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GLOBAL GENERALIZED BIANCHI IDENTITIES FOR INVARIANT VARIATIONAL PROBLEMS ON GAUGE-NATURAL BUNDLES

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Abstract. We derive both *local* and *global* generalized *Bianchi identities* for classical Lagrangian field theories on gauge-natural bundles. We show that globally defined generalized Bianchi identities can be found without the *a priori* introduction of a connection. The proof is based on a *global* decomposition of the *variational Lie derivative* of the generalized Euler-Lagrange morphism and the representation of the corresponding generalized Jacobi morphism on gauge-natural bundles. In particular, we show that *within* a gauge-natural invariant Lagrangian variational principle, the gauge-natural lift of infinitesimal principal automorphism *is not* intrinsically arbitrary. As a consequence the existence of *canonical* global superpotentials for gauge-natural Noether conserved currents is proved without resorting to additional structures.

1. Introduction

Local generalized Bianchi identities for geometric field theories were introduced [3, 5, 6, 19, 35] to get (after an integration by parts procedure) a consistent equation between *local* divergences within the first variation formula. It was also stressed that in the general theory of relativity these identities coincide with the contracted Bianchi identities for the curvature tensor of the pseudo-Riemannian metric. We recall that in the classical Lagrangian formulation of field theories the description of symmetries amounts to define suitable (vector) densities which generate the conserved currents; in all relevant physical theories this densities are found to be the divergence of skew-symmetric (tensor) densities, which are called *superpotentials* for the conserved currents. It is also well known that the importance of superpotentials relies on the fact that they can be integrated to provide conserved quantities associated with the conserved currents *via* the Stokes Theorem (see e.g. [11] and references quoted therein).

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Subsequently, many attempts to "covariantize" such a derivation of Bianchi identities and superpotentials have been made (see e.g. [4, 7, 9, 11, 12, 25, 26] and the wide literature quoted therein) by resorting to background metrics or (fibered) connections used to perform covariant integration by parts to get covariant (variations of) currents and superpotentials. In particular, in [10] such a covariant derivation was implicitly assumed to hold true for any choice of gauge-natural prolongations of principal connections equal to prolongations of principal connections with respect to a linear symmetric connection on the basis manifold in the sense of [27, 28, 33].

In the present paper, we derive both *local* and *global* generalized Bianchi identities for classical field theories by resorting to the gauge-natural invariance of the Lagrangian and *via* the application of the *Noether Theorems* [39]. In particular we show that invariant generalized Bianchi identities can be found without the *a priori* introduction of a connection. The proof is based on a *global* decomposition of the *variational Lie derivative* of the generalized Euler-Lagrange morphism involving the definition – and its representation – of a new morphism, the generalized gauge-natural Jacobi morphisms. It is in fact known that the second variation of a Lagrangian can be formulated in terms of Lie derivative of the corresponding Euler-Lagrange morphism [13, 14, 17, 20, 40]. As a consequence the existence of canonical, i.e. *completely determined by the variational problem and its invariance properties*, global superpotentials for gauge-natural Noether conserved currents is proved without resorting to additional structures.

Our general framework is the calculus of variations on finite order jet of fibered bundles. Fibered bundles will be assumed to be *gauge-natural bundles* (i.e. jet prolongations of fibered bundles associated to some gauge-natural prolongation of a principal bundle $\boldsymbol{P}$ [8, 22, 23, 27, 28, 32]) and variations of sections are (vertical) vector fields given by Lie derivatives of sections with respect to gauge-natural lifts of infinitesimal principal automorphisms (see e.g. [8, 24, 32]).

In this general geometric framework we shall in particular consider *finite order variational sequences on gauge-natural bundles.* The variational sequence on finite order jet prolongations of fibered manifolds was introduced by Krupka as the quotient of the de Rham sequence of differential forms (defined on the prolongation of the fibered manifold) with respect to a natural exact contact subsequence, chosen in such a way that the generalized Euler-Lagrange and Helmholtz-Sonin mappings can be recognized as some of its quotient mappings [36, 37]. The representation of the quotient sheaves of the variational sequence as sheaves of sections of tensor bundles given in [46] and previous results on *variational Lie derivatives* and Noether Theorems [10, 15] will be used. Furthermore, we relate the generalized Bianchi morphism to the second variation of the Lagrangian. A very fundamental abstract result due to Kolář concerning *global* decomposition formulae of vertical morphisms, involved with the integration by parts procedure [21, 29, 30], will be a key tool. In order to apply this results, we stress linearity properties of the Lie derivative operator acting on sections of the gauge-natural bundle, which in turn rely on properties of the gauge-natural lift of infinitesimal principal automorphisms. The gauge-natural lift enables one to define the generalized gauge-natural Jacobi morphism (i.e. a generalized Jacobi morphism where the variation vector fields – instead of general deformations – are Lie derivatives of sections of the gauge-natural bundle with respect to gauge-natural lifts of infinitesimal automorphisms of the underlying principal bundle), the kernel of which plays a very fundamental role.

The paper is structured as follows. In Section 2 we state the geometric framework by defining the variational sequence on gauge natural-bundles and by representing the Lie derivative of fibered morphisms on its quotient sheaves; Section 3 is dedicated to the definition and the representation of the generalized gauge-natural Jacobi morphism associated with a generalized gauge-natural Lagrangian. We stress some linearity properties of this morphism as a consequence of the properties of the gauge-natural lift of infinitesimal right-invariant automorphisms of the underlying structure bundle. In Section 4, by resorting to the Second Noether Theorem, we relate the generalized Bianchi identities with the kernel of the gauge-natural Jacobi morphism. We prove that the generalized Bianchi identities hold true globally if and only if the vertical part of jet prolongations of gauge-natural lifts of infinitesimal principal bundle automorphisms is in the kernel of the second variation, i.e. of the generalized gauge-natural Jacobi morphism.

Here, manifolds and maps between manifolds are C^∞. All morphisms of fibered manifolds (and hence bundles) will be morphisms over the identity of the base manifold, unless otherwise specified.

2. Variational sequences on gauge-natural bundles

2.1. **Jets of fibered manifolds.** In this subsection we recall some basic facts about jet spaces. We introduce jet spaces of a fibered manifold and the sheaves of forms on the s-th order jet space. Moreover, we recall the notion of horizontal and vertical differential [32, 38, 42].

Our framework is a fibered manifold $\pi : \boldsymbol{Y} \to \boldsymbol{X}$, with $\dim \boldsymbol{X} = n$ and $\dim \boldsymbol{Y} = n + m$.

For $s \geq q \geq 0$ integers we are concerned with the s-jet space $J_s\boldsymbol{Y}$ of s-jet prolongations of (local) sections of π; in particular, we set $J_0\boldsymbol{Y} \equiv \boldsymbol{Y}$. We recall the natural fiberings $\pi^s_q : J_s\boldsymbol{Y} \to J_q\boldsymbol{Y}$, $s \geq q$, $\pi^s : J_s\boldsymbol{Y} \to \boldsymbol{X}$, and, among these, the *affine* fiberings π^s_{s-1}. We denote with $V\boldsymbol{Y}$ the vector subbundle of the tangent bundle $T\boldsymbol{Y}$ of vectors on $\boldsymbol{Y}$ which are vertical with respect to the fibering π.

Charts on $\boldsymbol{Y}$ adapted to π are denoted by (x^σ, y^i). Greek indices $\sigma, \mu, \dots$ run from 1 to n and they label basis coordinates, while Latin indices $i, j, \dots$ run from 1 to m and label fibre coordinates, unless otherwise specified. We denote by $(\partial_\sigma, \partial_i)$ and (d^σ, d^i) the local basis of vector fields and 1-forms on $\boldsymbol{Y}$ induced by an adapted chart, respectively. We denote multi-indices of dimension n by boldface Greek letters such as $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, with $0 \leq \alpha_\mu$, $\mu = 1, \dots, n$; by an abuse of notation, we denote with σ the multi-index such that $\alpha_\mu = 0$, if $\mu \neq \sigma$, $\alpha_\mu = 1$, if $\mu = \sigma$. We also set $|\boldsymbol{\alpha}|{:=}\alpha_1 + \cdots + \alpha_n$ and $\boldsymbol{\alpha}!{:=}\alpha_1! \dots \alpha_n!$. The charts induced on $J_s\boldsymbol{Y}$ are denoted by $(x^\sigma, y^i_{\boldsymbol{\alpha}})$, with $0 \leq |\boldsymbol{\alpha}| \leq s$; in particular, we set $y^i_{\boldsymbol{0}} \equiv y^i$. The local vector fields and forms of $J_s\boldsymbol{Y}$ induced by the above coordinates are denoted by $(\partial^{\boldsymbol{\alpha}}_i)$ and $(d^i_{\boldsymbol{\alpha}})$, respectively.

In the theory of variational sequences a fundamental role is played by the *contact maps* on jet spaces (see [36, 37, 38, 46]). Namely, for $s \geq 1$, we consider the natural complementary fibered morphisms over $J_s\boldsymbol{Y} \to J_{s-1}\boldsymbol{Y}$

$$\mathcal{D} : J_s\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X} \to TJ_{s-1}\boldsymbol{Y}\,, \qquad \vartheta : J_s\boldsymbol{Y} \underset{J_{s-1}\boldsymbol{Y}}{\times} TJ_{s-1}\boldsymbol{Y} \to VJ_{s-1}\boldsymbol{Y}\,,$$

with coordinate expressions, for $0 \leq |\boldsymbol{\alpha}| \leq s-1$, given by

$$\mathcal{D} \;\; = d^\lambda \otimes \mathcal{D}_\lambda = d^\lambda \otimes (\partial_\lambda + y^j_{\boldsymbol{\alpha}+\lambda}\partial^{\boldsymbol{\alpha}}_j)\,, \vartheta \;\; = \vartheta^j_{\boldsymbol{\alpha}} \otimes \partial^{\boldsymbol{\alpha}}_j = (d^j_{\boldsymbol{\alpha}} - y^j_{\boldsymbol{\alpha}+\lambda}d^\lambda) \otimes \partial^{\boldsymbol{\alpha}}_j\,.$$

The morphisms above induce the following natural splitting (and its dual):

$$J_s\boldsymbol{Y} \underset{J_{s-1}\boldsymbol{Y}}{\times} T^*J_{s-1}\boldsymbol{Y} = \left(J_s\boldsymbol{Y} \underset{J_{s-1}\boldsymbol{Y}}{\times} T^*\boldsymbol{X}\right) \oplus \mathcal{C}^*_{s-1}[\boldsymbol{Y}]\,, \tag{1}$$

where $\mathcal{C}^*_{s-1}[\boldsymbol{Y}] := \operatorname{im} \vartheta^*_s$ and $\vartheta^*_s : J_s\boldsymbol{Y} \underset{J_{s-1}\boldsymbol{Y}}{\times} V^*J_{s-1}\boldsymbol{Y} \to J_s\boldsymbol{Y} \underset{J_{s-1}\boldsymbol{Y}}{\times} T^*J_{s-1}\boldsymbol{Y}$. We have the isomorphism $\mathcal{C}^*_{s-1}[\boldsymbol{Y}] \simeq J_s\boldsymbol{Y} \underset{J_{s-1}\boldsymbol{Y}}{\times} V^*J_{s-1}\boldsymbol{Y}$. The role of the splitting above will be fundamental in the present paper.

