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AN ALMOST-PERIODICITY CRITERION FOR SOLUTIONS OF THE OSCILLATORY DIFFERENTIAL EQUATION $y'' = q(t)y$ AND ITS APPLICATIONS

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ABSTRACT. The linear differential equation $(q) : y'' = q(t)y$ with the uniformly almost-periodic function q is considered. Necessary and sufficient conditions which guarantee that all bounded (on $\mathbb{R}$) solutions of (q) are uniformly almost-periodic functions are presented. The conditions are stated by a phase of (q). Next, a class of equations of the type (q) whose all non-trivial solutions are bounded and not uniformly almost-periodic is given. Finally, uniformly almost-periodic solutions of the non-homogeneous differential equations $y'' = q(t)y + f(t)$ are considered. The results are applied to the Appell and Kummer differential equations.

1. INTRODUCTION

In the paper we consider the differential equation

$$(q) \qquad y'' = q(t)y\,,$$

where q is either a real-valued continuous function on $\mathbb{R}$ or a real-valued uniformly almost-periodic function. At the same time we say that a (generally complex-valued) function f is *uniformly almost-periodic* (u.a.p.) or Bohr's almost-periodic if f is continuous on $\mathbb{R}$ and for each $\varepsilon > 0$ there exists a number $l > 0$ such that on every interval $[a, a+l]$ there is a τ such that $|f(t+\tau) - f(t)| < \varepsilon$ for $t \in \mathbb{R}$ (see e.g. [6], [8], [13]). Throughout the paper a bounded function (which is defined on $\mathbb{R}$) means that it is bounded on $\mathbb{R}$.

Let q be a u.a.p. function. Then (q) is either disconjugate (that is (q) has a positive solution on $\mathbb{R}$) or oscillatory (that is $\pm\infty$ are the cluster points of zeros of a non-trivial solution to (q)) (see e.g. [15]). The properties of solutions to the disconjugate equation (q) are usually considered by the associated Riccati equation $y' + y^2 = q(t)$. For the disconjugate equation (q) it is known that (q) can have

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a non-trivial u.a.p. solution y (and then $\{ky : k \in \mathbb{R}\}$ are all its u.a.p. solutions) only if (q) is the special disconjugate equation (that is (q) has the unique (up to the positive multiplicative constant) positive solution on $\mathbb{R}$) (see e.g. [15], [18]). Discussing u.a.p. solutions to the oscillatory equation (q) is more complicated since now the transformation to the associated Riccati equation is impossible on $\mathbb{R}$. But if q is a periodic function, then any bounded solution of (q) is u.a.p. (see e.g. [7], [10]).

In [16], the following question was put: If all solutions of (q) with a uniformly almost-periodic coefficient q are bounded, does it necessarily follow that all solutions are u.a.p. functions? The negative answer to this question is given in [12] even for n-order differential equations ($n \geq 2$)

$$y^{(n)} = p_1(t)y + \cdots + p_n(t)y^{(n-1)} \tag{1.1}$$

with u.a.p. coefficients p_j ($j = 1, \ldots, n$). The authors of [12] showed that, for each $n \geq 2$, there exists an equation of form (1.1) for which every solution is bounded but only the trivial solution is uniformly almost-periodic. The analogical result for the system $y' = A(t)y$ with the u.a.p. matrix $A(t)$ has been showed by Lillo [14] who considered the system $x' = f(t)y$, $y' = -f(t)x$ where the function f is u.a.p. whose mean value is zero and $\int_0^t f(s)\,ds$ is unbounded. The vectors $(\sin(\int_0^t f(s)\,ds), \cos(\int_0^t f(s)\,ds))$ and $(\cos(\int_0^t f(s)\,ds), -\sin(\int_0^t f(s)\,ds))$ form a base of the solution space. Any non-trivial solution is bounded but it is not a u.a.p. function.

Let q be a u.a.p. function and let all solutions of (q) are bounded (then (q) is oscillatory by Lemma 2.1). The aim of the paper is to discuss the existence and non-existence of u.a.p. solutions of (q) in detail. We present necessary and sufficient conditions which guarantee that all solutions of (q) are u.a.p. (Theorem 3.1). These conditions are stated by the notion of a (first) phase of (q) (see [3]). Theorem 3.9 gives a class of equations of the type (q) whose all non-trivial solutions are bounded but not u.a.p. functions. Finally, if f is a u.a.p. function, all solutions of (q) are u.a.p. and the non-homogeneous differential equation $y'' = q(t)y + f(t)$ has a bounded solution, then all its solutions are u.a.p. (Theorem 3.11). Results are applied to the linear Appell differential equation (Corollary 3.5) and the nonlinear Kummer differential equation (Corollary 3.6).

2. Definitions and auxiliary results

Throughout this section we assume that q is a real-valued continuous function on $\mathbb{R}$. We say that (u, v) is *a base of* (q) if u, v are linearly independent solution of (q).

Lemma 2.1. *Let q and all solutions of (q) be bounded. Then (q) is an oscillatory equation, the first and second derivatives of all its solutions are bounded and*

$$\inf\{u^2(t) + v^2(t) : t \in \mathbb{R}\} > 0 \tag{2.1}$$

for any base (u, v) of (q).

Proof. Let y be a non-trivial solution of (q). Then q and y'' $(= qy)$ are bounded and so y' is bounded which follows from the inequality $\|y'\|^2 \le 8\|y\|\|y''\|$ where $\|\cdot\|$ stands for the sup-norm in $C^0(\mathbb{R})$ (see e.g. [2], [17]).

