s)s^k\,ds \neq 0\,;$$

(v) *There are projectors $P : X \to X$ and $Q : Y \to Y$ such that* $\operatorname{Ker} L = \operatorname{Im} P$ *and* $\operatorname{Ker} Q = \operatorname{Im} L$. *Furthermore, let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap \operatorname{dom} L \neq \Phi$, then N is L-compact on $\overline{\Omega}$;*
(vi) *$x(t)$ is a solution of* BVP (3)–(4) *if and only if x is a solution of the operator equation $Lx = Nx$ in* $\operatorname{dom} L$.

In fact, we have

$$\begin{aligned}Px(t) &= \frac{x^{(n-2)}(0)}{(n-2)!} t^{n-2} \quad \text{for} \quad x \in X \cap \operatorname{dom} L\,,\\ Qy(t) &= \frac{1}{M}\Big(\int_0^1 (1-s)y(s)\,ds - \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s)y(s)\,ds\Big) \quad \text{for} \quad y \in Y\,,\\ K_p y(t) &= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)\,ds \quad \text{for} \quad y \in \operatorname{Im} L\,.\end{aligned}$$

Theorem 2.2. *For Case* (ii)*, assume the following conditions hold.*

(A$_1$) *There exist functions $a_i (i = 0, 1, \dots, n-1)$, b and $r \in L^1[0,1]$ and a constant $\theta \in [0,1)$ such that for all $x_i \in R (i = 0, 1, \dots, n-1)$, the following inequality hold:*

$$|f(t, x_0, x_1, x_2, \dots, x_{n-1})| \le \sum_{i-0}^{n-1} a_i(t)|x_i| + b(t)|x_{n-1}|^\theta + r(t)\,;$$

(A$_2$) *There is $M > 0$ such that for any $x \in \operatorname{dom} L/\operatorname{Ker} L$, if $|x^{(n-2)}(t)| > M$ for all $t \in [0,1]$, then*

$$\int_0^1 (1-s)\big(f(s, x(s), x'(s), \dots, x^{(n-1)}(s)) + e(s)\big)\,ds \\ - \sum_{i=1}^m \beta_i \int_0^{\eta_i} (\eta_i - s)\big(f(s, x(s), x'(s), \dots, x^{(n-1)}(s)) + e(s)\big)\,ds \ne 0\,;$$

(A$_3$) *There is $M^* > 0$ such that for any $c \in R$ if $|c| > M^*$ then either*

$$c\Big(\int_0^1 (1-s) f\big(s, cs^{n-2}, c(n-2)s^{n-3}, \dots, c(n-2)!, 0\big)\,ds \\ - \sum_{i=1}^m \alpha_i \int_0^{\xi_i} (\xi_i - s) f\big(s, cs^{n-2}, c(n-2)s^{n-3}, \dots, c(n-2)!, 0\big)\,ds\Big) < 0$$

or

$$c\Big(\int_0^1 (1-s) f\big(s, cs^{n-2}, c(n-2)s^{n-3}, \dots, c(n-2)!, 0\big)\,ds \\ - \sum_{i=1}^m \alpha_i \int_0^{\xi_i} (\xi_i - s) f\big(s, cs^{n-2}, c(n-2)s^{n-3}, \dots, c(n-2)!, 0\big)\,ds\Big) < 0\,;$$

(A$_4$) $\sum_{i=0}^{n-1} \|a_i\|_1 < \frac{1}{2}$.

Then BVP (3) *and* (4) *has at least one solution.*

For Case (iii), let

$$\operatorname{dom} L = \{x \in W^{n,1}(0,1),\ x^{(i)}(0) = 0 \text{ for } \ i = 0, 1, \dots, n-3,\ \ x^{(n-1)}(0) = x^{(n-1)}(\xi) \\ x^{(n-2)}(1) = \sum_{i=1}^m \beta_i x^{(n-2)}(\eta_i)\ \}\,.$$

We have the following results, whose proofs are similar to those of Lemma 2.1 and Theorem 2.1.

