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ON THE DEGENERATION OF HARMONIC SEQUENCES FROM SURFACES INTO COMPLEX GRASSMANN MANIFOLDS

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Abstract. Let $f : M \to G(m,n)$ be a harmonic map from surface into complex Grassmann manifold. In this paper, some sufficient conditions for the harmonic sequence generated by f to have degenerate ∂'-transform or ∂''-transform are given.

1. Introduction

Let $G(m,n)$ be the Grassmann manifold of all m-dimensional subspaces C^m in complex space C^n, M be a connected Riemannian surface. Given a harmonic map $f : M \to G(m,n)$, Chern-Wolfson obtain the following sequence of harmonic maps by using the ∂'-transforms and ∂''-transforms:

$$f = f_0 \xrightarrow{\partial'} f_1 \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} f_\alpha \xrightarrow{\partial'} \dots ,$$

$$f = f_0 \xrightarrow{\partial''} f_{-1} \xrightarrow{\partial''} \cdots \xrightarrow{\partial''} f_{-\alpha} \xrightarrow{\partial''} \dots , \tag{1.1}$$

(1.1) is called the harmonic sequence generated by $f = f_0$. It is important to ask when the harmonic sequence (1.1) includes degenerate ∂'-transform or ∂''-transform. If $m = 1$, then the degeneration of ∂'-transform or ∂''-transform is equivalent to the isotropy of f. We know that the harmonic sequence (1.1) must have degenerate ∂'-transform or ∂''-transform for an arbitrary harmonic map $f : M \to G(m,n)$ if one of the following conditions holds:

(i) $g = 0$, i.e., M is homeomorphic to the 2-sphere S^2 [2];
(ii) $g = 1$ and $\deg(f) \neq 0$ [2];
(iii) $m = 1, |\deg(f)| > (n-1)(g-1)$ [3, 4];
(iv) $m = 1$, $\mathrm{r}(\partial_0') + \mathrm{r}(\partial_0'') > 2n(g-1)$ [4, 5]; where g denotes the genus of M, $\deg(f)$ is the degree of the map and $\mathrm{r}(\partial_0')$ and $\mathrm{r}(\partial_0'')$ are the ramification indices of ∂_0' and ∂_0'' respectively.

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So far have we known above sufficient conditions to guarantee the existence of degenerate ∂'-transform or ∂''-transform, but when $m > 1$ and $g > 1$ or when M is non-compact, it seems that there aren't any results about it. The main purpose of the present paper is to find some sufficient conditions to ensure the existence of degenerate ∂'-transform or ∂''-transform. In order to do this, we establish the generalized Frenet formulae for harmonic maps and then use it to obtain some relative results.

2. Harmonic sequences

We equip C^n with the standard Hermitian inner product $\langle\, ,\, \rangle$, so that, for any two column vectors $X, Y \in C^n$, $\langle X, Y\rangle = Y^* X$, where Y^* denotes the conjugate and transpose of Y.

Let M be a connected Riemannian surface with Riemannian metric $ds^2_M = \varphi\overline{\varphi}$, where φ is a complex-valued 1-form defined up to a factor of norm one. The structure equations of M are

$$d\varphi = -\sqrt{-1}\rho \wedge \varphi\,, \qquad d\rho = -\frac{\sqrt{-1}}{2} K\varphi \wedge \overline{\varphi}\,, \tag{2.1}$$

where ρ is the real connection 1-form of M, and K the Gaussian curvature of M. Let $f : M \to G(m,n)$ be a harmonic map. Choose a local unitary frame $Z_1, \ldots, Z_n$ along f suitably such that $Z_1, \ldots, Z_m$ span f. We write