Let (u, v) be a phase of (q). Then the Wronskian $w = uv' - u'v$ of (u, v) is a non-zero constant function. If (2.1) is false, then there exists a sequence $\{t_n\} \subset \mathbb{R}$ such that $\lim_{n\to\infty} u(t_n) = \lim_{n\to\infty} v(t_n) = 0$. Hence $\lim_{n\to\infty}(u(t_n)v'(t_n) - u'(t_n)v(t_n)) = 0$ since u', v' are bounded, contrary to $u(t_n)v'(t_n) - u'(t_n)v(t_n) = w \ne 0$ for $n \in \mathbb{N}$. So (2.1) is true and then

$$\int_{-\infty}^{0} \frac{1}{u^2(t) + v^2(t)}\, dt = \int_{0}^{\infty} \frac{1}{u^2(t) + v^2(t)}\, dt = \infty\,.$$

Consequently (q) is oscillatory (see e.g. [11]). □

A function $\alpha \in C^0(\mathbb{R})$ is said to be a (*first*) *phase of the differential equation* (q) if there is a phase (u, v) of (q) such that

$$\tan\alpha(t) = \frac{u(t)}{v(t)} \quad \text{for} \quad t \in \mathbb{R} \setminus \{t : v(t) = 0\}\,.$$

If α is a phase of (q) then

(i) $\alpha \in C^3(\mathbb{R})$,
(ii) $\alpha'(t) \ne 0$ for $t \in \mathbb{R}$,
(iii) $\dfrac{\sin(\alpha(t))}{\sqrt{|\alpha'(t)|}}, \dfrac{\cos(\alpha(t))}{\sqrt{|\alpha'(t)|}}$ are linearly independent solutions of (q)

and any solution y of (q) can be written in the form

$$y(t) = c_1 \frac{\sin(\alpha(t) + c_2)}{\sqrt{|\alpha'(t)|}}, \quad t \in \mathbb{R}\,,$$

where $c_1, c_2 \in \mathbb{R}$.

A function α is a phase of (q) if and only if it is a solution of the nonlinear Kummer third-order differential equation

$$-\frac{1}{2}\frac{y'''}{y'} + \frac{3}{4}\Big(\frac{y''}{y'}\Big)^2 - (y')^2 = q(t)\,. \tag{2.2}$$

In addition, if (u, v) is a base of (q) and $|uv' - u'v| = 1$, then there exists a phase α of (q) such that

$$u(t) = \frac{\sin(\alpha(t))}{\sqrt{|\alpha'(t)|}}\,, \quad v(t) = \frac{\cos(\alpha(t))}{\sqrt{|\alpha'(t)|}} \quad \text{for} \quad t \in \mathbb{R}\,. \tag{2.3}$$

The definition of the phase of (q) and its properties are presented in [3].

Lemma 2.2. *Let q be bounded and α be a phase of (q).*

(j) *If all solutions of (q) are bounded then $\dfrac{1}{\alpha'}, \alpha', \alpha''$ and α''' are bounded, too.*

(jj) *If $\dfrac{1}{\alpha'}$ is bounded then all solutions of (q) are bounded, too.*

Proof. By (iii), the functions u, v defined by (2.3) are linearly independent solutions of (q). Set $\varepsilon = \operatorname{sign} \alpha'$ and

$$p(t) = \sqrt{u^2(t) + v^2(t)}\,, \quad s(t) = \sqrt{(u'(t))^2 + (v'(t))^2}\,, \quad t \in \mathbb{R}\,.$$

Then

$$\alpha' = \frac{\varepsilon}{p^2}\,, \quad \alpha'' = -\frac{2\varepsilon p'}{p^3}\,, \quad \alpha''' = -2\varepsilon\Big(\frac{q}{p^2} - \frac{3s^2}{p^4} + \frac{4}{p^6}\Big) \tag{2.4}$$

(see [3], p. 38). If all solutions of (q) are bounded then $u^{(j)}, v^{(j)}$ are bounded for $j = 0, 1, 2$ and $\inf\{u^2(t) + v^2(t) : t \in \mathbb{R}\} > 0$ by Lemma 2.1. The boundedness of $1/\alpha', \alpha', \alpha''$ and α''' now follows from (2.4).

If $1/\alpha'$ is bounded, then u, v are bounded and so all solutions of (q) are bounded, too. □

Denote by $\mathcal{A}$ the set of real-valued u.a.p. functions. For the remainder of this section we state here for the convenience of the reader some results from the theory of u.a.p. functions which will be used in our next considerations. A function $h \in \mathcal{A}$ if and only if $h \in C^0(\mathbb{R})$ and for every sequence $\{k_n\} \subset \mathbb{R}$ there exists a subsequence $\{k_{i_n}\}$ such that the sequence of functions $\{h(t + k_{i_n})\}$ is uniformly convergent on $\mathbb{R}$. The limit of any sequence $\{h_n\} \subset \mathcal{A}$ converging uniformly on $\mathbb{R}$ belongs to $\mathcal{A}$. If $h \in \mathcal{A}$ then h is bounded and equicontinuous on $\mathbb{R}$, $|h| \in \mathcal{A}$, $\gamma(h) \in \mathcal{A}$ for every γ being equicontinuous on the range of h, $h' \in \mathcal{A}$ provided h' is equicontinuous on $\mathbb{R}$ and $\int_0^t h(s)\,ds \in \mathcal{A}$ if and only if it is bounded. If $g, h \in \mathcal{A}$ then $gh \in \mathcal{A}$. Finally, each $h \in \mathcal{A}$ has the finite mean value

$$M[h] = \lim_{T\to\infty} \frac{1}{T} \int_0^T h(s)\,ds$$

(see e.g. [6], [8], [13]).

Lemma 2.3 (Bohr theorem, [13] p. 129)**.** *Let F be a u.a.p. function and let $\inf\{|F(t)| : t \in \mathbb{R}\} > 0$. Then*

$$\arg F(t) = at + \varphi(t)\,, \quad t \in \mathbb{R}\,,$$

where $a \in \mathbb{R}$ and $\varphi \in \mathcal{A}$.

