ARCHIVUM MATHEMATICUM (BRNO)
Tomus 41 (2005), 243 – 251

ON GENERALIZED Q.F.D. MODULES

MOHAMMAD SALEH, S. K. JAIN AND S. R. LÓPEZ-PERMOUTH

Abstract. A right R-module M is called a generalized *q.f.d.* module if every M-singular quotient has finitely generated socle. In this note we give several characterizations to this class of modules by means of weak injectivity, tightness, and weak tightness that generalizes the results in [25], Theorem 3. In particular, it is shown that a module M is *g.q.f.d.* iff every direct sum of M-singular M-injective modules in $\sigma[M]$ is weakly injective iff every direct sum of M-singular weakly tight is weakly tight iff every direct sum of the injective hulls of M-singular simples is weakly R-tight.

1. Introduction

Throughout this paper all rings are associative with identity and all modules are unitary. We denote the category of all right R-modules by Mod-R and for any $M \in$ Mod-R, $\sigma[M]$ stands for the full subcategory of Mod-R whose objects are submodules of M-generated modules (see [26]). Given a module X_R, the injective hull of X in Mod- R (resp., in $\sigma[M]$) is denoted by $E(X)$ (resp., $E_M(X)$). The M-injective hull $E_M(X)$ is the trace of M in $E(X)$, i.e. $E_M(X) = \sum\{f(M),\, f \in \mathrm{Hom}\big(M, E(X)\big)\}$ [3], [26].

The purpose of this paper is to further the study of the concepts of weak injectivity (tightness) in $\sigma[M]$ studied in [4], [6], [15], [21], [20], [24], [25], [27], [28]. In [25], Theorem 3 characterized finitely generated *g.q.f.d.* modules by means of weak injectivity. In this note we sharpen this result by characterizing any *g.q.f.d.*, not necessary finitely generated, by means of weak injectivity, tightness and weak tightness. It is shown that a module M is *g.q.f.d.* iff every direct sum of M-singular M-injective modules in $\sigma[M]$ is weakly injective iff every direct sum of M-singular weakly tight is weakly tight iff every direct sum of the injective hulls of M-singular simples is weakly R-tight.

Given two modules X and $N \in \sigma[M]$, we call X *weakly N-injective* in $\sigma[M]$ if for every homomorphism $\varphi : N \to E_M(X)$, there exist a homomorphism $\widehat{\varphi} : N \to X$ and a monomorphism $\sigma : X \to E_M(X)$ such that $\varphi = \sigma\widehat{\varphi}$. Equivalently, if there

---

2000 *Mathematics Subject Classification*: 16D50, 16D60, 16D70, 16P40.

Key words and phrases: tight, weakly tight, weakly injective, q.f.d., generalized q.f.d. modules, generalized weakly semisimple.

Received June 5, 2003, revised September 2004.

exists a submodule Y of $E_M(X)$ such that $\varphi(N) \subset Y \simeq X$. A module $X \in \sigma[M]$ is called *weakly injective* in $\sigma[M]$ if for every finitely generated submodule N of the M-injective hull $E_M(X)$, N is contained in a submodule Y of $E_M(X)$ such that $Y \simeq X$. Equivalently, if X is weakly N-injective for all finitely generated modules N in $\sigma[M]$. A module X is *N-tight* in $\sigma[M]$ if every quotient of N which is embeddable in the M-injective hull $E_M(X)$ of X is embeddable in X. A module is *tight* (*R-tight*) in $\sigma[M]$ if it is tight relative to all finitely generated (cyclic) submodules of its M-injective hull, and X is *weakly tight* (*weakly R-tight*) in $\sigma[M]$ if every finitely generated (cyclic) submodule N of $E_M(X)$ is embeddable in a direct sum of copies of X. It is clear that every weakly injective module in $\sigma[M]$ is tight in $\sigma[M]$, and every tight module in $\sigma[M]$ is weakly tight in $\sigma[M]$, but weak tightness does not imply tightness, (see [4], [28]).