If $f : J_s\boldsymbol{Y} \to \mathbb{R}$ is a function, then we set $D_\sigma f := \mathcal{D}_\sigma f$, $D_{\boldsymbol{\alpha}+\sigma} f := D_\sigma D_{\boldsymbol{\alpha}} f$, where D_σ is the standard *formal derivative*. Given a vector field $\Xi : J_s\boldsymbol{Y} \to TJ_s\boldsymbol{Y}$, the splitting (1) yields $\Xi \circ \pi^{s+1}_s = \Xi_H + \Xi_V$ where, if $\Xi = \Xi^\gamma\partial_\gamma + \Xi^i_{\boldsymbol{\alpha}}\partial^{\boldsymbol{\alpha}}_i$, then we have $\Xi_H = \Xi^\gamma D_\gamma$ and $\Xi_V = (\Xi^i_{\boldsymbol{\alpha}} - y^i_{\boldsymbol{\alpha}+\gamma}\Xi^\gamma)\partial^{\boldsymbol{\alpha}}_i$. We shall call Ξ_H and Ξ_V the horizontal and the vertical part of Ξ, respectively.

The splitting (1) induces also a decomposition of the exterior differential on $\boldsymbol{Y}$, $(\pi^s_{s-1})^* \circ d = d_H + d_V$, where d_H and d_V are defined to be the *horizontal* and *vertical differential*. The action of d_H and d_V on functions and 1-forms on $J_s\boldsymbol{Y}$ uniquely characterizes d_H and d_V (see, e.g., [42, 46] for more details). A *projectable vector field* on $\boldsymbol{Y}$ is defined to be a pair (Ξ, ξ), where $\Xi : \boldsymbol{Y} \to T\boldsymbol{Y}$ and $\xi : \boldsymbol{X} \to T\boldsymbol{X}$ are vector fields and Ξ is a fibered morphism over ξ. If there is no danger of confusion, we will denote simply by Ξ a projectable vector field (Ξ, ξ). A projectable vector field (Ξ, ξ), with coordinate expression $\Xi = \xi^\sigma\partial_\sigma + \xi^i\partial_i$, $\xi = \xi^\sigma\partial_\sigma$, can be conveniently prolonged to a projectable vector field $(j_s\Xi, \xi)$, whose coordinate expression turns out to be

$$j_s\Xi = \xi^\sigma\partial_\sigma + (D_{\boldsymbol{\alpha}}\xi^i - \sum_{\boldsymbol{\beta}+\boldsymbol{\gamma}=\boldsymbol{\alpha}} \frac{\boldsymbol{\alpha}!}{\boldsymbol{\beta}!\boldsymbol{\gamma}!} D_{\boldsymbol{\beta}}\xi^\mu\, y^i_{\boldsymbol{\gamma}+\mu})\, \partial^{\boldsymbol{\alpha}}_i\,,$$

where $\boldsymbol{\beta} \neq 0$ and $0 \leq |\boldsymbol{\alpha}| \leq s$ (see e.g. [36, 38, 42, 46]); in particular, we have the following expressions $(j_s\Xi)_H = \xi^\sigma D_\sigma$, $(j_s\Xi)_V = D_{\boldsymbol{\alpha}}(\Xi_V)^i\,\partial^{\boldsymbol{\alpha}}_i$, with $(\Xi_V)^i = \xi^i - y^i_\sigma\xi^\sigma$, for the horizontal and the vertical part of $j_s\Xi$, respectively. From now on, by an abuse of notation, we will write simply $j_s\Xi_H$ and $j_s\Xi_V$. In particular, $j_s\Xi_V : J_{s+1}\boldsymbol{Y} \underset{J_s\boldsymbol{Y}}{\times} J_s\boldsymbol{Y} \to J_{s+1}\boldsymbol{Y} \underset{J_s\boldsymbol{Y}}{\times} J_sV\boldsymbol{Y}$.

We are interested in the case in which physical fields are assumed to be sections of a fibered bundle and the variations of sections are generated by suitable vector fields. More precisely, fibered bundles will be assumed to be *gauge-natural bundles* and variations of sections are (vertical) vector fields given by Lie derivatives of sections with respect to gauge-natural lifts of infinitesimal principal automorphisms.

Such geometric structures have been widely recognized to suitably describe so-called gauge-natural field theories, i.e. physical theories in which right-invariant infinitesimal automorphisms of the structure bundle $\boldsymbol{P}$ uniquely define the transformation laws of the fields themselves (see e.g. [8, 32]). In the following, we shall develop a suitable geometrical setting which enables us to define and investigate the fundamental concept of conserved quantity in gauge-natural Lagrangian field theories.

2.2. **Gauge-natural prolongations.** First we shall recall some basic definitions and properties concerning gauge-natural prolongations of (structure) principal bundles (for an extensive exposition see e.g. [32] and references therein; in the interesting paper [23] fundamental reduction theorems for general linear connections on vector bundles are provided in the gauge-natural framework).

Let $\boldsymbol{P} \to \boldsymbol{X}$ be a principal bundle with structure group $\boldsymbol{G}$. Let $r \leq k$ be integers and $\boldsymbol{W}^{(r,k)}\boldsymbol{P} := J_r\boldsymbol{P} \underset{\boldsymbol{X}}{\times} L_k(\boldsymbol{X})$, where $L_k(\boldsymbol{X})$ is the bundle of k-frames in $\boldsymbol{X}$ [8, 22, 32], $\boldsymbol{W}^{(r,k)}\boldsymbol{G}:=J_r\boldsymbol{G} \odot GL_k(n)$ the semidirect product with respect to the action of $GL_k(n)$ on $J_r\boldsymbol{G}$ given by the jet composition and $GL_k(n)$ is the group of k-frames in $\mathbb{R}^n$. Here we denote by $J_r\boldsymbol{G}$ the space of (r,n)-velocities on $\boldsymbol{G}$.

Elements of $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$ are given by $(j^{\boldsymbol{x}}_r\gamma, j^0_k t)$, with $\gamma : \boldsymbol{X} \to \boldsymbol{P}$ a local section, $t : \mathbb{R}^n \to \boldsymbol{X}$ locally invertible at zero, with $t(0) = \boldsymbol{x}$, $\boldsymbol{x} \in \boldsymbol{X}$. Elements of $\boldsymbol{W}^{(r,k)}\boldsymbol{G}$ are $(j^0_r g, j^0_k\alpha)$, where $g : \mathbb{R}^n \to \boldsymbol{G}$, $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ locally invertible at zero, with $\alpha(0) = 0$.

Remark 1. The bundle $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$ is a principal bundle over $\boldsymbol{X}$ with structure group $\boldsymbol{W}^{(r,k)}\boldsymbol{G}$. The right action of $\boldsymbol{W}^{(r,k)}\boldsymbol{G}$ on the fibers of $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$ is defined by the composition of jets (see, e.g., [22, 32]).

Definition 1. The principal bundle $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$ (resp. the Lie group $\boldsymbol{W}^{(r,k)}\boldsymbol{G}$) is said to be the *gauge-natural prolongation of order* (r,k) *of* $\boldsymbol{P}$ (*resp. of* $\boldsymbol{G}$).

Remark 2. Let (Φ, ϕ) be a principal automorphism of $\boldsymbol{P}$ [32]. It can be prolonged in a natural way to a principal automorphism of $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$, defined by:

$$\boldsymbol{W}^{(r,k)}(\Phi,\phi) : (j^{\boldsymbol{x}}_r\gamma, j^0_k t) \mapsto (j^{\phi(\boldsymbol{x})}_r(\Phi \circ \gamma \circ \phi^{-1}), j^0_k(\phi \circ t)) \,.$$

The induced automorphism $\boldsymbol{W}^{(r,k)}(\Phi,\phi)$ is an equivariant automorphism of $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$ with respect to the action of the structure group $\boldsymbol{W}^{(r,k)}\boldsymbol{G}$. We shall simply denote it by the same symbol Φ, if there is no danger of confusion.

Definition 2. We define the *vector* bundle over $\boldsymbol{X}$ of right-invariant infinitesimal automorphisms of $\boldsymbol{P}$ by setting $\mathcal{A} = T\boldsymbol{P}/\boldsymbol{G}$.

We also define the *vector* bundle over $\boldsymbol{X}$ of right invariant infinitesimal automorphisms of $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$ by setting $\mathcal{A}^{(r,k)}:=T\boldsymbol{W}^{(r,k)}\boldsymbol{P}/\boldsymbol{W}^{(r,k)}\boldsymbol{G}$ $(r \leq k)$.

Remark 3. We have the following projections $\mathcal{A}^{(r,k)} \to \mathcal{A}^{(r',k')}$, $r \leq k$, $r' \leq k'$, with $r \geq r'$, $s \geq s'$.

2.3. **Gauge-natural bundles and lifts.** Let $\boldsymbol{F}$ be any manifold and $\zeta : \boldsymbol{W}^{(r,k)}\boldsymbol{G}\times \boldsymbol{F} \to \boldsymbol{F}$ be a left action of $\boldsymbol{W}^{(r,k)}\boldsymbol{G}$ on $\boldsymbol{F}$. There is a naturally defined right action of $\boldsymbol{W}^{(r,k)}\boldsymbol{G}$ on $\boldsymbol{W}^{(r,k)}\boldsymbol{P} \times \boldsymbol{F}$ so that we can associate in a standard way to $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$ the bundle, on the given basis $\boldsymbol{X}$, $\boldsymbol{Y}_\zeta := \boldsymbol{W}^{(r,k)}\boldsymbol{P} \times_\zeta \boldsymbol{F}$.

Definition 3. We say $(\boldsymbol{Y}_\zeta, \boldsymbol{X}, \pi_\zeta; \boldsymbol{F}, \boldsymbol{G})$ to be the *gauge-natural bundle* of order (r, k) associated to the principal bundle $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$ by means of the left action ζ of the group $\boldsymbol{W}^{(r,k)}\boldsymbol{G}$ on the manifold $\boldsymbol{F}$ [8, 32].

Remark 4. A principal automorphism Φ of $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$ induces an automorphism of the gauge-natural bundle by:

$$\Phi_\zeta : \boldsymbol{Y}_\zeta \to \boldsymbol{Y}_\zeta : [(j^x_r\gamma, j^0_k t), \hat{f}]_\zeta \mapsto [\Phi(j^x_r\gamma, j^0_k t), \hat{f}]_\zeta \,, \tag{2}$$

where $\hat{f} \in \boldsymbol{F}$ and $[\cdot, \cdot]_\zeta$ is the equivalence class induced by the action ζ.