Lemma 2.4. *Let $b \in \mathbb{R}$ and $\chi, \omega \in \mathcal{A}$. Then the composite function $\chi(bt + \omega(t))$ belongs to $\mathcal{A}$.*

Proof. If $b = 0$ then $\chi(\omega) \in \mathcal{A}$ since χ is equicontinuous on $\mathbb{R}$. Let $b \neq 0$ and ε be a positive number. Then there exists $\delta > 0$ such that

$$|\chi(t + \nu) - \chi(t)| < \frac{\varepsilon}{2} \quad \text{for} \quad t, \nu \in \mathbb{R}\,, \ \ |\nu| < \delta \tag{2.5}$$

which follows from χ being equicontinuous on $\mathbb{R}$. Let $\{k_n\} \subset \mathbb{R}$ be a sequence. Going if necessary to a subsequence, we can assume that the sequences $\{\chi(bt + bk_n)\}$ and $\{\omega(t + k_n)\}$ are uniformly convergent on $\mathbb{R}$. Let $\lim_{n\to\infty} \chi(bt + bk_n) = p(t)$, $\lim_{n\to\infty} \omega(t + k_n) = r(t)$. Then there exists $n_0 \in \mathbb{N}$ such that

$$|\chi(bt + bk_n) - p(t)| < \frac{\varepsilon}{2}\,, \quad |\omega(t + k_n) - r(t)| < \delta \quad \text{for} \quad t \in \mathbb{R}\,, \ \ n \geq n_0\,. \tag{2.6}$$

By (2.5) and (2.6),

$$\begin{aligned}
&\Big|\chi(bt+bk_n+\omega(t+k_n)) - p\Big(t+\frac{r(t)}{b}\Big)\Big| \\
&\quad \le |\chi(bt+bk_n+\omega(t+k_n)) - \chi(bt+bk_n+r(t))| \\
&\qquad + \Big|\chi(bt+bk_n+r(t)) - p\Big(t+\frac{r(t)}{b}\Big)\Big| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

for $t\in\mathbb{R}$ and $n\ge n_0$, which proves that

$$\lim_{n\to\infty} \chi(bt+bk_n+\omega(t+k_n)) = p\Big(t+\frac{r(t)}{b}\Big) \quad \text{uniformly on} \quad \mathbb{R}\,.$$

Hence $\chi(bt+\omega(t))\in\mathcal{A}$. $\square$

3. Main results

Theorem 3.1. *Let $q\in\mathcal{A}$ and α be a phase of (q). Then all solutions of (q) are u.a.p. functions if and only if*

$$\alpha(t) = at+\varphi(t) \quad \textit{for } t\in\mathbb{R}, \tag{3.1}$$

where $a\in\mathbb{R}$, $\varphi^{(i)}\in\mathcal{A}$ for $i=0,1$ and

$$\inf\{|a+\varphi'(t)| : t\in\mathbb{R}\} > 0\,. \tag{3.2}$$

Proof. Let u,v be defined by (2.3). Then (u,v) is a base of (q) and all solutions of (q) belong to $\mathcal{A}$ if and only if $u,v\in\mathcal{A}$.

First assume that $u,v\in\mathcal{A}$. Then all solutions of (q) are bounded and so the functions $1/\alpha',\alpha'$ and α'' are bounded by Lemma 2.2. Set

$$F(t) = v(t)+iu(t) \quad \text{for} \quad t\in\mathbb{R}\,. \tag{3.3}$$

Then F is u.a.p. and $|F(t)| = 1/\sqrt{|\alpha'(t)|}\ge l>0$ for $t\in\mathbb{R}$. Since

$$\arg F(t) = \arg\frac{e^{i\alpha(t)}}{\sqrt{|\alpha'(t)|}} = \alpha(t)+2k\pi\,, \quad t\in\mathbb{R}\,,$$

where k is an integer, Lemma 2.3 shows that $\alpha(t)=at+\varphi(t)$ where $a\in\mathbb{R}$ and $\varphi\in\mathcal{A}$. From the properties of the phase α we see that $\varphi\in C^3(\mathbb{R})$ and $\varphi^{(i)}$ is bounded for $i=1,2$. Hence φ' is equicontinuous on $\mathbb{R}$, and so $\varphi'\in\mathcal{A}$. In addition, the inequality $\inf\{|\alpha'(t)| : t\in\mathbb{R}\}>0$ implies (3.2).

Let $\alpha(t)=at+\varphi(t)$ for $t\in\mathbb{R}$ where $a\in\mathbb{R}$, $\varphi^{(i)}\in\mathcal{A}$ for $i=0,1$ and (3.2) be satisfied. Then $\sin(at+\varphi(t))$, $\cos(at+\varphi(t))$ and $1/\sqrt{\alpha'(t)}$ are u.a.p., and consequently $u,v\in\mathcal{A}$. $\square$

Remark 3.2. Let $q\in\mathcal{A}$ and α be a phase of (q). If all solutions of (q) belong to $\mathcal{A}$ and (3.1) is satisfied with an $a\in\mathbb{R}$ and $\varphi\in\mathcal{A}$ then $\varphi^{(i)}\in\mathcal{A}$ even for $i=2,3$. To prove this fact we note that from the boundedness of α''' by Lemma 2.2 we deduce that φ'' is equicontinuous on $\mathbb{R}$ and so $\varphi''\in\mathcal{A}$. Now from the equalities

$$q = -\frac{1}{2}\frac{\alpha'''}{\alpha'}+\frac{3}{4}\Big(\frac{\alpha''}{\alpha'}\Big)^2-(\alpha')^2 = -\frac{1}{2}\frac{\varphi'''}{a+\varphi'}+\frac{3}{4}\Big(\frac{\varphi''}{a+\varphi'}\Big)^2-(a+\varphi')^2$$

we have

$$\varphi''' = 2\Big[\frac{3}{4}\frac{(\varphi'')^2}{a+\varphi'} - q(a+\varphi') - (a+\varphi')^3\Big]$$

and so $\varphi''' \in \mathcal{A}$ since $\inf\{|a+\varphi'(t)| : t \in \mathbb{R}\} > 0$.