$$dZ_i = X_i\varphi + Y_i\overline{\varphi} \mod f\,, \tag{2.2}$$

where X_i, $Y_i \in f^\perp$, $i = 1, \ldots, m$. It follows from [1] that except at isolated points the ranks of $\mathrm{span}\{X_1, \ldots, X_m\}$ and $\mathrm{span}\{Y_1, \ldots, Y_m\}$ are constant, and they define two harmonic maps $f_1 = \partial' f : M \to G(m_1, n)$ and $f_{-1} : M \to G(m_{-1}, n)$, where m_1 and m_{-1} are the ranks of $\mathrm{span}\{X_1, \ldots, X_m\}$ and $\mathrm{span}\{Y_1, \ldots, Y_m\}$ respectively. Repeating in this way, we can get the harmonic sequence (1.1), where $f_{\pm\alpha} : M \to G(m_{\pm\alpha}, n)$. If $m_\alpha > m_{\alpha+1}$, then the ∂'-transform of f_α is called degenerate. Similarly, when $m_{-\alpha} > m_{-\alpha-1}$, then the ∂''-transform of $f_{-\alpha}$is called degenerate. In order to avoid confusion, sometimes we denote the ∂'-transform and ∂''-transform of f_k by ∂'_k and ∂''_k respectively, $k = 0, \pm 1, \ldots$ If $f_{-1} \perp f_1$ then $f = f_0$ is called strongly conformal [6]. If for any $\alpha > 0$, $f_\alpha \neq 0$, then the number

$$r = \max\{j : f_0 \perp f_i,\ \forall\, 1 \le i \le j\} \tag{2.3}$$

must be finite, and it is called the isotropy order of f [6]. It is known that when $\partial'_k \neq 0$ or $\partial''_k \neq 0$, then ∂'_k or ∂''_k has only isolated zeros, $k = 0, \pm 1, \ldots$ Hence, when M is compact, the number of zeros of ∂'_k or ∂''_k, counted according to multiplicity, is finite which is called the ramification index of ∂'_k or ∂''_k, and will be denoted by $\mathrm{r}(\partial'_k)$ or $\mathrm{r}(\partial''_k)$.

From [6] we know that there is a one-to-one correspondence between smooth map $f : M \to G(m,n)$ and the subbundle $\underline{f}$ of the trivial bundle $M \times C^n$ of rank m which has fiber at $x \in M$ given by $\underline{f_x} = f(x)$. Therefore, we can identify f with the Hermitian orthogonal projection from $M \times C^n$ onto $\underline{f}$. From this point of view, we have

Lemma 2.1 ([7]). *Let $f : M \to G(m,n) \subset U(n)$ be a smooth map. Then f is harmonic if and only if $d * A = 0$, where $A = \frac{1}{2}s^{-1}ds$, $s = f - f^{\perp}$.*

For the harmonic sequence (1.1), let $Z_1^{(k)}, \dots, Z_{m_k}^{(k)}$ be the local unitary frame for $\underline{f_k}, k = 0, \pm 1, \dots$ Then the Hermitian orthogonal projection f_k can be expressed by

$$f_k = W_k W_k^* \,, \tag{2.4}$$

where $W_k = (Z_1^{(k)}, \dots, Z_{m_k}^{(k)})$ is the $(n \times m_k)$-matrix.

3. The generalized Frenet formulae

Let $f = f_0 : M \to G(m,n)$ be a harmonic map which generates the harmonic sequence (1.1). The exterior derivative d has the decomposition $d = \partial + \overline{\partial}$. From [6] we see that $\partial'(\partial'' \underline{f_k}) \subset \underline{f_k}$ and $\partial''(\partial' \underline{f_k}) \subset \underline{f_k}, k = 0, \pm 1, \dots$ Hence, locally there exist $(m_k \times m_k)$-matrices B_k, D_k, $(m_{k+1} \times m_k)$-matrix A_k and $(m_{k-1} \times m_k)$-matrix C_k so that

$$\begin{aligned} \partial W_k &= (W_{k+1} A_k + W_k B_k)\varphi \,, \\ \overline{\partial} W_k &= (W_{k-1} C_k + W_k D_k)\overline{\varphi} \,. \end{aligned}$$

By the construction of the harmonic sequence (1.1), we have $\underline{f_k} \perp \underline{f_{k+1}}$ and so $W_k W_k^* = I_{m_k}$ and $W_k W_{k+1}^* = 0$. Operating $\overline{\partial}$ on them we obtain $B_k + D_k^* = 0$ and $A_k^* + C_{k+1} = 0$. Consequently we get the following generalized Frenet formulae:

$$\begin{aligned} \partial W_k &= (W_{k+1} A_k + W_k B_k)\varphi \,, \\ \overline{\partial} W_k &= -(W_{k-1} A_{k-1}^* + W_k B_k^*)\overline{\varphi} \,. \end{aligned} \tag{3.1}$$

Since rank$(\underline{f_k})$ is constant for each k except at isolated points, A_k is of full rank except at these isolated points. When f is an isometric immersion, we have [8]

$$|A_0|^2 + |A_{-1}|^2 = 1 \,, \quad \cos\alpha = |A_0|^2 - |A_{-1}|^2 \,, \tag{3.2}$$

here the norm $|Q|$ of a matrix Q is defined by $|Q|^2 = \mathrm{tr}(QQ^*)$ in a standard manner, and α is the Kaehler angle of f.