Denote by $\mathcal{T}_{\boldsymbol{X}}$ and $\mathcal{A}^{(r,k)}$ the sheaf of vector fields on $\boldsymbol{X}$ and the sheaf of right invariant vector fields on $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$, respectively. A functorial mapping $\mathfrak{G}$ is defined which lifts any right-invariant local automorphism (Φ, ϕ) of the principal bundle $W^{(r,k)}\boldsymbol{P}$ into a unique local automorphism (Φ_ζ, ϕ) of the associated bundle $\boldsymbol{Y}_\zeta$. Its infinitesimal version associates to any $\bar{\Xi} \in \mathcal{A}^{(r,k)}$, projectable over $\xi \in \mathcal{T}_{\boldsymbol{X}}$, a unique *projectable* vector field $\hat{\Xi} := \mathfrak{G}(\bar{\Xi})$ on $\boldsymbol{Y}_\zeta$ in the following way:

$$\mathfrak{G} : \boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} \mathcal{A}^{(r,k)} \to T\boldsymbol{Y}_\zeta \; : (\boldsymbol{y}, \bar{\Xi}) \mapsto \hat{\Xi}(\boldsymbol{y}) \,, \tag{3}$$

where, for any $\boldsymbol{y} \in \boldsymbol{Y}_\zeta$, one sets: $\hat{\Xi}(\boldsymbol{y}) = \frac{d}{dt}[(\Phi_{\zeta\, t})(\boldsymbol{y})]_{t=0}$, and $\Phi_{\zeta\, t}$ denotes the (local) flow corresponding to the gauge-natural lift of Φ_t.

This mapping fulfils the following properties:

(1) $\mathfrak{G}$ is linear over $\mathrm{id}_{\boldsymbol{Y}_\zeta}$;
(2) we have $T\pi_\zeta \circ \mathfrak{G} = \mathrm{id}_{T\boldsymbol{X}} \circ \bar{\pi}^{(r,k)}$, where $\bar{\pi}^{(r,k)}$ is the natural projection $\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} \mathcal{A}^{(r,k)} \to T\boldsymbol{X}$;
(3) for any pair $(\bar{\Lambda}, \bar{\Xi})$ of vector fields in $\mathcal{A}^{(r,k)}$, we have

$$\mathfrak{G}([\bar{\Lambda}, \bar{\Xi}]) = [\mathfrak{G}(\bar{\Lambda}), \mathfrak{G}(\bar{\Xi})] \,;$$

(4) we have the coordinate expression of $\mathfrak{G}$

$$\mathfrak{G} = d^\mu \otimes \partial_\mu + d^A_{\boldsymbol{\nu}} \otimes (\mathcal{Z}^{i\boldsymbol{\nu}}_A \partial_i) + d^\nu_{\boldsymbol{\lambda}} \otimes (\mathcal{Z}^{i\boldsymbol{\lambda}}_\nu \partial_i) \,, \tag{4}$$

with $0 < |\boldsymbol{\nu}| < k$, $1 < |\boldsymbol{\lambda}| < r$ and $\mathcal{Z}^{i\boldsymbol{\nu}}_A$, $\mathcal{Z}^{i\boldsymbol{\lambda}}_\nu \in C^\infty(\boldsymbol{Y}_\zeta)$ are suitable functions which depend on the bundle, precisely on the fibers (see [32]).

Definition 4. The map $\mathfrak{G}$ is called the *gauge-natural lifting functor*. The projectable vector field $(\hat{\Xi}, \xi) \equiv \mathfrak{G}((\bar{\Xi}, \xi))$ is called the *gauge-natural lift* of $(\bar{\Xi}, \xi)$ to the bundle $\boldsymbol{Y}_\zeta$.

2.4. **Lie derivative of sections of gauge-natural bundles.** Let γ be a (local) section of $\boldsymbol{Y}_\zeta$, $\bar{\Xi} \in \mathcal{A}^{(r,k)}$ and $\hat{\Xi}$ its gauge-natural lift. Following [32] we define a (local) section $\pounds_{\bar{\Xi}}\gamma : \boldsymbol{X} \to V\boldsymbol{Y}_\zeta$, by setting: $\pounds_{\bar{\Xi}}\gamma = T\gamma \circ \xi - \hat{\Xi} \circ \gamma$.

Definition 5. The (local) section $\pounds_{\bar{\Xi}}\gamma$ is called the *generalized Lie derivative* of γ along the vector field $\hat{\Xi}$.

Remark 5. This section is a vertical prolongation of γ, i.e. it satisfies the property: $\nu_{\boldsymbol{Y}_\zeta} \circ \pounds_{\Xi}\gamma = \gamma$, where $\nu_{\boldsymbol{Y}_\zeta}$ is the projection $\nu_{\boldsymbol{Y}_\zeta} : V\boldsymbol{Y}_\zeta \to \boldsymbol{Y}_\zeta$. Its coordinate expression is given by $(\pounds_{\bar{\Xi}}\gamma)^i = \xi^\sigma \partial_\sigma \gamma^i - \hat{\Xi}^i(\gamma)$.

Remark 6. The Lie derivative operator acting on sections of gauge-natural bundles satisfies the following properties:

(1) for any vector field $\bar{\Xi} \in \mathcal{A}^{(r,k)}$, the mapping $\gamma \mapsto \pounds_{\bar{\Xi}}\gamma$ is a first-order quasilinear differential operator;

(2) for any local section γ of $\boldsymbol{Y}_\zeta$, the mapping $\bar{\Xi} \mapsto \pounds_{\bar{\Xi}}\gamma$ is a linear differential operator;

(3) by using the canonical isomorphism $VJ_s\boldsymbol{Y}_\zeta \simeq J_sV\boldsymbol{Y}_\zeta$, we have $\pounds_{\bar{\Xi}}[j_r\gamma] = j_s[\pounds_{\bar{\Xi}}\gamma]$, for any (local) section γ of $\boldsymbol{Y}_\zeta$ and for any (local) vector field $\bar{\Xi} \in \mathcal{A}^{(r,k)}$.

We can regard $\pounds_{\bar{\Xi}} : J_1\boldsymbol{Y}_\zeta \to V\boldsymbol{Y}_\zeta$ as a morphism over the basis $\boldsymbol{X}$. In this case it is meaningful to consider the (standard) jet prolongation of $\pounds_{\bar{\Xi}}$, denoted by $j_s\pounds_{\bar{\Xi}} : J_{s+1}\boldsymbol{Y}_\zeta \to VJ_s\boldsymbol{Y}_\zeta$. Furthermore, we have $j_s\hat{\Xi}_V(\gamma) = -\pounds_{j_s\bar{\Xi}}\gamma$.

(4) we can consider $\pounds$ as a bundle morphism:

$$\pounds : J_{s+1}(\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} \mathcal{A}^{(r,k)}) \to J_{s+1}\boldsymbol{Y}_\zeta \underset{J_s\boldsymbol{Y}_\zeta}{\times} VJ_s\boldsymbol{Y}_\zeta \,. \tag{5}$$

2.5. **Variational sequences.** For the sake of simplifying notation, sometimes, we will omit the subscript ζ, so that all our considerations shall refer to $\boldsymbol{Y}$ as a gauge-natural bundle as defined above.

We shall be here concerned with some distinguished sheaves of forms on jet spaces [36, 37, 42, 46]. Due to the topological triviality of the fibre of $J_s\boldsymbol{Y} \to \boldsymbol{Y}$, we will consider sheaves on $J_s\boldsymbol{Y}$ with respect to the topology generated by open sets of the kind $(\pi^s_0)^{-1}(\boldsymbol{U})$, with $\boldsymbol{U} \subset \boldsymbol{Y}$ open in $\boldsymbol{Y}$.

i. For $s \geq 0$, we consider the standard sheaves Λ^p_s of p-forms on $J_s\boldsymbol{Y}$.

ii. For $0 \leq q \leq s$, we consider the sheaves $\mathcal{H}^p_{(s,q)}$ and $\mathcal{H}^p_s$ of *horizontal forms*, i.e. of local *fibered morphisms* (following the well known correspondence between forms and fibered morphisms over the basis manifold, see e.g. [32]) over π^s_q and π^s of the type $\alpha : J_s\boldsymbol{Y} \to \overset{p}{\wedge} T^*J_q\boldsymbol{Y}$ and $\beta : J_s\boldsymbol{Y} \to \overset{p}{\wedge} T^*\boldsymbol{X}$, respectively.

iii. For $0 \leq q < s$, we consider the subsheaf $\mathcal{C}^p_{(s,q)} \subset \mathcal{H}^p_{(s,q)}$ of *contact forms*, i.e. of sections $\alpha \in \mathcal{H}^p_{(s,q)}$ with values into $\overset{p}{\wedge}(\mathcal{C}^*_q[\boldsymbol{Y}])$. We have the distinguished subsheaf $\mathcal{C}^p{}_s \subset \mathcal{C}^p_{(s+1,s)}$ of local fibered morphisms $\alpha \in \mathcal{C}^p_{(s+1,s)}$ such that $\alpha = \overset{p}{\wedge}\vartheta^*_{s+1} \circ \tilde{\alpha}$, where $\tilde{\alpha}$ is a section of the fibration $J_{s+1}\boldsymbol{Y} \underset{J_s\boldsymbol{Y}}{\times} \overset{p}{\wedge} V^*J_s\boldsymbol{Y} \to J_{s+1}\boldsymbol{Y}$ which projects down onto $J_s\boldsymbol{Y}$.

Remark 7. *Notice* that according to [36, 37, 46], the fibered splitting (1) yields the *sheaf splitting* $\mathcal{H}^p_{(s+1,s)} = \bigoplus_{t=0}^p \mathcal{C}^{p-t}_{(s+1,s)} \wedge \mathcal{H}^t_{s+1}$, which restricts to the inclusion $\Lambda^p_s \subset \bigoplus_{t=0}^p \mathcal{C}^{p-t}{}_s \wedge \mathcal{H}^{t,h}{}_{s+1}$, where $\mathcal{H}^{p,h}{}_{s+1} := h(\Lambda^p_s)$ for $0 < p \leq n$ and the surjective map h is defined to be the restriction to Λ^p_s of the projection of the above splitting onto the non-trivial summand with the highest value of t.

The induced sheaf splitting above plays here a fundamental role. We stress again that, for any jet order s, it is induced by the natural contact structure on the affine bundle π^{s+1}_s. Since a variational problem (described by the corresponding action integral) is insensitive to the addition of any piece containing contact factors, such an affine structure has been pointed out in [36] to be fundamental for the description of the geometric structure of the Calculus of Variations on finite order jets of fibered manifolds. This property reflects on the intrinsic structure of all objects defined and represented in the *variational sequence* of a given order which we are just going to introduce. In particular this holds true for the generalized Jacobi morphism we will define and represent in Section 3.

We shortly recall now the theory of variational sequences on finite order jet spaces, as it was developed by D. Krupka in [36].

By an abuse of notation, let us denote by $d \ker h$ the sheaf generated by the presheaf $d \ker h$ in the standard way. We set $\Theta^*_s := \ker h + d \ker h$.