A right R-module M is called a generalized *q.f.d.* module if every M-singular quotient has finitely generated socle. A module M_R is called *S-WI* (*S-WRI*) if every M-singular module $N \in \sigma[M]$ is weakly injective (weakly R-injective) in $\sigma[M]$. *An essential* (*large*) submodule X of an R-module Y will be denoted by $X \subseteq' Y$. A right module N is said to be *compressible* if for every essential submodule $K \subset' N$, N is embeddable in K. A module Q is called *weakly injective* (resp., *tight, weakly tight*) [9], [10], [11], [12] if it is weakly injective (resp., tight, weakly tight) in $\sigma[R_R] = \text{Mod-}R$.

The socle of a module X is denoted by $\text{Soc}\,(X)$. A module $N \in \sigma[M]$ is called *singular* in $\sigma[M]$ or M-singular if there exists a module X in $\sigma[M]$ containing an essential submodule K such that $N \simeq X/K$ (see [26]). The class of all M-singular modules is closed under submodules, homomorphic images and direct sum ([26], 17.3, 17.4). Thus every module $N \in \sigma[M]$ contains a largest M-singular submodule which is denoted by $Z_M(N)$.

For a module X_R and a module property $\mathbb{P}$ X is said to be $\sum -\mathbb{P}$ in case every direct sum of copies of X enjoys the property $\mathbb{P}$. Also we call X *locally* $\mathbb{P}$ in case every finitely generated submodule of X enjoys the property $\mathbb{P}$ (see [1], [3], [13]).

The class of weak injectivity (tightness, weak tightness) in $\sigma[M]$ is closed under finite direct sum and essential extensions. Also, the domains of the class of weak injectivity (tightness, weak tightness) in $\sigma[M]$ are closed under submodules, and quotients.

We list below some of the basic results on weak injectivity (tightness, weak tightness) in $\sigma[M]$ that will be needed in this paper (see [12], [21], [20], [28]).

Lemma 1.1 ([28, Lemma 2.2], [20])**.** *Given modules $N, Q \in \sigma[M]$. If Q is uniform then Q is weakly N-tight in $\sigma[M]$ iff Q is weakly N-injective in $\sigma[M]$.*

Lemma 1.2 ([28, Lemma 2.2], [20, Corollary 3.5])**.** *For a right R-module M_R, every* (*uniform*) *cyclic in $\sigma[M]$ is weakly R-injective* (*R-tight, weakly R-tight*) *in $\sigma[M]$ iff every* (*uniform*) *cyclic is compressible.*

Lemma 1.3 ([28, Lemma 3.1], [20, Proposition 3.6]). *Given modules N, $Q \in \sigma[M]$. If Q is self-injective and weakly N-tight in $\sigma[M]$, then Q is N-injective in $\sigma[M]$.*

In [14], it is shown that any completely reducible module is a direct summand of a weakly injective module, the next lemma shows that any module is in fact a direct summand of a weakly injective module.

Lemma 1.4. *For every module X in $\sigma[M]$, $X \oplus E_M(X)^{(\alpha)}$, where α is an infinite cardinal number, is weakly injective in $\sigma[M]$.*

Lemma 1.5. *A finite direct sum of weakly injective (tight, weakly tight) in $\sigma[M]$ is weakly injective (tight, weakly tight) in $\sigma[M]$, and an essential extension of a weakly injective (tight, weakly tight) module in $\sigma[M]$ is weakly injective (tight, weakly tight) in $\sigma[M]$.*

Example 1.6. (i) [12, Example 2.11], [14]. Let R be the ring of endomorphisms of an infinite dimensional vector space V over a field F. Then $M = \mathrm{Soc}(R_R) \oplus R$ is tight but not weakly injective.

(ii) [4]. Let $R = Z$ and $X = (Q/Z) \oplus (Z/pZ)$, where p is a prime number. Then X is weakly tight in $\sigma[M]$ but not tight.

(iii) [12, Example 4.4(d)]. Let F be a field. Then $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ is weakly injective but the summand $S = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$ as an R-module is not weakly injective.

