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ASYMPTOTIC BEHAVIOUR OF A DIFFERENCE EQUATION WITH COMPLEX-VALUED COEFFICIENTS

JOSEF KALAS

Abstract. The asymptotic behaviour for solutions of a difference equation $\Delta z_n = f(n, z_n)$, where the complex-valued function $f(n, z)$ is in some meaning close to a holomorphic function h, and of a Riccati difference equation is studied using a Lyapunov function method. The paper is motivated by papers on the asymptotic behaviour of the solutions of differential equations with complex-valued right-hand sides.

1. Introduction

In [3]–[7], the asymptotic behaviour of solutions of a nonlinear differential equation

$$z' = G(t,z)[h(z) + g(t,z)] \tag{1}$$

is studied using Lyapunov function method. h is a holomorphic function defined in a complex simply connected region Ω containing 0, G is a real function and g is a complex function, t and z being a real and complex variable, respectively. It is supposed that the right-hand side of (1) is in a suitable meaning close to h. A Lyapunov-like function $V(z)$ for the equation (1) is suggested in a following manner under the assumption that $h'(0) \neq 0$ and $h(z) = 0 \iff z = 0$ in Ω:

$$V(z) = |v(z)|\,,$$

where

$$v(z) = z \exp \int_0^z r(\zeta)d\zeta\,, \qquad r(z) = \begin{cases} \dfrac{zh'(0) - h(z)}{zh(z)} & \text{for} \quad z \in \Omega, z \neq 0\,, \\[2ex] -\dfrac{h''(0)}{2h'(0)} & \text{for} \quad z = 0\,. \end{cases}$$

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The functions v, r are holomorphic in Ω. It is shown that there exists a number λ_0, $0 < \lambda_0 \leq \infty$ and a simply connected region $K(\lambda_0) \subseteq \Omega$ such that every set

$$\hat{K}(\lambda) := \{z \in K(\lambda_0) : V(z) = \lambda\}$$

is the geometric image of a certain Jordan curve for $0 < \lambda < \lambda_0$ and

$$\operatorname{Int} \hat{K}(\lambda) = \{z \in K(\lambda_0) : V(z) < \lambda\}\,.$$

If we define $\hat{K}(0) := \{0\}$,

$$K(\lambda) : = \bigcup_{0 \leq \mu < \lambda} \hat{K}(\mu) \quad \text{for} \quad 0 < \lambda \leq \lambda_0\,,$$
$$K(\lambda_1, \lambda_2) : = \bigcup_{\lambda_1 < \mu < \lambda_2} \hat{K}(\mu) \quad \text{for} \quad 0 \leq \lambda_1 < \lambda_2 \leq \lambda_0\,,$$

we have

$$K(\lambda) = \operatorname{Int} \hat{K}(\lambda) \quad \text{for} \quad 0 < \lambda < \lambda_0\,,$$
$$K(\lambda_1, \lambda_2) = K(\lambda_2) \setminus \operatorname{Cl} K(\lambda_1) \quad \text{for} \quad 0 < \lambda_1 < \lambda_2 \leq \lambda_0$$

and

$$K(0, \lambda) = K(\lambda) \setminus \{0\} \quad \text{for} \quad 0 < \lambda \leq \lambda_0\,.$$

For details see e. g. [4]. Notice that the functions V, v, r and the sets $\hat{K}(\lambda)$, $K(\lambda)$, $K(\lambda_1, \lambda_2)$ are defined only by means of the function h.

It was shown that the trajectories of the equation $z' = h(z)$ intersect the curves $\hat{K}(\lambda)$ for $0 < \lambda < \lambda_0$ from their exterior to their interior or reversely, if $\operatorname{Re} h'(0) \neq 0$ is assumed. Therefore the Lyapunov-like function V is useful for the investigation of the asymptotic behaviour of the solutions of the equation (1), provided that the right-hand side of (1) is in a suitable meaning close to the function h. The results on the asymptotic behaviour of (1) can be applied to Riccati differential equation and allow to generalize the most of results of earlier papers on the asymptotic properties of Riccati equations with complex-valued coefficients published in [15]–[18].