Lemma 2.3. *The following results hold.*

(i) $\operatorname{Ker} L = \{ct^{n-2},\ t \in [0,1],\ c \in R\}$*;*

(ii) $\operatorname{Im} L = \big\{\ y \in Y,\ \int_0^\xi y(s)\,ds = 0\big\}$*;*

(iii) *L is a Fredholm operator of index zero;*
(iv) *There is $k \in \{0, 1, \ldots, m\}$ such that*

$$\sum_{i=1}^{m} \beta_i \eta_i^{k-1} = 1 \quad \text{and} \quad \sum_{i=1}^{m} \beta_i \eta_i^{k} \neq 1\,;$$

(v) *There are projectors $P : X \to X$ and $Q : Y \to Y$ such that* $\operatorname{Ker} L = \operatorname{Im} P$ *and* $\operatorname{Ker} Q = \operatorname{Im} L$. *Furthermore, let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap \operatorname{dom} L \neq \Phi$, then N is L-compact on $\overline{\Omega}$;*
(vi) *$x(t)$ is a solution of* BVP (3)–(4) *if and only if x is a solution of the operator equation $Lx = Nx$ in* $\operatorname{dom} L$.

In fact, we have

$$Px(t) = \frac{x^{(n-2)}(0)}{(n-2)!} t^{n-2} \quad \text{for} \quad x \in X \cap \operatorname{dom} L\,,$$

$$Qy(t) = \frac{1}{\xi} \int_0^{\xi} y(s)\, ds \quad \text{for} \quad y \in Y\,,$$

$$K_p y(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)\, ds - \frac{t^{n-1}}{1 - \sum_{i=1}^{m} \beta_i \eta_i} \Big(\sum_{i=1}^{m} \int_{\eta_i}^{1} \int_0^s y(u)\, du\, ds \Big)$$

for $y \in \operatorname{Im} L$.

Theorem 2.3. *For Case* (iii), *assume the following conditions hold.*

(A_1) *There exist functions $a_i (i = 0, 1, \ldots, n-1)$, b and $r \in L^1[0,1]$ and a constant $\theta \in [0,1)$ such that for all $x_i \in R (i = 0, 1, \ldots, n-1)$, the following inequality hold:*

$$|f(t, x_0, x_1, x_2, \ldots, x_{n-1})| \leq \sum_{i-0}^{n-1} a_i(t)|x_i| + b(t)|x_{n-1}|^{\theta} + r(t)\,;$$

(A_2) *There is $M > 0$ such that for any $x \in \operatorname{dom} L / \operatorname{Ker} L$, if $|x^{(n-1)}(t)| > M$ for all $t \in [0,1]$, then*

$$\sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s) f\big(s, x(s), x'(s), \ldots, x^{(n-1)}(s)\big)\, ds \neq \int_0^1 (1-s) f\big(s, x(s), x'(s), \ldots, x^{(n-1)}(s)\big)\, ds\,;$$

(A_3) *There is $M^* > 0$ such that for any $c \in R$ if $|c| > M^*$ then either*

$$c\Big(\int_0^1 (1-s) f\big(s, cs^{n-2}, c(n-2)s^{n-3}, \ldots, c(n-2)!, 0\big)\, ds - \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} (\xi_i - s) f\big(s, cs^{n-2}, c(n-2)s^{n-3}, \ldots, c(n-2)!, 0\big)\, ds \Big) < 0$$

or

$$c\Big(\int_0^1 (1-s)f\big(s, cs^{n-2}, c(n-2)s^{n-3}, \dots, c(n-2)!, 0\big)\, ds \\ - \sum_{i=1}^m \alpha_i \int_0^{\xi_i} (\xi_i - s) f\big(s, cs^{n-2}, c(n-2)s^{n-3}, \dots, c(n-2)!, 0\big)\, ds\Big) < 0\,;$$

(A_4) *There exist $\alpha > 0$, $\beta \geq 0$ and $L_1 \geq 0$ such that*

$$|f(t, x_0, x_1, \dots, x_{n-1})| \geq \alpha |x_{n-2}| - \beta |x_{n-1}| - L_1$$

for all $t \in [0,1]$ and $x_i \in R$ for $i = 0, 1, \dots, n-1$;

(A_5) $\left(\frac{\beta}{\alpha} + 2\right) \sum_{i=0}^{n-1} \|a_i\|_1 < 1.$

Then BVP (3) *and* (4) *has at least one solution.*

For Case (iv), let

$$\operatorname{dom} L = \{x \in W^{n,1}(0,1), x^{(i)}(0) = 0 \text{ for } i = 0, 1, \dots, n-3,\ x^{(n-1)}(1) = x^{(n-1)}(\xi) \\ x^{(n-2)}(1) = \sum_{i=1}^m \beta_i x^{(n-2)}(\eta_i)\}\,.$$

We have the following results, whose proofs are similar to those of Lemma 2.1 and Theorem 2.1.