Lemma 1.7 ([25, Theorem 2]). *Let R be a ring and M be a right R-module. Then the following are equivalent:*

(i) *M is a generalized q.f.d. module;*
(ii) *every M-cyclic M-singular module is finite dimensional;*
(iii) *every finitely M-generated M-singular module is finite dimensional.*

Theorem 1.8. *For a module M_R, the following implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) always hold.*

(a) *every direct sum $\bigoplus_\Lambda M_\lambda$ of M-singular injective modules in $\sigma[M]$ is weakly injective in $\sigma[M]$;*
(b) *every direct sum $\bigoplus_\Lambda M_\lambda$ of M-singular weakly injective modules in $\sigma[M]$ is weakly injective in $\sigma[M]$;*
(c) *every direct sum $\bigoplus_\Lambda M_\lambda$ of M-singular weakly injective modules in $\sigma[M]$ is tight in $\sigma[M]$;*
(d) *every direct sum $\bigoplus_\Lambda M_\lambda$ of M-singular tight modules in $\sigma[M]$ is tight in $\sigma[M]$;*
(e) *every direct sum $\bigoplus_\Lambda M_\lambda$ of M-singular tight modules in $\sigma[M]$ is weakly tight in $\sigma[M]$;*
(f) *every direct sum $\bigoplus_\Lambda M_\lambda$ of M-singular weakly tight modules in $\sigma[M]$ is weakly tight in $\sigma[M]$.*

Proof. (a) $\Rightarrow$ (b) Consider the module $X = \bigoplus_\Lambda M_\lambda$ a direct sum of M-singular weakly injective modules in $\sigma[M]$. Let N be a finitely generated submodule of

$E_M(X)$. By (a) the direct sum $\bigoplus_\Lambda E_M(M_\lambda)$ is weakly injective in $\sigma[M]$ and $X = \bigoplus_\Lambda M_\lambda \subseteq' \bigoplus_\Lambda E_M(M_\lambda) \subseteq' E_M(\bigoplus_\Lambda E_M(M_\lambda))$. Thus by (a) there exists a submodule $Y \subseteq E_M(\bigoplus_\Lambda E_M(M_\lambda))$ such that $N \subseteq Y \cong \bigoplus_\Lambda E_M(M_\lambda)$. Write $Y = \bigoplus_\Lambda E_M(Y_\lambda)$, where $Y_i \cong M_i, i \in \Lambda$. Since N is finitely generated, there exists a finite subset $\Gamma = \{\lambda_1, \ldots, \lambda_m\} \subseteq \Lambda$ such that $N \subseteq \bigoplus_\Gamma E_M(Y_\lambda) = E_M(\bigoplus_\Gamma Y_\lambda)$. Since $Y_{\lambda_1}, \ldots, Y_{\lambda_m}$ are weakly injective in $\sigma[M]$, the finite direct sum $Y_{\lambda_1} \oplus \cdots \oplus Y_{\lambda_m}$ is weakly injective in $\sigma[M]$. Therefore, there exists $X_1 \cong \bigoplus_\Gamma Y_\lambda \cong \bigoplus_\Gamma M_\lambda$ such that $N \subseteq X_1 \subseteq E_M(\bigoplus_\Gamma Y_\lambda)$. Thus $N \subseteq X_1 \oplus \bigoplus_{\lambda \notin \Gamma} Y_\lambda \simeq X$, proving that X is weakly injective.

(c) $\Rightarrow$ (d) Consider the module $X = \bigoplus_\Lambda M_\lambda$ a direct sum of M-singular tight modules in $\sigma[M]$. Let N be a finitely generated submodule of $E_M(X) = E_M(\bigoplus_\Lambda E_M(M_\lambda))$. By (c) the direct sum $\bigoplus_\Lambda E_M(M_\lambda)$ is tight in $\sigma[M]$. Thus N embeds in $\bigoplus_\Lambda E_M(M_\lambda)$ via a monomorphism, say, φ . Also $\varphi(N)$ is finitely generated and thus $N \subset E_M(M_{\lambda_1}) \oplus \cdots \oplus E_M(M_{\lambda_m}) = E_M(\bigoplus_{i=1}^m M_{\lambda_i})$ for some finite $\{\lambda_1, \ldots, \lambda_m\} \subseteq \Lambda$. Since $M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m}$ is tight then $N \simeq \varphi(N)$ embeds in the finite direct sum $M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m}$, proving that X is tight.