In [36] it was proved that the following sequence is an exact resolution of the constant sheaf $\mathbb{R}_{\boldsymbol{Y}}$ over $\boldsymbol{Y}$:

$$0 \longrightarrow \mathbb{R}_Y \longrightarrow \Lambda^0_s \xrightarrow{\mathcal{E}_0} \Lambda^1_s/\Theta^1_s \xrightarrow{\mathcal{E}_1} \Lambda^2_s/\Theta^2_s \xrightarrow{\mathcal{E}_2} \dots \xrightarrow{\mathcal{E}_{I-1}} \Lambda^I_s/\Theta^I_s \xrightarrow{\mathcal{E}_I} \Lambda^{I+1}_s \xrightarrow{d} 0$$

Definition 6. The above sequence, where the highest integer I depends on the dimension of the fibers of $J_s\boldsymbol{Y} \to \boldsymbol{X}$ (see, in particular, [36]), is said to be the s-th order *variational sequence* associated with the fibered manifold $\boldsymbol{Y} \to \boldsymbol{X}$.

For practical purposes, specifically to deal with morphisms which have a well known interpretation within the Calculus of Variations, we shall limit ourselves to consider the truncated variational sequence:

$$0 \longrightarrow \mathbb{R}_Y \longrightarrow \mathcal{V}^0_s \xrightarrow{\mathcal{E}_0} \mathcal{V}^1_s \xrightarrow{\mathcal{E}_1} \dots \xrightarrow{\mathcal{E}_n} \mathcal{V}^{n+1}_s \xrightarrow{\mathcal{E}_{n+1}} \mathcal{E}_{n+1}(\mathcal{V}^{n+1}_s) \xrightarrow{\mathcal{E}_{n+2}} 0\,,$$

where, following [46], the sheaves $\mathcal{V}^p_s := \mathcal{C}^{p-n}_s \wedge \mathcal{H}^{n,h}{}_{s+1}/h(d \ker h)$ with $0 \leq p \leq n+2$ are suitable representations of the corresponding quotient sheaves in the variational sequence by means of sheaves of sections of vector bundles. We notice that in the following, to avoid confusion, sometimes (when the interpretation could be dubious) we shall denote with a subscript the relevant fibered bundle on which the variational sequence is defined; e.g. in the case above, we would write $(\mathcal{V}^p_s)_{\boldsymbol{Y}}$.

Let $\alpha \in \mathcal{C}^1_s \wedge \mathcal{H}^{n,h}{}_{s+1} \subset \mathcal{V}^{n+1}_{s+1}$. Then there is a unique pair of sheaf morphisms ([29, 34, 46])

$$(6) \qquad E_\alpha \in \mathcal{C}^1_{(2s,0)} \wedge \mathcal{H}^{n,h}{}_{2s+1}\,, \qquad F_\alpha \in \mathcal{C}^1_{(2s,s)} \wedge \mathcal{H}^{n,h}{}_{2s+1}\,,$$

such that $(\pi^{2s+1}_{s+1})^*\alpha = E_\alpha - F_\alpha$, and F_α is *locally* of the form $F_\alpha = d_H p_\alpha$, with $p_\alpha \in \mathcal{C}^1_{(2s-1,s-1)} \wedge \mathcal{H}^{n-1}{}_{2s}$.

Definition 7. Let $\gamma \in \Lambda^{n+1}_s$. The morphism $E_{h(\gamma)} \in \mathcal{V}^{n+1}_s$ is called the *generalized Euler-Lagrange morphism* associated with γ and the operator $\mathcal{E}_n$ is called the *generalized Euler-Lagrange operator*. Furthermore $p_{h(\gamma)}$ is a generalized momentum associated with $E_{h(\gamma)}$.

Let $\eta \in \mathcal{C}^1_s \wedge \mathcal{C}^1_{(s,0)} \wedge \mathcal{H}^{n,h}{}_{s+1} \subset \mathcal{V}^{n+2}_{s+1}$, then there is a unique morphism

$$K_\eta \in \mathcal{C}^1_{(2s,s)} \otimes \mathcal{C}^1_{(2s,0)} \wedge \mathcal{H}^{n,h}{}_{2s+1}$$

such that, for all $\Xi : \boldsymbol{Y} \to V\boldsymbol{Y}$, $E_{j_s\Xi \rfloor \eta} = C^1_1(j_{2s}\Xi \otimes K_\eta)$, where C^1_1 stands for tensor contraction on the first factor and $\rfloor$ denotes inner product (see [34, 46]). Furthermore, there is a unique pair of sheaf morphisms

$$H_\eta \in \mathcal{C}^1_{(2s,s)} \wedge \mathcal{C}^1_{(2s,0)} \wedge \mathcal{H}^{n,h}{}_{2s+1}\,, \quad G_\eta \in \mathcal{C}^2_{(2s,s)} \wedge \mathcal{H}^{n,h}{}_{2s+1}\,, \tag{7}$$

such that $(\pi^{2s+1}_{s+1})^* \eta = H_\eta - G_\eta$ and $H_\eta = \frac{1}{2} A(K_\eta)$, where A stands for antisymmetrisation. Moreover, G_η is *locally* of the type $G_\eta = d_H q_\eta$, where $q_\eta \in \mathcal{C}^2_{(2s-1,s-1)} \wedge \mathcal{H}^{n-1}{}_{2s}$, hence $[\eta] = [H_\eta]$ [34, 46].

Definition 8. Let $\gamma \in \Lambda^{n+1}_s$. The morphism $H_{hd\gamma} \equiv H_{[\mathcal{E}_{n+1}(\gamma)]}$, where square brackets denote equivalence class, is called the *generalized Helmholtz morphism* and the operator $\mathcal{E}_{n+1}$ is called the *generalized Helmholtz operator*. Furthermore $q_{hd\gamma} \equiv q_{[\mathcal{E}_{n+1}(\gamma)]}$ is a generalized momentum associated with the Helmholtz morphism.

Remark 8. A section $\lambda \in \mathcal{V}^n_s$ is just a Lagrangian of order $(s+1)$ of the standard literature. Furthermore, $\mathcal{E}_n(\lambda) \in \mathcal{V}^{n+1}_s$ coincides with the standard higher order Euler-Lagrange morphism associated with λ.

Remark 9. It is well known that it is always possible to find global morphisms $p_{h(\gamma)}$ and $q_{hd\gamma}$ satisfying decomposition formulae above; however, this possibility depends in general on the choice of a linear symmetric connection on the basis manifold (see [1, 2, 31]). In the present paper, we shall avoid to perform such a choice *a priori*, with the explicit intention of performing an invariant derivation of generalized Bianchi identities, which does not relay on an invariant decomposition involving (local) divergences; that is in fact possible when resorting to the representation of the Second Noether Theorem in the variational sequence, as shown by Theorem 3 below. Indeed, contact forms and horizontal differentials of contact forms of higher degree are factored out in the quotient sheaves of the variational sequence. It is also clear that this will give, at least, prescriptions on the meaningful (within a fully gauge-natural invariant variational problem) possible choices of connections to be used to derive covariantly generalized Bianchi identities in the classical way.

2.6. **Variational Lie derivative.** In this subsection, following essentially [15], we give a representation in the variational sequence of the standard Lie derivative operator acting on fibered morphisms. We consider a projectable vector field (Ξ, ξ) on $\boldsymbol{Y}$ and take into account the Lie derivative operator $L_{j_s\Xi}$ with respect to the jet prolongation $j_s\Xi$ of Ξ. In fact, as well known, such a prolonged vector field preserves the fiberings π^s_q, π^s; hence it preserves the splitting (1). Thus we have

$$\mathcal{L}_{j_s\Xi} : \mathcal{V}^p_s \to \mathcal{V}^p_s : [\alpha] \mapsto \mathcal{L}_{j_s\Xi}([\alpha]) = [L_{j_s\Xi}\alpha] \,.$$

Definition 9. Let (Ξ, ξ) be a projectable vector field. We call the map $\mathcal{L}_{j_s\Xi}$ defined above the *variational Lie derivative*.

Variational Lie derivatives allow us to calculate infinitesimal symmetries of forms in the variational sequence. In particular, we are interested in symmetries of generalized Lagrangians and Euler-Lagrange morphisms which will enable us to represent in this framework Noether Theorems as well as known results stated in the framework of geometric bundles (see e.g. the fundamental papers by Trautman [43, 44, 45]).

Remark 10. Let $s \leq q$. Then the inclusions $\Lambda^p_s \subset \Lambda^p_q$ and $\Theta^p_s \subset \Theta^p_q$ yield the injective sheaf morphisms (see [36]) $\chi^q_s : (\Lambda^p_s/\Theta^p_s) \to \left(\Lambda^p_q/\Theta^p_q\right) : [\alpha] \mapsto [(\pi^q_s)^*\alpha]$, hence the inclusions $\kappa^q_s : \mathcal{V}^p_s \to \mathcal{V}^p_q$ for $s \leq q$.

The inclusions κ^q_s of the variational sequence of order s in the variational sequence of order q give rise to new representations of $\mathcal{L}_{j_s\Xi}$ on $\mathcal{V}^p_q$. In particular, the following two results hold true [15].

Theorem 1. *Let $[\alpha] = h(\alpha) \in \mathcal{V}^n_s$. Then we have* locally

$$\kappa^{2s+1}_s \circ \mathcal{L}_{j_s\Xi}(h(\alpha)) = \Xi_V \rfloor \mathcal{E}_n(h(\alpha)) + d_H(j_{2s}\Xi_V \rfloor p_{d_V h(\alpha)} + \xi \rfloor h(\alpha)) \,.$$

Proof. We have

$$\begin{aligned}\kappa^{2s+1}_s \circ \mathcal{L}_{j_s\Xi}(h(\alpha)) &= h(L_{j_{s+1}\Xi}h(\alpha))) \\ &= d_H(j_{s+1}\Xi_H \rfloor h(\alpha)) + h(j_{s+2}\Xi_V \rfloor d_V h(\alpha)) \\ &= d_H(\xi \rfloor h(\alpha)) + h(j_{2s+1}\Xi_V \rfloor (E_{d_V h(\alpha)} + F_{d_V h(\alpha)})) \,.\end{aligned}$$

Since $F_{d_V h(\alpha)} = d_H p_{d_V h(\alpha)}$ *locally*, then

$$\kappa^{2s+1}_s \circ \mathcal{L}_{j_s\Xi}(h(\alpha)) = \Xi_V \rfloor \mathcal{E}_n(h(\alpha)) + d_H(j_{2s}\Xi_V \rfloor p_{d_V h(\alpha)} + \xi \rfloor h(\alpha)) \,.$$

□

Theorem 2. *Let $\alpha \in \Lambda^{n+1}_s$. Then we have* globally

$$\kappa^{2s+1}_s \circ \mathcal{L}_{j_s\Xi}[\alpha] = \mathcal{E}_n(j_{s+1}\Xi_V \rfloor h(\alpha)) + C^1_1(j_s\Xi_V \otimes K_{hd\alpha}) \,.$$

Proof. We have

$$\begin{aligned}\kappa^{2s+1}_s \circ \mathcal{L}_{j_s\Xi}[\alpha] &= [\mathcal{E}_n(j_{s+1}\Xi_V \rfloor h(\alpha)) + j_{s+1}\Xi_V \rfloor d_V h(\alpha)] \\ &= \mathcal{E}_n(j_{s+1}\Xi_V \rfloor h(\alpha)) + C^1_1(j_s\Xi_V \otimes K_{hd\alpha}) \,.\end{aligned}$$

□

3. Variations and generalized Jacobi morphisms

To proceed further, we now need to recall some previous results concerning the representation of *generalized Jacobi morphisms* in variational sequences and their relation with the second variation of a generalized Lagrangian ([13, 14, 17, 40], see also the fundamental paper [20]). In [14] the relation between the classical variations and Lie derivatives of a Lagrangian with respect to (vertical) variation vector fields was worked out and the *variational vertical derivative* was introduced as an operator acting on sections of sheaves of variational sequences, by showing that this operator is in fact a natural transformation, being a functor on the category of variational sequences. We shall here introduce *formal variations* of a morphism as *multiparameter deformations* showing that this is equivalent to take iterated variational Lie derivatives with respect to (vertical) variation vector fields. Our aim is to relate, on the basis of relations provided by Corollary 1 and Proposition 1 below, the second variation of the Lagrangian λ to the Lie derivative of the associated Euler-Lagrange morphism and to the generalized Bianchi morphism, defined by Eq. (21) in Subsection 4.1 below.