Lemma 2.4. *The following results hold.*

(i) $\operatorname{Ker} L = \{ct^{n-1},\ t \in [0,1],\ c \in R\}$*;*

(ii) $\operatorname{Im} L = \left\{y \in Y,\ \int_0^\xi y(s)\, ds = 0\right\}$*;*

(iii) *L is a Fredholm operator of index zero;*

(iv) *There are projectors $P : X \to X$ and $Q : Y \to Y$ such that* $\operatorname{Ker} L = \operatorname{Im} P$ *and* $\operatorname{Ker} Q = \operatorname{Im} L$. *Furthermore, let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap \operatorname{dom} L \neq \Phi$, then N is L-compact on $\overline{\Omega}$;*

(v) *$x(t)$ is a solution of* BVP (3)–(4) *if and only if x is a solution of the operator equation $Lx = Nx$ in* $\operatorname{dom} L$.

In fact, we have

$$Px(t) = \frac{x^{(n-1)}(0)}{(n-1)!} t^{n-1} \quad \text{for} \quad x \in X \cap \operatorname{dom} L\,,$$

$$Qy(t) = \frac{1}{\xi} \int_0^\xi y(s)\, ds \quad \text{for} \quad y \in Y\,,$$

$$K_p y(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)\, ds - \frac{t^{n-2}}{1 - \sum_{i=1}^m \beta_i \eta_i} \\ \times \Big(\int_0^1 (1-s) y(s)\, ds - \sum_{i=1}^m \beta_i \int_0^{\eta_i} (\eta_i - s) y(s)\, ds\Big) \quad \text{for } y \in \operatorname{Im} L.$$

Theorem 2.4. *For Case* (iv), *assume the following conditions hold.*

(A$_1$) *There exist functions $a_i (i = 0, 1, \ldots, n-1)$, b and $r \in L^1[0,1]$ and a constant $\theta \in [0,1)$ such that for all $x_i \in R (i = 0, 1, \ldots, n-1)$, the following inequality hold:*

$$|f(t, x_0, x_1, x_2, \ldots, x_{n-1})| \le \sum_{i-0}^{n-1} a_i(t)|x_i| + b(t)|x_{n-1}|^\theta + r(t)\,;$$

(A$_2$) *There is $M > 0$ such that for any $x \in \operatorname{dom} L / \operatorname{Ker} L$, if $|x^{(n-1)}(t)| > M$ for all $t \in [0,1]$, then*

$$\sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s) f\big(s, x(s), x'(s), \ldots, x^{(n-1)}(s)\big)\, ds \neq \int_0^1 (1-s) f\big(s, x(s), x'(s), \ldots, x^{(n-1)}(s)\big)\, ds;$$

(A$_3$) *There is $M^* > 0$ such that for any $c \in R$ if $|c| > M^*$ then either*

$$c\Big(\int_0^1 (1-s) f\big(s, cs^{n-1}, c(n-1)s^{n-2}, \ldots, c(n-1)!\big)\, ds - \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s) f\big(s, cs^{n-1}, c(n-1)s^{n-2}, \ldots, c(n-1)!\big)\, ds\Big) < 0$$

or

$$c\Big(\int_0^1 (1-s) f\big(s, cs^{n-1}, c(n-1)s^{n-2}, \ldots, c(n-1)!\big)\, ds - \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s) f\big(s, cs^{n-1}, c(n-1)s^{n-2}, \ldots, c(n-1)!\big)\, ds\Big) < 0\,;$$

(A$_4$) *There exist $\alpha > 0$, $\beta \ge 0$ and $L_1 \ge 0$ such that*

$$|f(t, x_0, x_1, \ldots, x_{n-1})| \ge \alpha|x_{n-2}| - \beta|x_{n-1}| - L_1$$

for all $t \in [0,1]$ and $x_i \in R$ for $i = 0, 1, \ldots, n-1$;

(A$_5$) $\left(\frac{\beta}{\alpha} + 2\right) \sum_{i=0}^{n-1} \|a_i\|_1 < 1.$

Then BVP (3) *and* (4) *has at least one solution.*

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