(e) $\Rightarrow$ (f) Consider the module $X = \bigoplus_\Lambda M_\lambda$ a direct sum of M-singular weakly tight modules in $\sigma[M]$. Let N be a finitely generated submodule of $E_M(X) = E_M(\bigoplus_\Lambda E_M(M_\lambda))$. By (e) the direct sum $\bigoplus_\Lambda E_M(M_\lambda)$ is weakly tight in $\sigma[M]$. Thus N embeds in $(\bigoplus_\Lambda E_M(M_\lambda))^{(\aleph_0)}$ via a monomorphism, say, φ. Also $\varphi(N)$ is finitely generated and thus $N \subset E_M(M_{\lambda_1}) \oplus \cdots \oplus E_M(M_{\lambda_m}) = E_M(\bigoplus_{i=1}^m M_{\lambda_i})$ for some finite $\{\lambda_1, \ldots, \lambda_m\} \subseteq \Lambda$. Since $M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m}$ is weakly tight then $N \simeq \varphi(N)$ embeds in a direct sum of copies of $(M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m})$ and thus embeds in a direct sum of X, proving that X is weakly tight.

Clearly, (b) $\Rightarrow$ (c), (d) $\Rightarrow$ (e) and (b) $\Rightarrow$ (a). □

In [25], several characterizations of finitely generated generalized *q.f.d.* modules are given using weak injectivity. In the next result we provide other characterizations of an arbitrary generalized *q.f.d.* modules using tightness and weak tightness, a generalization of the characterization given in [25].

Theorem 1.9. *For a module M_R with $Z_M(M) = 0$, the following conditions are equivalent:*

(a) *M is a generalized q.f.d. module;*
(b) *every direct sum $\bigoplus_\Lambda E_\lambda$ of M-singular injective modules in $\sigma[M]$ is weakly injective in $\sigma[M]$;*
(c) *every direct sum $\bigoplus_\Lambda M_\lambda$ of M-singular injective modules in $\sigma[M]$ is tight in $\sigma[M]$;*
(d) *every direct sum $\bigoplus_\Lambda M_\lambda$ of M-singular injective modules in $\sigma[M]$ is weakly tight in $\sigma[M]$;*
(e) *every direct sum $\bigoplus_\Lambda M_\lambda$ of M-singular weakly tight modules in $\sigma[M]$ is weakly tight in $\sigma[M]$;*
(f) *every direct sum $\bigoplus_\Lambda M_\lambda$ of M-singular tight modules in $\sigma[M]$ is weakly tight in $\sigma[M]$;*
(g) *every direct sum $\bigoplus_\Lambda M_\lambda$ of M-singular weakly injective modules in $\sigma[M]$ is weakly tight in $\sigma[M]$;*

(h) *every direct sum* $\bigoplus_\Lambda M_\lambda$ *of* M*-singular weakly injective modules in* $\sigma[M]$ *is weakly* N*-tight for every cyclic module* N *in* $\sigma[M]$*;*
(i) *every direct sum* $\bigoplus_\Lambda E_M(P_\lambda)$*, where* P_λ *is* M*-singular simple module in* $\sigma[M]$*, is weakly* N*-tight, for every cyclic module* N *in* $\sigma[M]$*;*
(j) *every direct sum* $\bigoplus_\Lambda E_M(P_\lambda)$*, where* P_λ *is* M*-singular simple module in* $\sigma[M]$*, is weakly* R*-tight in* $\sigma[M]$*.*