Consider a difference equation

$$\Delta z_n = f(n, z_n)\,, \tag{2}$$

where $f(n, z)$ is defined od $\mathbb{N}_0 \times \Omega$, $\Omega \subseteq \mathbb{C}$ being a simply connected region containing 0. Let h be a holomorphic function defined in Ω and satisfying the conditions $h'(0) \neq 0$, $h(0) = 0 \iff z = 0$. Define the functions V, v, r and the sets $\hat{K}(\lambda)$, $K(\lambda)$, $K(\lambda_1, \lambda_2)$ as before. If the function $f(n, z)$ is in some sense close to $h(z)$, it can be expected then the function V and the sets $\hat{K}(\lambda)$, $K(\lambda)$, $K(\lambda_1, \lambda_2)$ might be also suitable for the investigation of the asymptotic behaviour of the solutions of (2). In the present paper we attempt to give several results on the asymptotic behaviour of the solutions of (2) and apply some of these results to a special type of difference equation − Riccati difference equation. The exact meanig of the closeness of f to h will be given by the assumptions of results. Notice that the scalar or matrix Riccati or generalized Riccati difference equation in real domain is studied in many papers, such as [1]–[2], [11]–[12] and [20], mainly in the connection with the investigation of the oscillation and asymptotic properties of linear difference equations of the second order. Observe that the method of Lyapunov functions for difference equations is described in several monographs, such as [14] and [13].

2. Results

Theorem 1. *Suppose* $0 < \nu \le \lambda_0$, $f(n,0) = 0$ *for* $n \in \mathbb{N}_0$. *Assume that there is a sequence* $\{\alpha_n\}_{n=0}^{\infty}$ *such that* $\alpha_n \ge 0$ *for* $n \in \mathbb{N}_0$,

$$\sup_{n\in\mathbb{N}_0} \prod_{k=0}^{n} \alpha_k = \varkappa < \infty\,, \tag{3}$$

$$\Big|1 + \frac{f(n,z)}{z}\Big|\Big|\exp\int_z^{z+f(n,z)} r(\vartheta)d\vartheta\Big| \le \alpha_n \tag{4}$$

for $n \in \mathbb{N}_0$, $z \in K(0,\nu)$. *If a solution* $\{z_n\}_{n=0}^{\infty}$ *of the equation* (2) *satisfies*

$$z_n \in K(\lambda_0) \quad \textit{for} \quad n \in \mathbb{N}_0\,, \tag{5}$$

$$z_0 \in \operatorname{Cl} K(\gamma_0)\,, \tag{6}$$

where $0 < \gamma_0 \max(1,\varkappa) < \nu$, *then*

$$z_n \in \operatorname{Cl} K(\gamma_0\varkappa)$$

for $n \in \mathbb{N}$.

Proof. Let $\{z_n\}_{n=0}^{\infty}$ be any solution of (2) satisfying (5), (6). With respect to (4) we have

$$\begin{aligned}
\Delta V(z_n) &= V(z_{n+1}) - V(z_n) = |v(z_{n+1})| - |v(z_n)| \\
&= |z_n + f(n,z_n)|\,\Big|\exp\int_0^{z_n+f(n,z_n)} r(\vartheta)d\vartheta\Big| - |z_n|\,\Big|\exp\int_0^{z_n} r(\vartheta)d\vartheta\Big| \\
&= V(z_n)\Big[\Big|1 + \frac{f(n,z_n)}{z_n}\Big|\Big|\exp\int_{z_n}^{z_n+f(n,z_n)} r(\vartheta)d\vartheta\Big| - 1\Big] \\
&\le (\alpha_n - 1)V(z_n)\,,
\end{aligned}$$

for any $n \in \mathbb{N}_0$ such that $z_n \in K(0,\nu)$. Hence

$$V(z_{n+1}) \le \alpha_n V(z_n)\,. \tag{7}$$

In view of $f(n,0) = 0$ the inequality (7) holds also if $z_n = 0$. Put $\beta = \gamma_0\varkappa$. We shall prove that $z_n \in \operatorname{Cl} K(\beta)$ for $n \in \mathbb{N}$. It holds $z_0 \in K(\nu)$. In view of (7) we obtain $V(z_1) \le \alpha_0 V(z_0) \le \alpha_0\gamma_0 \le \beta$. This implies $z_1 \in \operatorname{Cl} K(\beta)$. Suppose now for the contrary that there is an $n_1 \in \mathbb{N}$ such that $z_n \in \operatorname{Cl} K(\beta)$ for $n = 1, 2, \dots, n_1$, $z_{n_1+1} \notin \operatorname{Cl} K(\beta)$. The inequality (7) yields

$$V(z_{n_1+1}) \le \alpha_{n_1} V(z_{n_1}) \le \alpha_{n_1}\alpha_{n_1-1} V(z_{n_1-1}) \le \dots \le \Big(\prod_{j=0}^{n_1} \alpha_j\Big) V(z_0) \le \varkappa\gamma_0 = \beta\,,$$

which is a contradiction with $z_{n_1+1} \notin \operatorname{Cl} K(\beta)$. □

Remark 1. If $V(z) \ge \lambda_0$ for $z \in \Omega\backslash K(\lambda_0)$, then the condition (5) can be omitted, because (7) together with (3) imply $z_n \in K(\lambda_0)$ for $n \in \mathbb{N}$.