We recall (see [13, 14, 17]) a Lemma which relates the i-th variation with the iterated Lie derivative of the morphism itself. Furthermore, following [29], we recall the relation between the variation of the morphism and the vertical exterior differential.

Definition 10. Let $\alpha : J_s\boldsymbol{Y} \to \overset{p}{\wedge} T^* J_s \boldsymbol{Y}$. Let $\psi^k_{t_k}$, with $1 \leq k \leq i$, be the flows generated by an i-tuple $(\Xi_1, \dots, \Xi_i)$ of (vertical, although actually it is enough that they are projectable) vector fields on $\boldsymbol{Y}$ and let Γ_i be the i-th *formal variation* generated by the Ξ_k's (to which we shall refer as variation vector fields) and defined, for each $\boldsymbol{y} \in \boldsymbol{Y}$, by $\Gamma_i(t_1, \dots, t_i)(\boldsymbol{y}) = \psi^i_{t_i} \circ \dots \circ \psi^1_{t_1}(\boldsymbol{y})$. We define the i-th formal variation of the morphism α to be

$$\delta^i \alpha := \frac{\partial^i}{\partial t_1 \dots \partial t_i}\Big|_{t_1,\dots,t_i=0} (\alpha \circ j_s \Gamma_i(t_1, \dots, t_i)(\boldsymbol{y})) \,. \tag{8}$$

The following two Lemmas state the relation between the i-th formal variation of a morphism and its iterated Lie derivative [13, 14, 17, 20].

Lemma 1. *Let $\alpha : J_s\boldsymbol{Y} \to \overset{p}{\wedge} T^* J_s \boldsymbol{Y}$ and $L_{j_s\Xi_k}$ be the Lie derivative operator acting on differential fibered morphism.*

Let Γ_i be the i-th formal variation generated by variation vector fields Ξ_k, $1 \leq k \leq i$ on $\boldsymbol{Y}$. Then we have

$$\delta^i \alpha = L_{j_s\Xi_1} \dots L_{j_s\Xi_i} \alpha \,. \tag{9}$$

Lemma 2. *Let Ξ be a variation vector field on $\boldsymbol{Y}$ and $\lambda \in \Lambda^n_s$. Then we have $\delta\lambda = j_s\Xi \rfloor d_V \lambda$ [29].*

Remark 11. Owing to the linearity properties of $d_V\lambda$, we can think of the operator δ as a *linear* morphism with respect to the vector bundle structure $J_sV\boldsymbol{Y} \to \boldsymbol{X}$, so that we can write $\delta\alpha : J_s\boldsymbol{Y} \to J_sV^*\boldsymbol{Y} \wedge \overset{n}{\wedge} T^*\boldsymbol{X}$. This property can be obviously iterated for each integer i, so that one can analogously define an i-linear morphism δ^i. In particular, we have $\delta^2\alpha : J_s\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} J_sV\boldsymbol{Y} \to J_sV^*\boldsymbol{Y} \otimes J_sV^*\boldsymbol{Y} \wedge \overset{n}{\wedge} T^*\boldsymbol{X}$.

For notational convenience and by an abuse of notation, in the sequel we shall denote with the same symbol an object defined on the vertical prolongation $V\boldsymbol{Y}$ as well as the corresponding one defined on the iterated vertical prolongation $V(V\boldsymbol{Y})$, whenever there is no danger of confusion.

Corollary 1. *Let $\lambda \in (\Lambda^n_s)_{\boldsymbol{Y}}$. Let Ξ_1, Ξ_2 be two variation vector fields on $\boldsymbol{Y}$ generating the formal variation Γ_2. Then we have*

$$\delta^2\lambda = j_{2s}\Xi_2 \rfloor \mathcal{E}_n(\delta\lambda) + d_H(j_{2s}\Xi_2 \rfloor p_{d_V\delta\lambda}) \tag{10}$$

$$= \delta(j_{2s}\Xi_1 \rfloor \mathcal{E}_n(\lambda) + d_H(j_{2s}\Xi_1 \rfloor p_{d_V\lambda})) \tag{11}$$

$$= j_{2s}\Xi_2 \rfloor \mathcal{E}_n((j_{2s}\Xi_1 \rfloor \mathcal{E}_n(\lambda)) + d_H(j_{2s+1}\Xi_2 \rfloor p_{d_V(\Xi_1 \rfloor \mathcal{E}_n(\lambda))} \tag{12}$$

$$+ \delta(j_{2s}\Xi_1 \rfloor p_{d_V\lambda}))\,. \tag{13}$$

Proof. We apply Lemma 2 and decomposition provided by Theorem 1. Furthermore, $d_H\delta = \delta d_H$, which follows directly from the analogous naturality property of the Lie derivative operator. □

Remark 12. From the relations above we also infer, of course, that

$$j_{2s}\Xi_2 \rfloor \mathcal{E}_n((j_{2s}\Xi_1 \rfloor \mathcal{E}_n(\lambda)) + d_H(j_{2s+1}\Xi_2 \rfloor p_{d_V(\Xi_1 \rfloor \mathcal{E}_n(\lambda))}) = \delta(j_{2s}\Xi_1 \rfloor \mathcal{E}_n(\lambda)) = \delta^2\lambda\,,$$

$$j_{2s}\Xi_2 \rfloor \mathcal{E}_n((j_{2s}\Xi_1 \rfloor \mathcal{E}_n(\lambda)) + d_H(\delta(j_{2s}\Xi_1 \rfloor p_{d_V\lambda})) = j_{2s}\Xi_2 \rfloor \mathcal{E}_n(\delta\lambda)\,.$$

3.1. **Variational vertical derivatives and generalized Jacobi morphisms.** In this section we restrict our attention to morphisms which are (identified with) sections of sheaves in the variational sequence. We shall recall some results of ours [13, 14] by defining the i-th *variational vertical derivative* of morphisms.

Let $\alpha \in (\mathcal{V}^n_s)_{\boldsymbol{Y}}$. We have

$$\delta^i[\alpha] := [\delta^i\alpha] = [L_{\Xi_i} \dots L_{\Xi_1}\alpha] = \mathcal{L}_{\Xi_i} \dots \mathcal{L}_{\Xi_1}[\alpha]\,.$$

Definition 11. We call the operator δ^i the *i-th variational vertical derivative.*

In [14] the variational vertical derivative was introduced as an operator acting on sections of sheaves of variational sequences, by showing that this operator is in fact a natural transformation, being a functor on the category of variational sequences as it can be summarized by the following commutative diagram.

$$\begin{array}{ccccccccc}
\dots & \xrightarrow{d_H} & (\mathcal{V}^n_s)_{\boldsymbol{Y}} & \xrightarrow{\mathcal{E}_n} & (\mathcal{V}^{n+1}_s)_{\boldsymbol{Y}} & \xrightarrow{\mathcal{E}_{n+1}} & \mathcal{E}_{n+1}(\mathcal{V}^{n+1}_s)_{\boldsymbol{Y}} & \xrightarrow{\mathcal{E}_{n+2}} & 0 \\
 & & \delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \\
\dots & \xrightarrow{d_H} & (\mathcal{V}^n_s)_{\boldsymbol{Y}\underset{\boldsymbol{X}}{\times}V\boldsymbol{Y}} & \xrightarrow{\mathcal{E}_n} & (\mathcal{V}^{n+1}_s)_{\boldsymbol{Y}\underset{\boldsymbol{X}}{\times}V\boldsymbol{Y}} & \xrightarrow{\mathcal{E}_{n+1}} & \mathcal{E}_{n+1}((\mathcal{V}^{n+1}_s)_{\boldsymbol{Y}\underset{\boldsymbol{X}}{\times}V\boldsymbol{Y}}) & \xrightarrow{\mathcal{E}_{n+2}} & 0 \\
 & & \delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \\
\dots & \xrightarrow{d_H} & (\mathcal{V}^n_s)_{\boldsymbol{Y}\underset{\boldsymbol{X}}{\times}V(V\boldsymbol{Y})} & \xrightarrow{\mathcal{E}_n} & (\mathcal{V}^{n+1}_s)_{\boldsymbol{Y}\underset{\boldsymbol{X}}{\times}V(V\boldsymbol{Y})} & \xrightarrow{\mathcal{E}_{n+1}} & \mathcal{E}_{n+1}((\mathcal{V}^{n+1}_s)_{\boldsymbol{Y}\underset{\boldsymbol{X}}{\times}V(V\boldsymbol{Y})}) & \xrightarrow{\mathcal{E}_{n+2}} & 0 \\
 & & \delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{array}$$

As a straightforward consequence we have the following characterization of the second variation of a generalized Lagrangian in the variational sequence.

Proposition 1. *Let $\lambda \in (\mathcal{V}^n_s)_{\boldsymbol{Y}}$ and let Ξ be a variation vector field; then we have*

$$\delta^2\lambda = [\mathcal{E}_n(j_{2s}\Xi\rfloor h\delta\lambda) + C^1_1(j_{2s}\Xi\otimes K_{hd\delta\lambda})]\,. \tag{14}$$

Proof. Since $\delta\lambda \in (\mathcal{V}^n_s)_{V\boldsymbol{Y}}$, by Remark 11 we have that $\delta\lambda \in (\mathcal{V}^{n+1}_s)_{\boldsymbol{Y}}$ and then $hd\delta\lambda \in \mathcal{E}_{n+1}(\mathcal{V}^{n+1}_s)_{\boldsymbol{Y}} \subset (\mathcal{V}^{n+2}_s)_{\boldsymbol{Y}}$; thus the assertion follows by a straightforward application of Theorem 2. Notice that here $C^1_1(j_{2s}\Xi\otimes K_{hd\delta\lambda}) : J_{2s}\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} J_{2s}V\boldsymbol{Y} \to V^*\boldsymbol{Y} \wedge \overset{n}{\wedge}T^*\boldsymbol{X}$. □

Remark 13. Let $l \geq 0$ and let $\boldsymbol{F}$ be any vector bundle over $\boldsymbol{X}$. Let $\alpha : J_l(\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} \boldsymbol{F}) \to \overset{p}{\wedge}T^*\boldsymbol{X}$ be a linear morphism with respect to the fibering $J_l\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} J_l\boldsymbol{F} \to J_l\boldsymbol{Y}$ and let $\hat{D}_H$ be the horizontal differential on $\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} \boldsymbol{F}$. We can uniquely write α as

$$\overline{\alpha} \equiv \alpha : J_l\boldsymbol{Y} \to \mathcal{C}^*_l[\boldsymbol{F}] \wedge (\overset{p}{\wedge}T^*\boldsymbol{X})\,.$$

Then $\overline{\hat{D}_H\alpha} = \hat{D}_H\overline{\alpha}$ (this property was pointed out in [16]).