Proof. (a) $\Rightarrow$ (b) Consider the module $X = \bigoplus_\Lambda M_\lambda$ a direct sum of M-singular injective modules in $\sigma[M]$. Let N be a finitely generated submodule of $E_M(X)$. Since M is non M-singular, by ([7], 4.1), $E_M(X)$ is M-singular and thus N is M-singular. By (a), N contains as an essential submodule a finite direct sum of uniform submodules $\bigoplus_\Lambda U_\lambda$. Since X is essential in $E_M(X) = E_M(\bigoplus_\Lambda E_M(M_\lambda))$, for each i, choose $0 \neq x_i \in U_i \cap X$. Then $\bigoplus_{i=1}^n x_iR \subseteq \bigoplus_{i=1}^n M_{\lambda_i}$ for some λ_i's. It follows that $\bigoplus_{i=1}^n M_{\lambda_i}$ contains an M-injective hull E of $\bigoplus_{i=1}^n x_iR$. Since E is M-injective and contained in X, we may write $X = E \oplus K$, for some submodule K of X. On the other hand, let $E_M(N)$ be an M-injective hull of N in $E_M(X)$. Then $E_M(N) = E_M(\bigoplus_{i=1}^n x_iR) \cong E$ Since $\bigoplus_{i=1}^n x_iR$ is essential in $E_M(N)$, it follows that $E_M(N) \cap K = 0$. So let $Y = E_M(N) \oplus K \cong E \oplus K = X$. Then $N \subseteq Y \cong X$, proving that X is weakly injective.

Clearly, (b) $\Rightarrow$ (c) $\Rightarrow$ (d), (e) $\Rightarrow$ (f) $\Rightarrow$ (g) $\Rightarrow$ (h) $\Rightarrow$ (i) $\Rightarrow$ (j).

(d) $\Rightarrow$ (e) Consider the module $X = \bigoplus_\Lambda M_\lambda$ a direct sum of M-singular weakly tight modules in $\sigma[M]$. Let N be a finitely generated submodule of $E_M(X)$. By (d) the direct sum $\bigoplus_\Lambda E_M(M_\lambda)$ is weakly tight in $\sigma[M]$. Thus N embeds in $\big(\bigoplus_\Lambda E_M(M_\lambda)\big)^{(\aleph_0)}$ via a monomorphism, say, φ. Also $\varphi(N)$ is finitely generated and thus $N \subset E_M(M_{\lambda_1}) \oplus \cdots \oplus E_M(M_{\lambda_m}) = E_M(\bigoplus_{i=1}^m M_{\lambda_i})$ for some finite $\{\lambda_1, \ldots, \lambda_m\} \subseteq \Lambda$. Since $M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m}$ is weakly tight then $N \simeq \varphi(N)$ embeds in a direct sum of $(M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m})$ and thus embeds in a direct sum of X, proving that X is weakly tight.

(j) $\Rightarrow$ (a) Let K be a cyclic submodule of M. If $\mathrm{Soc}\,(K) = 0$, we are done. Suppoe $0 \neq \mathrm{Soc}\,(K) = \bigoplus_\Lambda P_\lambda$. We show that $\mathrm{Soc}\,(K)$ is finitely generated. For this consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus_\Lambda P_\lambda \arrow[r, "\gamma"] \arrow[d, "\varphi"] & K \\ & E_M\big(\bigoplus_\Lambda E_M(P_\lambda)\big) & \end{tikzcd}$$

where φ and γ are the inclusion homomorphisms. By M-injectivity of $E_M\big(\bigoplus_\Lambda E_M(P_\lambda)\big)$, there exists $\psi : K \to E_M(\bigoplus_\Lambda E_M(P_\lambda))$ such that $\psi\gamma = \varphi$. By our hypothesis, $\bigoplus_\Lambda E_M(P_\lambda)$ is weakly R-tight in $\sigma[M]$, hence $Im\varphi \subset Im\psi$ is embeddable in $(\bigoplus_\Lambda E_M(P_\lambda))^{(\aleph_0)}$. Therefore, $\mathrm{Soc}\,(K)$ is embeddable in $E_M(P_{\lambda_1}) \oplus \cdots \oplus E_M(P_{\lambda_m})$ for some finite $\{\lambda_1, \ldots, \lambda_m\} \subseteq \Lambda$. Since each $E_M(P_{\lambda_i})$ is uniform, $\mathrm{Soc}\,(K)$ has finite uniform dimension and is therefore finitely generated. □

Notice that Theorem 1.9 implies that the conditions of Theorem 1.8 are all equivalent. From Lemma 1.1, Theorems 1.8 and 1.9, we get the following characterizations of generalized *q.f.d.* modules.