Lemma 3.1. *In the generalized Frenet formulae* (3.1)*, we have*

$$\begin{aligned} & dA_k + (A_k B_k^* - B_{k+1}^* A_k)\overline{\varphi} - \sqrt{-1}\rho A_k \equiv 0 \mod \varphi \,, \\ & d(B_k^* \overline{\varphi}) - d(B_k \varphi) = (A_k^* A_k - A_{k-1} A_{k-1}^* + B_k^* B_k - B_k B_k^*)\varphi \wedge \overline{\varphi} \,. \end{aligned} \tag{3.3}$$

Proof. Set $s_k = f_k - f_k^{\perp}$, $A^{(k)} = \frac{1}{2}s_k^{-1} ds_k$. Thus $A^{(k)}$ is one half of the pull-back of Maurer- Cartan form of $U(n)$ by f_k, and it satisfies

$$dA^{(k)} + 2A^{(k)} \wedge A^{(k)} = 0 \,. \tag{3.4}$$

From Lemma 2.1. we see that

$$d * A^{(k)} = 0 \,. \tag{3.5}$$

On the other hand, by virtue of (2.4), (3.1) and the definition of $A^{(k)}$ we get

$$
\begin{aligned}
A^{(k)} &= (- W_{k+1}A_kW_k^* - W_kA_{k-1}W_{k-1}^*)\varphi \\
&\quad + (W_kA_k^*W_{k+1}^* + W_{k-1}A_{k-1}^*W_k^*)\overline{\varphi}\,.
\end{aligned} \tag{3.6}
$$

Combining (3.4)–(3.6) one can obtain $(3.3)_1$. Substituting (3.1) into $W_k^*d^2W_k = 0$ yields $(3.3)_2$. □

Lemma 3.2. *If $m_k = m_{k+1}$, then at points where $\det A_k \neq 0$, we have*

$$
\Delta \log |\det A_k| = m_kK + 2(|A_{k-1}|^2 - 2|A_k|^2 + |A_{k+1}|^2)\,. \tag{3.7}
$$

Moreover, if M is a compact surface with genus g, then

$$
\mathrm{r}(\det A_k) = 2m_k(g-1) + \deg(f_k) - \deg(f_{k+1})\,, \tag{3.8}
$$

where $\mathrm{r}(\det A_k)$ denotes the number of zeros of $\det A_k$ counted according to multiplicity.

Proof. Note that $d\log(\det A_k) = \mathrm{tr}(A_k^{-1}dA_k)$, by $(3.3)_1$ we get

$$
d\log(\det A_k) + \mathrm{tr}(B_k^* - B_{k+1}^*)\overline{\varphi} - \sqrt{-1}m_k\rho \equiv 0 \mod \varphi\,. \tag{3.9}
$$

A standard computation together with $(3.3)_2$ and (3.9) yields (3.7). Integrating (3.7) on M and using Lemma 4.1 of [9], the Gauss-Bonnet theorem together with the definition of $\deg(f_k)$ and $\deg(f_{k+1})$ [2] we finally get (3.8). □

4. The main results

In this section we shall study the sufficient conditions to ensure the existence of degenerate ∂'-transform or ∂''-transform in harmonic sequence (1.1). First we have

Theorem 4.1. *Let M be a connected and complete Riemannian surface with non-negative Gaussian curvature K and $f : M \to G(m,n)$ be a harmonic isometric immersion. If there exists a positive number $\varepsilon > 0$ so that $|\cos\alpha| \geq \varepsilon$, where α is the Kaehler angle of the immersion f, then at least one of the ∂'-transforms or ∂''-transforms in harmonic sequence (1.1) generated by f is degenerate.*

Proof. Suppose that under the conditions of the theorem, none of the ∂'-transforms and ∂''-transforms in (1.1) generated by f is degenerate, that is to say, $m = m_0 = m_{\pm 1} = \dots$ Equivalently speaking, square matrices $A_0, A_{\pm 1}, \dots$ are all non-singular. Without loss of generality, we may assume that $\cos\alpha \geq \varepsilon > 0$. Then, for any positive integer p, a direct computation together with (3.2) and (3.7) yields (c.f. [8])

$$
\begin{aligned}
&\Delta \log \prod_{k=-1}^{-p} \prod_{j=-1}^{k} |\det A_j| \\
&\qquad = \frac{1}{2}mp(p+1)K + (2p+1)\cos\alpha - 1 + 2|A_{-p-1}|^2\,.
\end{aligned} \tag{4.1}
$$