Lemma 3. *Let Ξ be a variation vector field. Let $\chi(\lambda,\Xi) := C^1_1(j_{2s}\Xi\otimes K_{hd\delta\lambda}) \equiv E_{j_s\Xi\rfloor h(d\delta\lambda)}$ and let $\tilde{D}_H$ be the horizontal differential on $\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} V\boldsymbol{Y}$. We can see $\chi(\lambda,\Xi)$ as an extended morphism $\chi(\lambda,\Xi) : J_{2s}(\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} V\boldsymbol{Y}) \to J_{2s}V^*(\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} V\boldsymbol{Y})\otimes V^*\boldsymbol{Y} \wedge (\overset{n}{\wedge}T^*\boldsymbol{X})$ satisfying $\tilde{D}_H\chi(\lambda,\Xi) = 0$.*

Proof. The morphism $\chi(\lambda,\Xi) : J_{2s}(\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} V\boldsymbol{Y}) \to V^*\boldsymbol{Y} \wedge (\overset{n}{\wedge}T^*\boldsymbol{X})$ is a linear morphism with respect to the projection $J_{2s}(\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} V\boldsymbol{Y}) \to J_{2s}\boldsymbol{Y}$ (see Remark 11), then we can apply the Remark above, so that $\chi(\lambda,\Xi) : J_{2s}\boldsymbol{Y} \to J_{2s}V^*(V\boldsymbol{Y})\otimes V^*\boldsymbol{Y} \wedge (\overset{n}{\wedge}T^*\boldsymbol{X}) \simeq J_{2s}V^*\boldsymbol{Y}\otimes J_{2s}V^*\boldsymbol{Y}\otimes V^*\boldsymbol{Y} \wedge (\overset{n}{\wedge}T^*\boldsymbol{X})$ and again by linearity we get $\chi(\lambda,\Xi) : J_{2s}(\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} V\boldsymbol{Y}) \to J_{2s}V^*\boldsymbol{Y}\otimes V^*\boldsymbol{Y} \wedge (\overset{n}{\wedge}T^*\boldsymbol{X})$. □

The following Lemma is an application of an abstract result, due to Horák and Kolář [21, 29, 30], concerning a global decomposition formula for vertical morphisms.

Lemma 4. *Let Ξ be a variation vector field.*
Let $\chi(\lambda,\Xi)$ as in the above Lemma. Then we have $(\pi^{4s+1}_{2s+1})^*\chi(\lambda,\Xi) = E_{\chi(\lambda,\Xi)} + F_{\chi(\lambda,\Xi)}$, where

$$E_{\chi(\lambda,\Xi)} : J_{4s}(\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} V\boldsymbol{Y}) \to \mathcal{C}^*_0[\boldsymbol{Y}]\otimes\mathcal{C}^*_0[\boldsymbol{Y}] \wedge \overset{n}{\wedge}T^*\boldsymbol{X}\,, \tag{15}$$

and locally, $F_{\chi(\lambda,\Xi)} = \tilde{D}_H M_{\chi(\lambda,\Xi)}$, *with*

$$M_{\chi(\lambda,\Xi)} : J_{4s-1}(\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} V\boldsymbol{Y}) \to \mathcal{C}^*_{2s-1}[\boldsymbol{Y}]\otimes\mathcal{C}^*_0[\boldsymbol{Y}] \wedge \overset{n-1}{\wedge}T^*\boldsymbol{X}\,.$$

Proof. Following e.g. [29, 30, 46], the *global* morphisms $E_{\chi(\lambda,\Xi)}$ and $F_{\chi(\lambda,\Xi)}$ can be evaluated by means of a backwards procedure. $\square$

Definition 12. We call the morphism $\mathcal{J}(\lambda,\Xi){:=}E_{\chi(\lambda,\Xi)}$ the *generalized Jacobi morphism* associated with the Lagrangian λ.

Example 1. Let us write explicitly the coordinate expression of $\mathcal{J}(\lambda,\Xi)$. By functoriality of δ, we have $h(d\delta\lambda) = h(\delta d\lambda)$. Let now locally $\lambda = L\omega$, where L is a function of $J_s\boldsymbol{Y}$ and ω a volume form on $\boldsymbol{X}$, then $d\lambda = \partial_i^{\boldsymbol{\alpha}}(L)d^i_{\boldsymbol{\alpha}} \wedge \omega$ and $\delta d\lambda = \partial_j^{\boldsymbol{\sigma}}(\partial_i^{\boldsymbol{\alpha}}L)d^j_{\boldsymbol{\sigma}}\otimes d^i_{\boldsymbol{\alpha}} \wedge \omega$, thus finally $h(\delta d\lambda) = h(d\delta\lambda) = \partial_j(\partial_i^{\boldsymbol{\alpha}}L)d^j_{\boldsymbol{\alpha}}\otimes d^i \wedge \omega$. As a consequence we have, with $0 \leq |\boldsymbol{\mu}|, |\boldsymbol{\alpha}|, |\boldsymbol{\sigma}| \leq 2s+1$:

$$\begin{aligned}\chi(\lambda,\Xi) &= D_{\boldsymbol{\sigma}}\Xi^l_V\Big(\partial_j(\partial_i^{\boldsymbol{\mu}}L) - \sum_{|\boldsymbol{\alpha}|=0}^{s-|\boldsymbol{\mu}|}(-1)^{|\boldsymbol{\mu}+\boldsymbol{\alpha}|}\frac{(\boldsymbol{\mu}+\boldsymbol{\alpha})!}{\boldsymbol{\mu}!\boldsymbol{\alpha}!}D_{\boldsymbol{\alpha}}\partial_j^{\boldsymbol{\alpha}}(\partial_i^{\boldsymbol{\mu}}L)\Big)\partial_l^{\boldsymbol{\sigma}}\otimes\vartheta^j_{\boldsymbol{\mu}}\otimes\vartheta^i\wedge\omega \\ &\doteq \chi^{l\boldsymbol{\mu}}_{\boldsymbol{\sigma}ji}\partial_l^{\boldsymbol{\sigma}}\otimes\vartheta^j_{\boldsymbol{\mu}}\otimes\vartheta^i\wedge\omega\,;\end{aligned}$$

and by the Lemma above, we get (up to divergences):

$$\mathcal{J}(\lambda,\Xi) = (-1)^{|\boldsymbol{\alpha}|}D_{\boldsymbol{\alpha}}\,\chi^{l\boldsymbol{\alpha}}_{\boldsymbol{\sigma}ji}\partial_l^{\boldsymbol{\sigma}}\otimes\vartheta^j\otimes\vartheta^i\wedge\omega\,. \tag{16}$$

3.2. **Generalized *gauge-natural* Jacobi morphisms.** We intend now to specify the just mentioned results and definitions concerning the Jacobi morphism by considering as variation vector fields the vertical parts of prolongations of gauge-natural lifts of infinitesimal principal automorphisms to the gauge-natural bundle $\boldsymbol{Y}_\zeta$. Owing to linearity properties of the Lie derivative of sections and taking into account the fact that, as we already recalled, $j_s\hat{\Xi}_V = -\pounds_{j_s\bar{\Xi}}$, we can state the following important results.

Recall (see [32], Proposition 15.5) that the jet prolongation of order s of $\boldsymbol{Y}_\zeta$ is a gauge-natural bundle itself associated to some principal prolongation of order $(r+s, k+s)$ of the underling principal bundle $\boldsymbol{P}$. Let $\bar{\Xi} \in \mathcal{A}^{(r,k)}$ and $\hat{\Xi}{:=}\mathfrak{G}(\bar{\Xi})$ the corresponding gauge-natural lift to $\boldsymbol{Y}_\zeta$. Let $j_s\hat{\Xi}$ be the s-jet prolongation of $\hat{\Xi}$ which is a vector field on $J_s\boldsymbol{Y}_\zeta$. It turns out then that it is a gauge natural lift of $\bar{\Xi}$ too, i.e. $j_s\mathfrak{G}(\bar{\Xi}) = \mathfrak{G}(j_s\bar{\Xi})$. Let us consider $j_s\hat{\Xi}_V$, i.e. the vertical part according to the splitting (1). We shall denote by $j_s\bar{\Xi}_V$ the induced section of the vector bundle $\mathcal{A}^{(r+s,k+s)}$. The set of all sections of this kind defines a vector subbundle of $J_s\mathcal{A}^{(r,k)}$ which we shall denote, by a slight abuse of notation (since we are speaking about vertical parts with respect to the splitting (1)), by $VJ_s\mathcal{A}^{(r,k)}$.