Theorem 1.10. *For a module M_R with $Z_M(M) = 0$, the following conditions are equivalent:*

(a) *M is a generalized q.f.d.;*
(b) *every direct sum $\bigoplus_\Lambda E_\lambda$ of injective modules in $\sigma[M]$, where each E_λ is M-singular, is weakly injective (or tight, weakly tight) in $\sigma[M]$;*
(c) *every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly injective modules in $\sigma[M]$, where each M_λ is M-singular, is weakly injective (or tight, weakly tight) in $\sigma[M]$;*
(d) *every direct sum $\bigoplus_\Lambda M_\lambda$ of tight modules in $\sigma[M]$, where each M_λ is M-singular, is tight (or weakly tight) in $\sigma[M]$;*
(e) *every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly tight modules in $\sigma[M]$, where each M_λ is M-singular, is weakly tight in $\sigma[M]$;*
(f) *every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly tight modules in $\sigma[M]$, where each M_λ is M-singular, is weakly N-tight, for every cyclic module N in $\sigma[M]$;*
(g) *every direct sum $\bigoplus_\Lambda E_M(P_\lambda)$, where P_λ is M-singular simple, is N-tight for every cyclic module N in $\sigma[M]$;*
(h) *every direct sum $\bigoplus_\Lambda E_M(P_\lambda)$, where P_λ is M-singular simple, is weakly R-tight in $\sigma[M]$.*

In case $M_R = R_R$, we get

Corollary 1.11. *For a nonsingular ring R, the following conditions are equivalent:*

(a) *R is a g.q.f.d.;*
(b) *every direct sum $\bigoplus_\Lambda E_\lambda$ of injective R-singular modules is weakly injective;*
(c) *every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly injective R-singular modules is weakly injective (or tight, weakly tight);*
(d) *every direct sum $\bigoplus_\Lambda M_\lambda$ of tight R-singular modules is tight (or weakly tight);*
(e) *every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly tight R-singular modules is weakly tight (or weakly R-tight);*
(f) *every direct sum $\bigoplus_\Lambda E_M(P_\lambda)$, where each P_λ is R-singular simple, is weakly R-tight.*

Theorem 1.12. *A g.q.f.d. module M_R over which every uniform cyclic M-singular right module in $\sigma[M]$ is weakly injective (tight, weakly tight) in $\sigma[M]$ is right S-WI.*

Proof. Let $N \in \sigma[M]$ be M-singular. Then N contains an essential submodule $X = \bigoplus_I X_i$ which is a direct sum of cyclic uniform M-singular submodules. It follows by our hypothesis that each X_i is weakly injective in $\sigma[M]$ and thus $\bigoplus_I X_i$ is weakly injective in $\sigma[M]$. Thus N is weakly injective in $\sigma[M]$, proving that M is S-WI. □

Theorem 1.13. *For a module M_R, the following are equivalent:*

(a) *M is S-WI;*
(b) *M is g.q.f.d. and every finitely generated M-singular module in $\sigma[M]$ is weakly injective (or tight, weakly tight) in $\sigma[M]$;*
(c) *M is g.q.f.d. and every cyclic M-singular module in $\sigma[M]$ is weakly injective (or tight, weakly tight) in $\sigma[M]$;*

(d) *M is g.q.f.d. and every uniform cyclic M-singular module in $\sigma[M]$ is weakly injective (or tight, weakly tight) in $\sigma[M]$;*
(e) *M is g.q.f.d. and every finitely generated M-singular module in $\sigma[M]$ is compressible.*

Proof. (a) $\Rightarrow$ (b) Follows from Theorem 1.9.

Clearly, (b) $\Rightarrow$ (c), (c) $\Rightarrow$ (d).