Remark 3.3. If $q \in \mathcal{A}$ and all solutions of (q) belong to $\mathcal{A}$ then the derivatives of all solutions of (q) are equicontinuous functions on $\mathbb{R}$, and therefore they belong to $\mathcal{A}$.

Corollary 3.4. *Let $q \in \mathcal{A}$ and α be a phase of (q). If all solutions of (q) are u.a.p. then all solutions of the differential equation*

$$y'' = [q(t) - \lambda(\alpha'(t))^2]y \tag{3.4}$$

are u.a.p. for each $\lambda \in (-1,\infty)$.

Proof. Let all solutions of (q) belong to $\mathcal{A}$ and fix $\lambda \in (-1,\infty)$. By Theorem 3.1, $\alpha(t) = at + \varphi(t)$ for $t \in \mathbb{R}$, where $a \in \mathbb{R}$, $\varphi, \varphi' \in \mathcal{A}$ and (3.2) is satisfied. Hence $(\alpha')^2 = (a+\varphi')^2 \in \mathcal{A}$. Set $\beta = \sqrt{1+\lambda}\alpha$. Now from the equalities

$$-\frac{1}{2}\frac{\beta'''}{\beta'} + \frac{3}{4}\Big(\frac{\beta''}{\beta'}\Big)^2 - (\beta')^2 = -\frac{1}{2}\frac{\alpha'''}{\alpha'} + \frac{3}{4}\Big(\frac{\alpha''}{\alpha'}\Big)^2 - (1+\lambda)(\alpha')^2 = q - \lambda(\alpha')^2$$

we see that β is a phase of (3.4) and since $\beta(t) = \sqrt{1+\lambda}at + \sqrt{1+\lambda}\varphi(t)$, all solutions of (3.4) belong to $\mathcal{A}$ by Theorem 3.1. $\square$

Corollary 3.5. *Let $q, q' \in \mathcal{A}$ and α be a phase of (q). Then all solutions of the Appell equation*

$$y''' = 4q(t)y' + 2q'(t)y \tag{3.5}$$

belong to $\mathcal{A}$ if and only if (3.1) is satisfied with some $a \in \mathbb{R}$ and $\varphi \in \mathcal{A}$ such that $\varphi' \in \mathcal{A}$ and $\inf\{|a+\varphi'(t)| : t \in \mathbb{R}\} > 0$.

Proof. Let u, v be defined by (2.3). Then (u,v) is a base of (q) and therefore u^2, uv, v^2 are linearly independent solutions of (3.5) (see e.g. [1], [9]).

If (3.1) is satisfied with some $a \in \mathbb{R}$ and $\varphi \in \mathcal{A}$ such that $\varphi' \in \mathcal{A}$ and $\inf\{|a+\varphi'(t)| : t \in \mathbb{R}\} > 0$, then $u, v \in \mathcal{A}$ by Theorem 3.1 and so $u^2, uv, v^2 \in \mathcal{A}$ which implies that all solutions of (3.5) belong to $\mathcal{A}$.

Assume that all solutions of (3.5) are u.a.p. functions. Then $u^2, uv, v^2 \in \mathcal{A}$ and so

$$2uv = \frac{\sin(2\alpha)}{|\alpha'|}, \quad v^2 - u^2 = \frac{\cos(2\alpha)}{|\alpha'|}$$

are u.a.p. functions. Set

$$F(t) = \frac{\cos(2\alpha(t))}{|\alpha'(t)|} + i\frac{\sin(2\alpha(t)}{|\alpha'(t)|}, \quad t \in \mathbb{R}.$$

Then F is a u.a.p. function and $|F(t)| = 1/|\alpha'(t)| \geq l > 0$ for $t \in \mathbb{R}$. By Lemma 2.3, $\arg F(t) = 2\alpha(t) + 2k\pi = a_1 + \varphi_1(t)$ for $t \in \mathbb{R}$, where k is an integer, $a_1 \in \mathbb{R}$ and $\varphi_1 \in \mathcal{A}$. Arguing as in the proof of Theorem 3.1 we have $\varphi_1' \in \mathcal{A}$ and

$\inf\{|a_1+\varphi_1'(t)| : t \in \mathbb{R}\} > 0$. Hence (3.1) is satisfied with $a = a_1/2$ and $\varphi = \varphi_1/2$. ☐

Corollary 3.6. *Let $q, Q \in \mathcal{A}$ and let all solutions of (q) and (Q) belong to $\mathcal{A}$. Then the derivative of all regular solutions (that is solutions in $C^3(\mathbb{R})$ with non-vanishing derivative on $\mathbb{R}$) to the Kummer differential equation*

$$-\frac{1}{2}\frac{y'''}{y'} + \frac{3}{4}\Big(\frac{y''}{y'}\Big)^2 + Q(y)(y')^2 = q(t) \tag{3.6}$$

are u.a.p. functions.

Proof. Denote by $\mathcal{G}$ the set of the phases of $y'' + y = 0$ and let α and Λ be a phase of (q) and (Q), respectively. Then $\Lambda^{-1}\mathcal{G}\alpha = \{\Lambda^{-1}(\gamma(\alpha)) : \gamma \in \mathcal{G}\}$ is the set of all regular solutions to (3.6) (see [3]). Here Λ^{-1} stands for the inverse function to Λ.