Since $\cos\alpha \geq \varepsilon > 0$, we can choose p such that $(2p+1)\cos\alpha - 1 > 0$. Thus from (4.1) we conclude that

$$\Delta \log \prod_{k=-1}^{-p} \prod_{j=-1}^{k} |\det A_j| > 0\,, \tag{4.2}$$

from which it follows that the function

$$\prod_{k=-1}^{-p} \prod_{j=-1}^{k} |\det A_j|$$

is a subharmonic function on a complete surface M with non-negative Gaussian curvature, and it must be a constant. But this is in contradiction with (4.2). So the theorem is proved. □

Corollary 4.2. *Let $f : M \to CP^n$ be an isometric minimal immersion of a complete and connected surface M with non-negative Gaussian curvature into CP^n. If the Kaehler angle α of the immersion f satisfies $|\cos\alpha| \geq \varepsilon$, where ε is a positive number, then f must be isotropy.*

Proposition 4.3. *Let $f : M \to G(m,(p+1)m)$ be a harmonic map of a compact surface with genus g into $G(m,(p+1)m)$. If none of the ∂'-transforms and ∂''-transforms in harmonic sequence (1.1) generated by f is degenerate, and the isotropy order of f is p. Then $|\deg(f)| \leq mp(g-1)$.*

Proof. It is easy to see that under the assumption of the proposition, $\underline{f_0}, \underline{f_1}, \ldots, \underline{f_p}$ are mutually orthogonal, and that $\underline{f_k} = \underline{f_{k+p+1}}$, $k = 0, \pm 1, \ldots$ Therefore, from the definition of $\deg(\cdot)$ it is clear that

$$\sum_{k=0}^{p} \deg(f_k) = 0\,, \tag{4.3}$$

from which together with (3.8) yields

$$\begin{aligned}
&\sum_{k=0}^{p} \mathrm{r}(\det A_k) = 2m(g-1)(p+1)\,,\\
&\sum_{k=0}^{p} (p-k)\mathrm{r}(\det A_k) = (p+1)\deg(f_0) + mp(p+1)(g-1)\,,\\
&\sum_{k=0}^{p} k\mathrm{r}(\det A_k) = -(p+1)\deg(f_0) + mp\,(p-1)(g-1)\ .
\end{aligned} \tag{4.4}$$

From (4.4) we get

$$\begin{aligned}
|\deg(f)| = |\deg(f_0)| &= \frac{1}{2}\Big|\sum_{k=0}^{p}\Big(\frac{p-2k}{p+1}\Big)\mathrm{r}(\det A_k)\Big|\\
&\leq \frac{1}{2}\sum_{k=0}^{p}\Big(\Big|\frac{p-2k}{p+1}\Big|\mathrm{r}(\det A_k)\Big) \leq \frac{p}{2(p+1)}\sum_{k=0}^{p}\mathrm{r}(\det A_k) = mp(g-1)\,.
\end{aligned}$$

Thus the proposition is proved. $\square$

The following two theorems are the direct consequences of Proposition 4.3.

Theorem 4.4. *Let $f : M \to G(m,2m)$ be a harmonic map of a compact surface M with genus g into $G(m,2m)$. If $|\deg(f)| > m(g-1)$, then at least one of the ∂'-transforms or ∂''-transforms in* (1.1) *is degenerate.*

Theorem 4.5. *Let $f : M \to G(m,3m)$ be a strongly conformal harmonic map of a compact surface M with genus g into $G(m,3m)$. If $|\deg(f)| > 2m(g-1)$, then at least one of the ∂'-transforms or ∂''-transforms in* (1.1) *is degenerate.*

Remark. Theorem 4.5 generalizes Proposition 7.8 of [10].