(d) $\Rightarrow$ (e) Let N be a finitely generated M-singular module in $\sigma[M]$ and let $K \subset' N$. Since M is *g.q.f.d.*, N has finite Goldie dimension. Thus there exist cyclic uniform M-singular submodules U_i, $i = 1, \ldots, n$, of N such that $\oplus_{i=1}^{n} U_i \subset' K \subset N$. Since each U_i is uniform M-singular it follows that each U_i is compressible and thus weakly injective in $\sigma[M]$ and thus $\oplus_{i=1}^{n} U_i$ is weakly injective in $\sigma[M]$. Thus K is weakly injective in $\sigma[M]$ and thus N embeds in K, proving that N is compressible.

(e) $\Rightarrow$ (d) Every compressible module is tight and thus weakly injective over a *q.f.d.* module.

(d) $\Rightarrow$ (a) Follows from Theorem 1.12.

In case $M_R = R_R$, we obtain the following characterization of W-SI rings. □

Corollary 1.14. *For a ring R, the following are equivalent:*

(a) *R is right W-SI ring;*
(b) *R is g.q.f.d. and every finitely generated R-singular right module is weakly injective (or tight, weakly tight);*
(c) *R is g.q.f.d. and every cyclic R-singular right module is weakly injective (or tight, weakly tight);*
(d) *R is g.q.f.d. and every uniform cyclic R-singular right module is weakly injective (or tight, weakly tight);*
(e) *R is g.q.f.d. and every uniform finitely generated R-singular right module is compressible;*
(f) *every finitely generated R-singular right module is compressible.*

Following a similar proof as in Theorem 1.12, we get the following

Theorem 1.15. *For a module M_R, the following are equivalent:*

(a) *Every M-singular $N \in \sigma[M]$ is weakly R-injective in $\sigma[M]$;*
(b) *M is g.q.f.d. and every finitely generate M-singular module in $\sigma[M]$ is weakly R-injective (or R-tight, weakly R-tight) in $\sigma[M]$;*
(c) *M is g.q.f.d. and every M-singular cyclic module in $\sigma[M]$ is weakly R-injective (or R-tight, weakly R-tight) in $\sigma[M]$;*
(d) *M is g.q.f.d. and every uniform M-singular cyclic module in $\sigma[M]$ is weakly R-injective (or R-tight, weakly R-tight) in $\sigma[M]$;*
(e) *M is g.q.f.d. and every cyclic M-singular module in $\sigma[M]$ is compressible.*

In case $M_R = R_R$, we obtain the following characterization of *S-WRI*-rings.

Corollary 1.16. *For a ring R, the following are equivalent:*

(a) *R is right S-WRI ring;*

(b) *R is g.q.f.d. and every finitely generated R-singular right module is weakly R-injective* (*or R-tight, weakly R-tight*)*;*
(c) *R is g.q.f.d. and every R-singular cyclic right module is weakly R-injective* (*or R-tight, weakly R-tight*)*;*
(d) *R is g.q.f.d. and every uniform R-singular cyclic right module is weakly R-injective* (*or R-tight, weakly R-tight*)*;*
(e) *R is g.q.f.d. and every R-singular cyclic right module is compressible.*

Acknowledgment. The authors wish to thank the learned referee for his valuable comments that improved the writing of this paper. Part of this paper was written during the stay of the first author at Ohio University under a Fulbright scholarship. The first author wishes to thank the Department of Mathematics and the Ohio University Center for Ring Theory and its Applications for the warm hospitality and the Fulbright for the financial support.

References

[1] Albu, T. and Nastasescu, C., *Relative finiteness in module theory*, Marcel Dekker 1984.

[2] Al-Huzali, A., Jain, S. K. and López-Permouth, S. R., *Rings whose cyclics have finite Goldie dimension*, J. Algebra **153** (1992), 37–40.

[3] Berry, D., *Modules whose cyclic submodules have finite dimension*, Canad. Math. Bull. **19** (1976), 1–6.

[4] Brodskii, G., Saleh, M., Thuyet, L. and Wisbauer, R., *On weak injectivity of direct sums of modules*, Vietnam J. Math. **26** (1998), 121–127.