Let $\gamma \in \mathcal{G}$ and set $X = \Lambda^{-1}(\gamma(\alpha))$. To prove the statement of our corollary we have to show that $X' \in \mathcal{A}$. We know, by Theorem 3.1,

$$\Lambda(t) = At + \Psi(t), \quad \alpha(t) = at + \psi(t)\,, \quad t \in \mathbb{R}\,, \tag{3.7}$$

where $A, a \in \mathbb{R}$, $\Psi^{(i)}, \psi^{(i)} \in \mathcal{A}$ for $i = 0, 1$ and

$$\inf\{|A + \Psi'(t)| : t \in \mathbb{R}\} > 0\,, \quad \inf\{|a + \psi'(t)| : t \in \mathbb{R}\} > 0\,.$$

Since (Q) and (q) are oscillatory by Lemma 2.1, we have

$$\lim_{|t|\to\infty} |\Lambda(t)| = \infty\,, \quad \lim_{|t|\to\infty} |\alpha(t)| = \infty$$

and then (3.7) and the boundedness of Ψ and ψ imply $A \neq 0$, $a \neq 0$. From $\gamma(t+\pi) = \gamma(t) + \mu\pi$ for $t \in \mathbb{R}$, where $\mu = \operatorname{sign}\gamma'$ (see [3]), we deduce that

$$\gamma(t) = \mu t + \eta(t)\,, \quad t \in \mathbb{R}\,, \tag{3.8}$$

where η is a π-periodic function. Of course, $\eta \in \mathcal{A}$ and $\inf\{|\mu + \eta'(t)| : t \in \mathbb{R}\} > 0$. We claim that

$$\Lambda^{-1}(t) = \frac{t}{A} + \Psi_*(t) \quad \text{for} \quad t \in \mathbb{R}\,, \tag{3.9}$$

where $\Psi_*, \Psi_*' \in \mathcal{A}$ and $\inf\{|1/A + \Psi_*'(t)| : t \in \mathbb{R}\} > 0$. We first have

$$t = \Lambda^{-1}(\Lambda(t)) = t + \frac{\Psi(t)}{A} + \Psi_*(At + \Psi(t))\,,$$

and so

$$\Psi(t) = -A\Psi_*(At + \Psi(t)) \quad \text{for} \quad t \in \mathbb{R}\,. \tag{3.10}$$

Let $\{k_n\} \subset \mathbb{R}$ be a sequence. Since $\Psi \in \mathcal{A}$, we can assume without restriction of generality that

$$\lim_{n\to\infty} \Psi(t + k_n) = r(t) \quad \text{uniformly on} \quad \mathbb{R}\,. \tag{3.11}$$

Obviously, $r \in \mathcal{A}$. Let ε be a positive number. By (3.10) and (3.11), there exists $n_0 \in \mathbb{N}$ such that $|\Psi_*(At + Ak_n + \Psi(t+k_n)) + r(t)/A| < \varepsilon/2$ and then

$$\Big|\Psi_*(At + Ak_n) + \frac{1}{A}r\Big(t - \frac{\Psi(t+k_n)}{A}\Big)\Big| < \frac{\varepsilon}{2} \tag{3.12}$$

for $t \in \mathbb{R}$ and $n \geq n_0$. From r being equicontinuous on $\mathbb{R}$ and (3.11), it follows that there is an $n_1 \in \mathbb{N}$ such that

$$\Big|r\Big(t - \frac{\Psi(t+k_n)}{A}\Big) - r\Big(t - \frac{r(t)}{A}\Big)\Big| < \frac{\varepsilon|A|}{2} \quad \text{for} \quad t \in \mathbb{R} \text{ and } n \geq n_1 . \tag{3.13}$$

Then (3.12) and (3.13) give

$$\begin{aligned}
&\Big|\Psi_*(At + Ak_n) + \frac{1}{A} r\Big(t - \frac{r(t)}{A}\Big)\Big| \\
&\quad \leq \Big|\Psi_*(At + Ak_n) + \frac{1}{A} r\Big(t - \frac{\Psi(t+k_n)}{A}\Big)\Big| \\
&\qquad + \frac{1}{|A|}\Big|r\Big(t - \frac{\Psi(t+k_n)}{A}\Big) - r\Big(t - \frac{r(t)}{A}\Big)\Big| \\
&\quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

for $t \in \mathbb{R}$ and $n \geq \max\{n_0, n_1\}$. We have proved that $\{\Psi_*(At + Ak_n)\}$ is uniformly convergent on $\mathbb{R}$. Hence $\Psi_*(At) \in \mathcal{A}$ and then $\Psi_* \in \mathcal{A}$. Since $\Lambda', 1/\Lambda'$ and Λ'' are bounded, we see that $\Lambda^{-1'}, 1/\Lambda^{-1'}$ and $\Lambda^{-1''}$ are also bounded which implies the boundedness of Ψ'_*, Ψ''_* and $\inf\{|1/A + \Psi'_*(t)| : t \in \mathbb{R}\} > 0$. Clearly, $\Psi'_* \in \mathcal{A}$. Now, applying Lemma 2.4, we see that $\eta(at + \psi(t))$ and $\eta'(at + \psi(t))$ belong to $\mathcal{A}$. Hence $f(t) = \mu\psi(t) + \eta(at + \psi(t)) \in \mathcal{A}$ and repeated the above lemma we deduce that

$$\Psi_*(\mu a t + f(t)), \quad \Psi'\Big(\frac{\mu a}{A} t + \frac{f(t)}{A} + \Psi_*(\mu a t + f(t))\Big)$$

are u.a.p. functions. From the equalities (see (3.7)–(3.9)),

$$\Lambda^{-1'}(\gamma(\alpha(t))) = \frac{1}{\Lambda'(\Lambda^{-1}(\gamma(\alpha(t))))} = \frac{1}{A + \Psi'\Big(\frac{\mu a}{A} t + \frac{f(t)}{A} + \Psi_*(\mu a t + f(t))\Big)},$$

$$\gamma'(\alpha(t)) = \mu + \eta'(at + \psi(t)), \quad \alpha'(t) = a + \psi'(t)$$

and the inequality $\inf\{|A + \Psi'(t)| : t \in \mathbb{R}\} > 0$, we deduce that $X'(t) = \Lambda^{-1'}(\gamma(\alpha(t)))\gamma'(\alpha(t))\alpha'(t)$ belongs to $\mathcal{A}$. This completes the proof. $\square$