Lemma 5. *Let* $\chi(\lambda,\mathfrak{G}(\bar{\Xi})_V){:=}C^1_1(j_{2s}\hat{\Xi}\otimes K_{hd\mathcal{L}_{j_{2s}\bar{\Xi}_V}}\lambda) \equiv E_{j_s\hat{\Xi}\rfloor hd\mathcal{L}_{j_{2s+1}\bar{\Xi}_V}\lambda}$. *Let* D_H *be the horizontal differential on* $\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} V\mathcal{A}^{(r,k)}$. *Then we have:*

$$(\pi^{4s+1}_{2s+1})^*\chi(\lambda,\mathfrak{G}(\bar{\Xi})_V) = E_{\chi(\lambda,\mathfrak{G}(\bar{\Xi})_V)} + F_{\chi(\lambda,\mathfrak{G}(\bar{\Xi})_V)}\,,$$

where

$$E_{\chi(\lambda,\mathfrak{G}(\bar{\Xi})_V} : J_{4s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{4s}\mathcal{A}^{(r,k)} \to \mathcal{C}^*_0[\mathcal{A}^{(r,k)}]\otimes\mathcal{C}^*_0[\mathcal{A}^{(r,k)}]\wedge(\overset{n}{\wedge}T^*\boldsymbol{X})\,, \tag{17}$$

and locally, $F_{\chi(\lambda,\mathfrak{G}(\bar{\Xi})_V)} = D_H M_{\chi(\lambda,\mathfrak{G}(\bar{\Xi})_V)}$, *with*

$$M_{\chi(\lambda,\mathfrak{G}(\bar{\Xi})_V)} : J_{4s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{4s}\mathcal{A}^{(r,k)} \to \mathcal{C}^*_{2s-1}[\mathcal{A}^{(r,k)}]\otimes\mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \wedge (\overset{n-1}{\wedge} T^*\boldsymbol{X})\,.$$

Proof. Notice that, since $\chi(\lambda,\mathfrak{G}(\bar{\Xi})_V) \equiv E_{(-\pounds_{j_s\bar{\Xi}_V}\rfloor hd\mathcal{L}_{j_{2s+1}\bar{\Xi}_V}\lambda)}$, as a consequence of the linearity properties of $\chi(\lambda,\Xi)$ and of linearity properties of the Lie derivative operator $\pounds$ (see Subsections 2.3 and 2.4) we have $\chi(\lambda,\mathfrak{G}(\bar{\Xi})_V) : J_{2s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{2s}\mathcal{A}^{(r,k)} \to \mathcal{C}^*_{2s}[\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} V\mathcal{A}^{(r,k)}]\otimes\mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \wedge (\overset{n}{\wedge} T^*\boldsymbol{X})$ and $D_H\chi(\lambda,\mathfrak{G}(\bar{\Xi})_V) = 0$. Thus the decomposition Lemma 4 can be applied. $\square$

Definition 13. Let $\bar{\Xi} \in \mathcal{A}^{(r,k)}$.
We call the morphism $\mathcal{J}(\lambda,\mathfrak{G}(\bar{\Xi})_V) := E_{\chi(\lambda,\mathfrak{G}(\bar{\Xi})_V)}$ the *gauge-natural generalized Jacobi morphism* associated with the Lagrangian λ and the gauge-natural lift $\mathfrak{G}(\bar{\Xi})_V$.

We have the following:

Proposition 2. *The morphism* $\mathcal{J}(\lambda,\mathfrak{G}(\bar{\Xi})_V)$ *is a* linear *morphism with respect to the projection* $J_{4s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{4s}\mathcal{A}^{(r,k)} \to J_{4s}\boldsymbol{Y}_\zeta$.

We are now able to provide an important specialization of Theorem 2.

Proposition 3. *Let* $[\mathcal{L}_{j_{s+1}\bar{\Xi}_V}\lambda] \in (\mathcal{V}^{n+1}_s)_{\boldsymbol{Y}}$. *Then we have*

$$\kappa^{4s+1}_{s+1}\circ\mathcal{L}_{j_s\bar{\Xi}}[\mathcal{L}_{j_{s+1}\bar{\Xi}_V}\lambda] = \mathcal{E}_n(j_s\bar{\Xi}_V\rfloor h(\mathcal{L}_{j_{s+1}\bar{\Xi}_V}\lambda)) + \mathcal{J}(\lambda,\mathfrak{G}(\bar{\Xi})_V)\,.$$

Proof. By Theorem 2 and the Lemma above we have:

$$\begin{aligned}\kappa^{4s+1}_{s+1}\circ\mathcal{L}_{j_s\bar{\Xi}}[\mathcal{L}_{j_{s+1}\bar{\Xi}_V}\lambda] &= \mathcal{E}_n(j_s\bar{\Xi}_V\rfloor h(\mathcal{L}_{j_{s+1}\bar{\Xi}_V}\lambda)) + [C^1_1(j_{2s}\bar{\Xi}\otimes K_{hd\mathcal{L}_{j_{2s}\bar{\Xi}_V}}\lambda)]\\ &= \mathcal{E}_n(j_s\bar{\Xi}_V\rfloor h(\mathcal{L}_{j_{s+1}\bar{\Xi}_V}\lambda)) + \mathcal{E}_n(j_s\bar{\Xi}_V\rfloor h(d\mathcal{L}_{j_{s+1}\bar{\Xi}_V}\lambda))\\ &= \mathcal{E}_n(j_s\bar{\Xi}_V\rfloor h(\mathcal{L}_{j_{s+1}\bar{\Xi}_V}\lambda)) + \mathcal{J}(\lambda,\mathfrak{G}(\bar{\Xi})_V)\,.\end{aligned}$$ $\square$

Remark 14. Theorem 1 in Subsection 2.6 provides an *invariant* decomposition, where both pieces are globally defined. However, the second one is only *locally* a divergence, unless some further geometric structures such as linear symmetric connections on the basis manifold or suitable gauge-natural principal (or prolongations with respect to linear symmetric connections of principal) connections are introduced [29, 30, 46]. Proposition 3 above, *instead*, provides an invariant decomposition into two pieces which are globally defined and no one of them, *seen as a section of* $(\mathcal{V}^{n+1}_s)_{\boldsymbol{Y}}$, is a (local) divergence. As we shall see, this fact has very important consequences concerning conserved quantities in gauge-natural Lagrangian field theories.

A simple comparison of Remark 12, Proposition 1 and the Proposition above gives us the following.

Corollary 2. *Let $\delta^2_{\mathfrak{G}}\lambda$ be the variation of λ with respect to vertical parts of gauge-natural lifts of infinitesimal principal automorphisms. We have:*

$$\mathfrak{G}(\bar{\Xi})_V \rfloor \mathcal{E}_n(\mathfrak{G}(\bar{\Xi})_V \rfloor \mathcal{E}_n(\lambda)) = \delta^2_{\mathfrak{G}}\lambda = \mathcal{E}_n(\mathfrak{G}(\bar{\Xi})_V \rfloor h(d\delta\lambda))\,. \tag{18}$$

The reader should notice that, *seen as a section of* $(\mathcal{V}^n_s)_{\boldsymbol{Y}\underset{\boldsymbol{X}}{\times}V\boldsymbol{Y}}$, the equivalence class $[\mathcal{E}_n(j_s\bar{\Xi}_V \rfloor h(\delta\lambda))]$ vanishes being a local divergence of higher degree contact forms. This result can also be compared with [17].

4. Noether Theorems and conserved currents for gauge-natural invariant Lagrangians

In the following we assume that the field equations are generated by means of a variational principle from a Lagrangian which is gauge-natural invariant, i.e. invariant with respect to any gauge-natural lift of infinitesimal right invariant vector fields. We consider now a projectable vector field $(\hat{\Xi}, \xi)$ on $\boldsymbol{Y}_\zeta$ and take into account the Lie derivative with respect to its prolongation $j_s\hat{\Xi}$.

Definition 14. Let $(\hat{\Xi}, \xi)$ be a projectable vector field on $\boldsymbol{Y}_\zeta$. Let $\lambda \in \mathcal{V}^n_s$ be a generalized Lagrangian. We say $\hat{\Xi}$ to be a *symmetry* of λ if $\mathcal{L}_{j_{s+1}\hat{\Xi}}\lambda = 0$.

We say λ to be a *gauge-natural invariant Lagrangian* if the gauge-natural lift $(\hat{\Xi}, \xi)$ of *any* vector field $\bar{\Xi} \in \mathcal{A}^{(r,k)}$ is a symmetry for λ, *i.e.* if $\mathcal{L}_{j_{s+1}\bar{\Xi}}\lambda = 0$. In this case the projectable vector field $\hat{\Xi} \equiv \mathfrak{G}(\bar{\Xi})$ is called a *gauge-natural symmetry* of λ.

Remark 15. Due to $\mathcal{E}_n\mathcal{L}_{j_s\bar{\Xi}} = \mathcal{L}_{j_s\bar{\Xi}}\mathcal{E}_n$, a symmetry of a Lagrangian λ is also a symmetry of its Euler-Lagrange morphism E_λ (but the converse is not true, see e.g. [44]).

Symmetries of a Lagrangian λ are calculated by means of Noether Theorems, which takes a particularly interesting form in the case of gauge-natural Lagrangians.

Proposition 4. *Let $\lambda \in \mathcal{V}^n_s$ be a gauge-natural Lagrangian and $(\hat{\Xi}, \xi)$ a gauge-natural symmetry of λ. Then we have*

$$0 = -\pounds_{\bar{\Xi}} \rfloor \mathcal{E}_n(\lambda) + d_H(-j_s\pounds_{\bar{\Xi}} \rfloor p_{d_V\lambda} + \xi \rfloor \lambda)\,. \tag{19}$$

Suppose that the section σ fulfills the condition $(j_{2s+1}\sigma)^*(-\pounds_{\bar{\Xi}} \rfloor \mathcal{E}_n(\lambda)) = 0$. Then, the $(n-1)$-form

$$\epsilon = -j_s\pounds_{\bar{\Xi}} \rfloor p_{d_V\lambda} + \xi \rfloor \lambda\,, \tag{20}$$

fulfills the equation $d((j_{2s}\sigma)^*(\epsilon)) = 0$.

Remark 16. If σ is a critical section for $\mathcal{E}_n(\lambda)$, i.e. $(j_{2s+1}\sigma)^*\mathcal{E}_n(\lambda) = 0$, the above equation admits a physical interpretation as a so-called *weak conservation law* for the density associated with ϵ.

Definition 15. Let $\lambda \in \mathcal{V}^n_s$ be a gauge-natural Lagrangian and $\bar{\Xi} \in \mathcal{A}^{(r,k)}$. Then the sheaf morphism $\epsilon : J_{2s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{2s}\mathcal{A}^{(r,k)} \to \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \wedge (\overset{n-1}{\wedge} T^*\boldsymbol{X})$ is said to be a *gauge-natural weakly conserved current*.

Remark 17. In general, this conserved current is not uniquely defined. In fact, it depends on the choice of $p_{d_V\lambda}$, which is not unique (see [46] and references quoted therein). Moreover, we could add to the conserved current any form $\mu \in \mathcal{V}^{n-1}_{2s}$ which is variationally closed, i.e. such that $\mathcal{E}_{n-1}(\mu) = 0$ holds. The form μ is locally of the type $\mu = d_H\gamma$, where $\gamma \in \mathcal{V}^{n-2}_{2s-1}$.

Corollary 3. *Let* $\epsilon : J_{2s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{2s}\mathcal{A}^{(r,k)} \to \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \wedge (\overset{n-1}{\wedge} T^*\boldsymbol{X})$ *be a conserved current. As an immediate consequence of Remark 13 we can regard* ϵ *as the equivalent morphism* $\epsilon \equiv \bar{\epsilon} : J_{2s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{2s}\mathcal{A}^{(r,k)} \to \mathcal{C}^*_{2s}[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \wedge (\overset{n-1}{\wedge} T^*\boldsymbol{X})$.

Remark 18. Let $\eta \in \mathcal{V}^{n+1}_s$ and let Ξ be a symmetry of η. Then, as a special case of Theorem 2, we have $0 = \mathcal{E}_n(\Xi_V \rfloor \eta) + [C^1_1(j_{2s+1}\bar{\Xi}_V \otimes K_{hd\eta})]$. Suppose that $K_{hd\eta} = 0$; then we have $\mathcal{E}_n(\bar{\Xi}_V \rfloor \eta) = 0$. This implies that $\bar{\Xi}_V \rfloor \eta$ is variationally trivial, i.e. it is locally of the type $\bar{\Xi}_V \rfloor \eta = d_H\mu$, where $\mu \in \mathcal{V}^{n-1}_{s-1}$.