[5] Camillo, V. P., *Modules whose quotients have finite Goldie dimension*, Pacific J. Math. **69** (1977), 337–338.

[6] Dhompong, S., Sanwong, J., Plubtieng, S. and Tansee, H., *On modules whose singular subgenerated modules are weakly injective*, Algebra Colloq. **8** (2001), 227–236.

[7] Dung, N. V., Huynh, D. V., Smith, P. and Wisbauer, R., *Extending modules*, Pitman, 1994.

[8] Golan, J. S. and López-Permouth, S. R., *QI-filters and tight modules*, Comm. Algebra **19** (1991), 2217–2229.

[9] Jain, S. K., López-Permouth, S. R., *Rings whose cyclics are essentially embeddable in projective modules*, J. Algebra **128** (1990), 257–269.

[10] Jain, S. K., López-Permouth, S. R., Oshiro, K. and Saleh, M., *Weakly projective and weakly injective modules*, Canad. J. Math. **34** (1994), 972–981.

[11] Jain, S. K., López-Permouth, S. R. and Singh, S., *On a class of QI-rings*, Glasgow J. Math. **34** (1992), 75–81.

[12] Jain, S. K. and López-Permouth, S. R., *A survey on the theory of weakly injective modules*, In: Computational Algebra, 205–233, Lecture notes in pure and applied mathematics, Marcel Dekker, Inc., New York, 1994.

[13] Kurshan, A. P., *Rings whose cyclic modules have finitely generated socle*, J. Algebra **14** (1970), 376–386.

[14] López-Permouth, S. R., *Rings characterized by their weakly injective modules*, Glasgow Math. J. **34** (1992), 349–353.

[15] Malik, S. and Vanaja, N., *Weak relative injective M-subgenerated modules*, Advances in Ring Theory, Birkhauser, 1997, 221–239.

[16] Page, S. and Zhou, Y., *When direct sums of singular injectives are injective*, In: Ring theory, Proceedings of the Ohio State-Denison Conference, World Scientific Publishing Co., 1993.

[17] Page, S. and Zhou, Y., *On direct sums of injective modules and chain conditions*, Canad. J. Math. **46** (1994), 634–647.

[18] Page, S. and Zhou, Y., *Direct sums of quasi-injective modules, injective hulls, and natural classes*, Comm. Alg. **22** (1994), 2911–2923.

[19] Saleh, M., *A note on tightness*, Glasgow Math. J. **41** (1999), 43–44.

[20] Saleh, M. and Abdel-Mohsen, A., *On weak injectivity and weak projectivity*, In: Proceedings of the Mathematics Conference, World Scientific Press, New Jersey, 2000, 196–207.

[21] Saleh, M. and Abdel-Mohsen, A., *A note on weak injectivity*, FJMS **11**(2) (2003), 199–206.

[22] Saleh, M., *On q.f.d. modules and q.f.d. rings*, Houston J. Math. **30** (2004), 629–636.

[23] Saleh, M., *On weakly projective and weakly injective modules*, Comment. Math. Univ. Corolin. **45** (2004), 389–402.

[24] Sanh, N. V., Shum, K. P., Dhompongsa, S. and Wongwai, S., *On quasi-principally injective modules*, Algebra Colloq. **6** (1999), 296–276.

[25] Sanh, N. V., Dhompongsa, S. and Wongwai, S., *On generalized q.f.d. modules and rings*, Algebra and Combinatorics (1999), 367–372.

[26] Wisbauer, R., *Foundations of module and ring theory*, Gordon and Breach, 1991.

[27] Zhou, Y., *Notes on weakly semisimple rings*, Bull. Austral. Math. Soc. **52** (1996), 517–525.

[28] Zhou, Y., *Weak injectivity and module classes*, Comm. Algebra **25** (1997), 2395–2407.

Department of Mathematics, Birzeit University
P.O.Box 14, West Bank, Palestine
E-mail: msaleh@birzeit.edu

Department of Mathematics, Ohio University
Athens, OH 45701, USA
E-mail: jain@math.ohiou.edu, slopez@math.ohiou.edu