Remark 3.7. Let q and all solutions of (q) belong to $\mathcal{A}$. If α is a phase of (q) then, by Theorem 3.1, $\alpha(t) = at + \varphi(t)$ for $t \in \mathbb{R}$ where $a \in \mathbb{R}$, $\varphi, \varphi' \in \mathcal{A}$ and $\inf\{|a + \varphi'(t)| : t \in \mathbb{R}\} > 0$. As in the proof of Corollary 3.6 we may verify that $a \neq 0$ and $\alpha^{-1}(t) = t/a + \varphi_*(t)$ for $t \in \mathbb{R}$, where $\varphi_*, \varphi'_* \in \mathcal{A}$ and $\inf\{|1/a + \varphi'_*(t)| : t \in \mathbb{R}\} > 0$. Let the inverse function α^{-1} to α be a phase of (q_*). It is a simple calculation to show that

$$q_*(t) = -1 - \Big(\frac{d}{dt}\alpha^{-1}(t)\Big)^2 (1 + q(\alpha^{-1}(t)), \quad t \in \mathbb{R}.$$

Then $q_* \in \mathcal{A}$ and Theorem 3.1 shows that all solutions of (q_*) belong to $\mathcal{A}$. Hence for each phase α of (q), all solutions of the differential equation

$$y'' = -\Big[1 + \Big(\frac{d}{dt}\alpha^{-1}(t)\Big)^2 (1 + q(\alpha^{-1}(t))\Big] y$$

are u.a.p. functions.

Remark 3.8. If q is a π-periodic continuous function then all solutions of (q) are bounded if and only if there is a phase α of (q) such that $\alpha(t+\pi) = \alpha(t) + a$ for $t \in \mathbb{R}$, where $a \in \mathbb{R}$ (see [4], [5]). In this case $\alpha(t) = (a/\pi)t + \psi(t)$ where ψ is a π-periodic function and consequently all solutions of (q) belong to $\mathcal{A}$ by Theorem 3.1.

Theorem 3.9. *Let $\varphi \in C^2(\mathbb{R})$, $\varphi^{(j)} \in \mathcal{A}$ for $j = 0, 1, 2$, the mean value $M[\varphi] = 0$, $\int_0^t \varphi(s)\,ds$ be unbounded and $\inf\{|a + \varphi(t)| : t \in \mathbb{R}\} > 0$ with some $a \in \mathbb{R}$. Set*

$$q(t) = -\frac{1}{2}\frac{\varphi''(t)}{a+\varphi(t)} + \frac{3}{4}\Big(\frac{\varphi''(t)}{a+\varphi(t)}\Big)^2 - (a+\varphi(t))^2 \quad \text{for } t \in \mathbb{R}. \tag{3.14}$$

Then $q \in \mathcal{A}$, all solutions of (q) are bounded and only the trivial solution of (q) belongs to $\mathcal{A}$.

Proof. It is easily seen that $q \in \mathcal{A}$. Set

$$\alpha(t) = at + \int_0^t \varphi(s)\,ds\,, \quad t \in \mathbb{R}\,.$$

Then α is a phase of (q) with q given by (3.14) and since $\alpha' = a + \varphi \in \mathcal{A}$ and $\sup\{1/|\alpha'(t)| : t \in \mathbb{R}\} = \sup\{1/|a+\varphi(t)| : t \in \mathbb{R}\} < \infty$, all solutions of (q) are bounded by Lemma 2.2. By our assumption $\int_0^t \varphi(s)\,ds \notin \mathcal{A}$ and so Theorem 3.1 shows that there is a solution u of (q) such that $u \notin \mathcal{A}$. Assume now that there exists a non-trivial solutions v of (q) such that $v \in \mathcal{A}$. Clearly, (u, v) is a base of (q). We know that v can be written in the form

$$v(t) = c_1 \frac{\sin(\alpha(t) + c_2)}{\sqrt{|\alpha'(t)|}}\,, \quad t \in \mathbb{R}\,,$$

where $c_1, c_2 \in \mathbb{R}$. Since $\sqrt{|\alpha'|} \in \mathcal{A}$, we have $\sin(\alpha + c_2) \in \mathcal{A}$. Set $w(t) = \sin(\alpha(t) + c_2)$ for $t \in \mathbb{R}$. Then from the equalities

$$w' = \alpha' \cos(\alpha + c_2), \quad w'' = \alpha'' \cos(\alpha + c_2) - (\alpha')^2 \sin(\alpha + c_2)$$

we deduce that w'' is bounded, so w' is equicontinuous and then $w' \in \mathcal{A}$. Therefore $\cos(\alpha + c_2) \in \mathcal{A}$ since $1/\alpha' \in \mathcal{A}$. Let

$$F_1(t) = \frac{\cos(\alpha(t) + c_2) + i \sin(\alpha(t) + c_2)}{\sqrt{|\alpha'(t)|}}\,, \quad t \in \mathbb{R}\,.$$

Then F_1 is a u.a.p. function, $|F_1(t)| = 1/\sqrt{|\alpha'(t)|} = 1/\sqrt{|a + \varphi(t)|} \geq l_1 > 0$ for $t \in \mathbb{R}$ with a constant l_1, and consequently Lemma 2.3 gives $\arg F_1(t) = \alpha(t) + c_2 + 2k\pi = a_1 t + \varphi_1(t)$ for $t \in \mathbb{R}$ where k is an integer, $a_1 \in \mathbb{R}$ and $\varphi_1 \in \mathcal{A}$. Thus

$$at + \int_0^t \varphi(s)\,ds + c_2 + 2k\pi = a_1 t + \varphi_1(t)\,, \quad t \in \mathbb{R}$$

and then $a = a_1$ and $\varphi_1(t) = \int_0^t \varphi(s)\,ds + c_2 + 2k\pi$ for $t \in \mathbb{R}$ since $M[\varphi] = 0$, contrary to $\varphi_1 \in \mathcal{A}$. We have proved that only the trivial solution of (q) belongs to $\mathcal{A}$. $\square$