Now let M be a compact surface with genus g and $f : M \to G(m,n)$ be a harmonic map which generates (1.1) with non-degenerate ∂'-transforms and ∂''-transforms. This implies that $m = m_0 = m_{\pm 1} = \ldots$ Suppose that the isotropy order of f is p so that

$$
\begin{aligned}
&W^*_{k+i+1} W_k = 0\,, \qquad 0 \le i \le p-1\,,\\
&W^*_{k+p+1} W_k \ne 0\,,
\end{aligned}
\tag{4.5}
$$

here $k = 0, \pm 1, \ldots$ Set $P_k = W^*_{k+p+1} W_k$, then from (3.1) and (4.5) we get

$$
A^*_{k+p} P_k = P_{k-1} A^*_{k-1}\,, \qquad \partial P_k = (P_k B_k - B_{k+p+1} P_k)\varphi\,.
\tag{4.6}
$$

It follows from (4.6) that rank(P_0)=rank($P_{\pm 1}$)=… except at isolated points. Thus we can choose the local frame $W_k, k = 0, 1, \ldots, 2p+1$ suitably such that

$$
P_k = \begin{pmatrix} Q_k & 0 \\ 0 & 0 \end{pmatrix}, \qquad k = 0, 1, \ldots, p\,,
\tag{4.7}
$$

where Q_k's are non-singular $(t \times t)$-matrices except at isolated points, and $t = \operatorname{rank}(P_0) = \operatorname{rank}(P_1) = \ldots$ Assume the corresponding blocks of the matrices A_k and B_k are

$$
A_k = \begin{pmatrix} A_{k11} & A_{k12} \\ A_{k21} & A_{k22} \end{pmatrix}, \qquad B_k = \begin{pmatrix} B_{k11} & B_{k12} \\ B_{k21} & B_{k22} \end{pmatrix}.
\tag{4.8}
$$

Combining (4.6)–(4.8) it follows that

$$
\begin{aligned}
&A_{k21} = 0\,, \qquad 0 \le k \le p-1\,,\\
&B_{k12} = 0\,, \qquad 0 \le k \le p\,,\\
&B_{k21} = 0\,, \qquad p+1 \le k \le 2p+1\,,
\end{aligned}
\tag{4.9}
$$

from which together with (3.3) and (4.6) yields

$$
\begin{aligned}
&dA_{k11} + (A_{k11} B^*_{k11} - B^*_{k+1,11} A_{k11})\overline{\varphi} - \sqrt{-1}\rho A_{k11} \equiv 0 \mod \varphi\,, \quad 0 \le k \le p\,,\\
&\partial Q_0 = (Q_0 B_{011} - B_{p+1,11} Q_0)\varphi\,.
\end{aligned}
\tag{4.10}
$$

From (4.10) we can calculate out that

$$
d \log\big(\det(A_{011} \ldots A_{p11} Q^*_0)\big) - \sqrt{-1}t(p+1)\rho \equiv 0 \mod \varphi\,,
\tag{4.11}
$$

and consequently,

$$\Delta \log|\det(A_{011}\dots A_{p11}Q_0^*)| = t(p+1)K\,. \tag{4.12}$$

Integrating (4.12) on M and making use of Gauss-Bonnet theorem, Lemma 4.1 of [9] and Noticing the fact that

$$\mathrm{r}(\partial'_k) = \mathrm{r}(|A_k|) \le \mathrm{r}(|A_{k11}|) \le \frac{1}{t}\mathrm{r}(\det A_{k11})\,, \tag{4.13}$$

we get

$$\sum_{k=0}^{p} \mathrm{r}(\partial'_k) \le 2(p+1)(g-1)\,. \tag{4.14}$$

Similarly we can prove

$$\sum_{k=-1}^{p-1} \mathrm{r}(\partial'_k) \le 2(p+1)(g-1)\,. \tag{4.15}$$

Note that $\mathrm{r}(\partial'_{-1})$=$\mathrm{r}(|A_{-1}|)$=$\mathrm{r}(\partial''_0)$, from (4.15) we get

$$\mathrm{r}(\partial'_0) + \mathrm{r}(\partial''_0) \le 2(p+1)(g-1) \le 2\frac{n}{m}(g-1)\,. \tag{4.16}$$

By (4.16) we can easily obtain the following theorem.

Theorem 4.6. *Let $f : M \to G(m,n)$ be a harmonic map of a compact surface M with genus g into $G(m,n)$ which generates the harmonic sequence* (1.1)*. If*

$$\mathrm{r}(\partial'_0) + \mathrm{r}(\partial''_0) > 2\frac{n}{m}(g-1)\,,$$

then at least one of the ∂'-transforms or ∂''-transforms in (1.1) *is degenerate.*

Remark. Theorem 4.6 generalizes the corresponding results in [4,5].

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