Suppose that the section $\sigma : \boldsymbol{X} \to \boldsymbol{Y}_\zeta$ fulfils $(j_{2s+1}\sigma)^*(\bar{\Xi}_V \rfloor \eta) = 0$. Then we have $d((j_{2s}\sigma)^*\mu) = 0$ so that, as in the case of symmetries of Lagrangians, μ is a conserved current along σ.

As in the case of Lagrangians, a conserved current for an Euler-Lagrange type morphism is not uniquely defined. In fact, we could add to $\Xi_V \rfloor \eta$ any variationally trivial Lagrangian, obtaining different conserved currents. Moreover, such conserved currents are defined up to variationally trivial $(n-1)$-forms.

4.1. **The Bianchi morphism.** In gauge-natural Lagrangian theories it is a well known procedure to perform suitable integrations by parts to decompose the conserved current ϵ into the sum of a conserved current vanishing along solutions of the Euler-Lagrange equations, the so-called *reduced current*, and the formal divergence of a skew-symmetric (tensor) density called a *superpotential* (which is defined modulo a divergence). Within such a procedure, the generalized Bianchi identities play a very fundamental role: they are in fact necessary and (locally) sufficient conditions for the conserved current ϵ to be not only closed but also the divergence of a skew-symmetric (tensor) density along solutions of the Euler-Lagrange equations.

In the following we shall perform such an integration by part of the conserved current by resorting to Kolář's invariant decomposition formula of vertical morphisms we already used to define the Jacobi morphism. We will also make an extensive use of Remark 13.

Remark 19. Let λ be a gauge-natural Lagrangian. By the linearity of $\pounds$ we have

$$\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V) = \pounds_{\bar{\Xi}} \rfloor \mathcal{E}_n(\lambda) : J_{2s}\boldsymbol{Y}_\zeta \to \mathcal{C}^*_{2s}[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \wedge (\overset{n}{\wedge} T^*\boldsymbol{X}) .$$

We have $D_H\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V) = 0$. We can regard $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ as the extended morphism $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V) : J_{2s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{2s}\mathcal{A}^{(r,k)} \to \mathcal{C}^*_{2s}[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_{2s}[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \wedge (\overset{n}{\wedge} T^*\boldsymbol{X})$. Thus we can state the following.

Lemma 6. *Let* $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ *be as in the above Remark. Then we have* globally

$$(\pi^{4s+1}_{s+1})^* \omega(\lambda, \mathfrak{G}(\bar{\Xi})_V) = \beta(\lambda, \mathfrak{G}(\bar{\Xi})_V) + F_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)} \,,$$

where

$$\beta(\lambda, \mathfrak{G}(\bar{\Xi})_V) \equiv E_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)} : \tag{21}$$

$$: J_{4s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{4s}\mathcal{A}^{(r,k)} \to \mathcal{C}^*_{2s}[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \wedge (\overset{n}{\wedge} T^*\boldsymbol{X}) \tag{22}$$

and locally*,* $F_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)} = D_H M_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)}$*, with*

$$M_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)} : J_{4s-1}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{4s-1}\mathcal{A}^{(r,k)} \to \tag{23}$$

$$\to \mathcal{C}^*_{2s}[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_{2s-1}[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \wedge (\overset{n-1}{\wedge} T^*\boldsymbol{X}) \,. \tag{24}$$

In particular, we get the following *local* decomposition of $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)$:

$$\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V) = \beta(\lambda, \mathfrak{G}(\bar{\Xi})_V) + D_H \tilde{\epsilon}(\lambda, \mathfrak{G}(\bar{\Xi})_V) \,, \tag{25}$$

Proof. We take into account that $D_H \omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ is obviously vanishing, then the result is a straightforward consequence of Lemma 4. $\square$

Definition 16. We call the global morphism $\beta(\lambda, \mathfrak{G}(\bar{\Xi})_V) := E_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)}$ the *generalized Bianchi morphism* associated with the Lagrangian λ.

Remark 20. For any $(\bar{\Xi}, \xi) \in \mathcal{A}^{(r,k)}$, as a consequence of the gauge-natural invariance of the Lagrangian, the morphism $\beta(\lambda, \mathfrak{G}(\bar{\Xi})_V) \equiv \mathcal{E}_n(\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V))$ is *locally* identically vanishing. We stress that these are just *local generalized Bianchi identities*. In particular, we have $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V) = D_H \tilde{\epsilon}(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ *locally* [3, 5, 6, 19, 35].

Definition 17. The form $\tilde{\epsilon}(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ is called a *local reduced current.*

4.2. **Global generalized Bianchi identities.** We are now able to state our main result providing necessary and sufficient conditions on the gauge-natural lift of infinitesimal right-invariant automorphisms of the principal bundle $\boldsymbol{P}$ in order to get globally defined generalized Bianchi identities. Let $\mathfrak{K} := \mathrm{Ker}_{\mathcal{J}(\lambda, \mathfrak{G}(\bar{\Xi})_V)}$ be the *kernel* of the generalized gauge-natural morphism $\mathcal{J}(\lambda, \mathfrak{G}(\bar{\Xi})_V)$. As a consequence of the considerations above, we have the following important result.

Theorem 3. *The generalized Bianchi morphism is globally vanishing if and only if* $\delta^2_{\mathfrak{G}} \lambda \equiv \mathcal{J}(\lambda, \mathfrak{G}(\bar{\Xi})_V) = 0$*, i.e. if and only if* $\mathfrak{G}(\bar{\Xi})_V \in \mathfrak{K}$*.*

Proof. By Corollary 2 we get

$$\mathfrak{G}(\bar{\Xi})_V \rfloor \beta(\lambda, \mathfrak{G}(\bar{\Xi})_V) = \delta^2_{\mathfrak{G}} \lambda \equiv \mathcal{J}(\lambda, \mathfrak{G}(\bar{\Xi})_V) \,.$$

Now, if $\mathfrak{G}(\bar{\Xi})_V \in \mathfrak{K}$ then $\beta(\lambda, \mathfrak{G}(\bar{\Xi})_V) \equiv 0$, which are *global generalized Bianchi identities. Vice versa*, if $\mathfrak{G}(\bar{\Xi})_V$ is such that $\beta(\lambda, \mathfrak{G}(\bar{\Xi})_V) = 0$, then $\mathcal{J}(\lambda, \mathfrak{G}(\bar{\Xi})_V) = 0$ and $\mathfrak{G}(\bar{\Xi})_V \in \mathfrak{K}$. *Notice* that $\mathfrak{G}(\bar{\Xi})_V \rfloor \beta(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ is nothing but the *Hessian morphism* associated with λ (see [17]). $\square$

Remark 21. We recall that given a vector field $j_s\hat{\Xi} : J_s\boldsymbol{Y}_\zeta \to TJ_s\boldsymbol{Y}_\zeta$, the splitting (1) yields $j_s\hat{\Xi} \circ \pi^{s+1}_s = j_s\hat{\Xi}_H + j_s\hat{\Xi}_V$ where, if $j_s\hat{\Xi} = \hat{\Xi}^\gamma \partial_\gamma + \hat{\Xi}^i_{\boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}}_i$, then we have $j_s\hat{\Xi}_H = \hat{\Xi}^\gamma D_\gamma$ and $j_s\hat{\Xi}_V = D_{\boldsymbol{\alpha}}(\hat{\Xi}^i - y^i_\gamma \hat{\Xi}^\gamma)\partial^{\boldsymbol{\alpha}}_i$. Analogous considerations hold true of course also for the unique corresponding invariant vector field $j_s\bar{\Xi}$ on $W^{(r,k)}\boldsymbol{P}$. In particular, the condition $j_s\bar{\Xi}_V \in \mathfrak{K}$ implies, of course, that the components $\bar{\Xi}^i_{\boldsymbol{\alpha}}$ and $\bar{\Xi}^\gamma$ *are not* independent, but they are *related* in such a way that $j_s\bar{\Xi}_V$ must be a solution of generalized gauge-natural Jacobi equations for the Lagrangian λ (see coordinate expression (16)). The geometric interpretation of this condition will be the subject of a separate paper [41]. Our results are quite evidently related to the theory of G-reductive Lie derivatives developed in [18]. It is in fact our opinion that the Kosmann lift (the kind of gauge-natural lift used to correctly define the Lie derivative of spinors, in [18] interpreted as a special kind of reductive lift) can be recognized as a kind of gauge-natural Jacobi vector field. Even more, we believe that the kernel of the generalized gauge-natural Jacobi morphism induces a canonical reductive pair on $W^{(r,k)}\boldsymbol{P}$. We also remark that, for each $\bar{\Xi} \in \mathcal{A}^{(r,k)}$ such that $\bar{\Xi}_V \in \mathfrak{K}$, we have $\mathcal{L}_{j_s\bar{\Xi}_H}\omega(\lambda, \mathfrak{K}) = 0$. Here it is enough to stress that, *within* a gauge-natural *invariant Lagrangian variational principle* the gauge-natural lift of infinitesimal principal automorphism *is not* intrinsically arbitrary. It would be also interesting to compare such results with reduction theorems stated in [23].

In the following we shall refer to *canonical* globally defined objects (such as currents or corresponding superpotentials) by their explicit dependence on $\mathfrak{K}$.

Corollary 4. *Let $\lambda \in \mathcal{V}^n_s$ be a gauge-natural Lagrangian and $j_s\hat{\Xi}_V \in \mathfrak{K}$ a gauge-natural symmetry of λ. Being $\beta(\lambda, \mathfrak{K}) \equiv 0$, we have,* globally, *$\omega(\lambda, \mathfrak{K}) = D_H\epsilon(\lambda, \mathfrak{K})$, then the following holds:*

$$D_H(\epsilon(\lambda, \mathfrak{K}) - \tilde{\epsilon}(\lambda, \mathfrak{K}) = 0 \,. \tag{26}$$

Eq. (26) is referred as a gauge-natural 'strong' conservation law for the *global* density $\epsilon(\lambda, \mathfrak{K}) - \tilde{\epsilon}(\lambda, \mathfrak{K})$.

We can now state the following fundamental result about the existence and *globality* of gauge-natural superpotentials in the framework of variational sequences.

Theorem 4. *Let $\lambda \in \mathcal{V}^n_s$ be a gauge-natural Lagrangian and $(j_s\hat{\Xi}, \xi)$ a gauge-natural symmetry of λ. Then there exists a global sheaf morphism $\nu(\lambda, \mathfrak{K}) \in \left(\mathcal{V}^{n-2}_{2s-1}\right)_{\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} \mathfrak{K}}$ such that*

$$D_H\nu(\lambda, \mathfrak{K}) = \epsilon(\lambda, \mathfrak{K}) - \tilde{\epsilon}(\lambda, \mathfrak{K}) \,.$$

Definition 18. We define the sheaf morphism $\nu(\lambda, \mathfrak{K})$ to be a *canonical gauge-natural superpotential* associated with λ.

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