Example 3.10. Let

$$\varphi(t) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(\frac{t}{n^2}), \quad t \in \mathbb{R}.$$

Then $\varphi^{(j)} \in \mathcal{A}$ for $j = 0, 1, 2$, $M[\varphi] = 0$ and $\int_0^t \varphi(s)\, ds$ is unbounded (see [13]). Set $\gamma = \sup\{|\varphi(t)| : t \in \mathbb{R}\}$ and q be defined by (3.14) with $|a| > \gamma$. Applying Theorem 3.9, all solutions of (q) are bounded but there is no (non-trivial) solution in $\mathcal{A}$.

Theorem 3.11. *Let $q, f \in \mathcal{A}$ and let all solutions of (q) belong to $\mathcal{A}$. If a solution of the non-homogeneous differential equation*

$$y'' = q(t)y + f(t) \tag{3.15}$$

is bounded, then all solutions of (3.15) *belong to $\mathcal{A}$.*

Proof. Let y be a bounded solution of (3.15) and let (u, v) be a base of (q), $u'v - uv' = 1$. Then

$$y(t) = c_1 u(t) + c_2 v(t) + u(t) \int_0^t v(s) f(s)\, ds - v(t) \int_0^t u(s) f(s)\, ds, \quad t \in \mathbb{R},$$

where $c_1, c_2 \in \mathbb{R}$. Since $u, v \in \mathcal{A}$, they are bounded and the function

$$z(t) = u(t) \int_0^t v(s) f(s)\, ds - v(t) \int_0^t u(s) f(s)\, ds, \quad t \in \mathbb{R}$$

is a bounded solution of (3.15). To prove our theorem it suffices to show that the functions $\int_0^t v(s)f(s)\, ds$ and $\int_0^t u(s)f(s)\, ds$ are bounded. Really, since $vf, uf \in \mathcal{A}$ we conclude that $\int_0^t v(s)f(s)\, ds$, $\int_0^t u(s)f(s)\, ds$ belong to $\mathcal{A}$ if they are bounded, and so $z \in \mathcal{A}$. Then from the structure of the solution space of (3.15) we deduce that all solutions of (3.15) belong to $\mathcal{A}$. We are going to prove that the functions $\int_0^t v(s)f(s)\, ds$, $\int_0^t u(s)f(s)\, ds$ are bounded. First, note that $z''\,(= qz + f)$ is bounded which implies the boundedness of z' (see the proof of Lemma 2.1). Now, let α be a phase of (q) such that the equalities (2.3) are satisfied. Then

$$\begin{aligned} u' &= \alpha' \frac{\cos\alpha}{\sqrt{|\alpha'|}} - \frac{\alpha''}{2\alpha'} \frac{\sin\alpha}{\sqrt{|\alpha'|}} = \alpha' v - \frac{\alpha''}{2\alpha'} u, \\ v' &= -\alpha' \frac{\sin\alpha}{\sqrt{|\alpha'|}} - \frac{\alpha''}{2\alpha'} \frac{\cos\alpha}{\sqrt{|\alpha'|}} = -\alpha' u - \frac{\alpha''}{2\alpha'} v \end{aligned}$$

and

$$\begin{aligned} z' &= \Big(\alpha' v - \frac{\alpha''}{2\alpha'} u\Big) \int_0^t vf\, ds + \Big(\alpha' u + \frac{\alpha''}{2\alpha'} v\Big) \int_0^t uf\, ds \\ &= \alpha' \Big(v \int_0^t vf\, ds + u \int_0^t uf\, ds\Big) - \frac{\alpha''}{2\alpha'} z. \end{aligned}$$

Since $1/\alpha'$, α' and α'' are bounded by Lemma 2.2 and we know that z, z' are bounded, from the equality

$$v\int_0^t vf\,ds + u\int_0^t uf\,ds = \frac{\alpha''}{2(\alpha')^2}z + \frac{1}{\alpha'}z'$$

we see that $v\int_0^t vf\,ds + u\int_0^t uf\,ds$ is bounded. As

$$\begin{aligned} &z^2 + \Big(v\int_0^t vf\,ds + u\int_0^t uf\,ds\Big)^2 \\ &= (u^2+v^2)\Big[\Big(\int_0^t vf\,ds\Big)^2 + \Big(\int_0^t uf\,ds\Big)^2\Big] \\ &= \frac{1}{|\alpha'|}\Big[\Big(\int_0^t vf\,ds\Big)^2 + \Big(\int_0^t uf\,ds\Big)^2\Big], \end{aligned}$$

$\Big(\int_0^t vf\,ds\Big)^2 + \Big(\int_0^t uf\,ds\Big)^2$ is bounded. Consequently the functions $\int_0^t v(s)f(s)\,ds$ and $\int_0^t u(s)f(s)\,ds$ are bounded, which completes the proof. □

Corollary 3.12. *Let $q \in \mathcal{A}$ and suppose that all solutions of (q) belong to $\mathcal{A}$. Let (u, v) be a base of (q). Then for each non-zero constants a, b, all solutions of the differential equation*

$$y'' = q(t)y + \frac{1}{(a^2u^2(t) + b^2v^2(t))^{\frac{3}{2}}} \tag{3.16}$$

belong to $\mathcal{A}$.

Proof. Fix $a, b \in \mathbb{R}$, $a \neq 0$, $b \neq 0$. Set

$$u_1(t) = \operatorname{sign} w\sqrt{\Big|\frac{a}{bw}\Big|}\,u(t), \quad v_1(t) = \sqrt{\Big|\frac{b}{aw}\Big|}\,v(t)\,, \quad t \in \mathbb{R}\,, \tag{3.17}$$

where $w = u'v - uv'$. Then (u_1, v_1) is a base of (q) and $u_1'v_1 - u_1v_1' = 1$. Therefore there exists a phase α of (q), $\alpha' > 0$, such that

$$u_1(t) = \frac{\sin\alpha(t)}{\sqrt{\alpha'(t)}}\,, \quad v_1(t) = \frac{\cos\alpha(t)}{\sqrt{\alpha'(t)}}\,, \quad t \in \mathbb{R}\,.$$

Hence (see (3.17))

$$\frac{1}{\alpha'(t)} = u_1^2(t) + v_1^2(t) = \Big|\frac{a}{bw}\Big|\,u^2(t) + \Big|\frac{b}{aw}\Big|\,v^2(t) = \frac{a^2u^2(t) + b^2v^2(t)}{|abw|}$$

and

$$a^2u^2(t) + b^2v^2(t) = \frac{|abw|}{\alpha'(t)}\,, \quad t \in \mathbb{R}\,. \tag{3.18}$$

Since

$$p(t) = u_1(t)\int_0^t \frac{v_1(s)}{(a^2u^2(s) + b^2v^2(s))^{\frac{3}{2}}}\,ds - v_1(t)\int_0^t \frac{u_1(s)}{(a^2u^2(s) + b^2v^2(s))^{\frac{3}{2}}}\,ds$$

is a solution of (3.16) and $(a^2u^2(t)+b^2v^2(t))^{-3/2}\in\mathcal{A}$, to prove our corollary it suffices to verify that p is bounded (see Theorem 3.11). From (3.17) and (3.18) it follows that

$$\begin{aligned}\int_0^t \frac{v_1(s)}{(a^2u^2(s)+b^2v^2(s))^{\frac{3}{2}}}\,ds &= \int_0^t \frac{\cos\alpha(s)}{\sqrt{\alpha'(s)}}\Big(\frac{\alpha'(s)}{|abw|}\Big)^{\frac{3}{2}}\,ds\\ &= \frac{1}{|abw|^{\frac{3}{2}}}\int_0^t \alpha'(s)\cos\alpha(s)\,ds\\ &= \frac{\sin\alpha(t)-\sin\alpha(0)}{|abw|^{\frac{3}{2}}}\end{aligned}$$

and

$$\begin{aligned}\int_0^t \frac{u_1(s)}{(a^2u^2(s)+b^2v^2(s))^{\frac{3}{2}}}\,ds &= \int_0^t \frac{\sin\alpha(s)}{\sqrt{\alpha'(s)}}\Big(\frac{\alpha'(s)}{|abw|}\Big)^{\frac{3}{2}}\,ds\\ &= \frac{1}{|abw|^{\frac{3}{2}}}\int_0^t \alpha'(s)\sin\alpha(s)\,ds\\ &= \frac{\cos\alpha(0)-\cos\alpha(t)}{|abw|^{\frac{3}{2}}}\,.\end{aligned}$$

Hence $\int_0^t \frac{v_1(s)}{(a^2u^2(s)+b^2v^2(s))^{3/2}}\,ds$ and $\int_0^t \frac{u_1(s)}{(a^2u^2(s)+b^2v^2(s))^{3/2}}\,ds$ are bounded functions and since $u_1, v_1\in\mathcal{A}$, the function p is bounded. □

References

[1] Appell, P., *Sur les transformations des équations différentielles linéaires*, C. R. Acad. Sci. Paris **91** (1880), 211–214.

[2] Beckenbach, E. F. and Bellman, R., *Inequalities*, Springer 1961.

[3] Borůvka, O., *Linear differential transformations of the second order*, The English Univ. Press, London 1971.

[4] Borůvka, O., *Sur les blocs des équations différentielles $y''=q(t)y$ aux coefficients périodiques*, Rend. Mat. **8** (1975), 519–532.

[5] Borůvka, O., *The theory of global properties of second-order ordinary differential equations*, Differentsial'nye Uravneniya, **12** (1976), 1347–1383. (in Russian).

[6] Corduneanu, C., *Almost Periodic Functions*, Wiley, New York 1968.

[7] Fink, M. A., *Almost periodic differential equations*, Springer, New York – Berlin 1974.

[8] Guter, R. S., Kudryavtsev, L. D. and Levitan, B. M., *Elements of the theory of functions*, Pergamon Press, Oxford 1966.

[9] Greguš, M., *Linear differential equations of the third order*, North Holland, Reider Co., Dordrecht-Boston-Lancaster 1986.

[10] Haraux, A., *A simple almost-periodicity criterion and applications*, J. Differential Equations **66** (1987), 51–61.

[11] Hartman, P., *Ordinary differential equations*, J. Wiley, New York 1964.

[12] Hu, Z. S. and Mingarelli, A. B., *On a question in the theory of almost periodic differential equations*, Proc. Amer. Math. Soc. **127** (1999), 2665–2670.

[13] Levitan, B. M., *Almost-periodic functions*, G.I.T.-T.L., Moscow 1953 (in Russian).

[14] Lillo, J. C., *Approximate similarity and almost periodic matrices*, Proc. Amer. Math. Soc. **12** (1961), 400-407.

[15] Markus, L. and Moore, R. A., *Oscillation and disconjugacy for linear differential equations with almost periodic coefficients*, Acta mathematica **96** (1956), 99-123.

[16] Mingarelli, A. B., Pu, P. Q. and Zheng, L., *A Counter-example in the theory of almost periodic differential equations*, Rocky Mountain J. Math. **25** (1995), 437–440.

[17] Rudin, W., *Principles of Mathematical Analysis*, McGraw-Hill, New York 1964.

[18] Staněk, S., *On some properties of solutions of the disconjugate equation $y'' = q(t)y$ with an almost periodic coefficient q*, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. **25** (1986), 31–56.

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