Theorem 2. *Suppose $0 \le \delta < \nu \le \lambda_0$. Assume there is a sequence $\{\alpha_n\}_{n=0}^{\infty}$ such that $\alpha_n \ge 0$ for $n \in \mathbb{N}_0$,*

$$\inf_{n\in\mathbb{N}_0} \prod_{k=0}^{n} \alpha_k = \varkappa > 0\,, \tag{8}$$

$$\Big|1 + \frac{f(n,z)}{z}\Big|\,\Big|\exp \int_z^{z+f(n,z)} r(\vartheta)d\vartheta\Big| \ge \alpha_n \tag{9}$$

for $n \in \mathbb{N}_0$, $z \in K(\delta,\nu)$. If a solution $\{z_n\}_{n=0}^{\infty}$ of (2) satisfies

$$z_n \in K(\delta,\nu) \quad \textit{for} \quad n \in \mathbb{N}_0\,, \tag{10}$$

$$z_0 \in \hat{K}(\gamma_0)\,, \tag{11}$$

where $\delta < \gamma_0 < \nu$, $\delta < \gamma_0\varkappa < \nu$, then

$$z_n \notin K(\gamma_0\varkappa) \quad \textit{for} \quad n \in \mathbb{N}\,. \tag{12}$$

Proof. Let $\{z_n\}$ be any solution of (2) satisfying (10), (11). Similarly as before we have

$$\begin{aligned} \Delta V(z_n) &= V(z_n)\Big[\Big|1 + \frac{f(n,z_n)}{z_n}\Big|\,\Big|\exp\int_{z_n}^{z_n+f(n,z_n)} r(\vartheta)d\vartheta\Big| - 1\Big] \\ &\ge (\alpha_n - 1)V(z_n) \end{aligned}$$

and subsequently

$$V(z_{n+1}) \ge \alpha_n V(z_n)$$

for any $n \in \mathbb{N}_0$ such that $z_n \in K(\delta,\nu)$. Put $\beta = \gamma_0\varkappa$. Then $V(z_1) \ge \alpha_0 V(z_0) \ge \alpha_0\gamma_0 \ge \varkappa\gamma_0$ and $z_1 \notin K(\beta)$. We have to prove that (12) holds. Suppose on the contrary that (12) is not true. Then there exists an $n_1 \in \mathbb{N}$ such that $z_n \in K(\delta,\nu)$ and $z_n \notin K(\beta)$ for $n = 1, 2, \dots, n_1$, and, $z_{n_1+1} \in K(\delta,\beta)$. Now

$$V(z_{n_1+1}) \ge \alpha_{n_1} V(z_{n_1}) \ge \alpha_{n_1}\alpha_{n_1-1} V(z_{n_1-1}) \ge \dots \ge \Big(\prod_{j=0}^{n_1} \alpha_j\Big) V(z_0) \ge \varkappa\gamma_0 = \beta\,.$$

This contradicts to $z_{n_1+1} \in K(\delta,\beta)$. □

Theorem 3. *Suppose* $0 \le \delta \le \beta < \gamma_0 < \nu \le \lambda_0$, $N \in \mathbb{N}$. *Assume that there is a sequence* $\{\alpha_n\}_{n=0}^{\infty}$ *such that* $0 \le \alpha_n \le 1$ *for* $n \in \mathbb{N}_0$,

$$\prod_{k=0}^{N} \alpha_k < \gamma_0^{-1}\beta\,, \tag{13}$$

$$\Big|1 + \frac{f(n,z)}{z}\Big|\Big|\exp\int_z^{z+f(n,z)} r(\vartheta)d\vartheta\Big| \le \alpha_n \tag{14}$$

for $n = 0, 1, \dots, N$, $z \in K(\delta, \nu)$. *If a solution* $\{z_n\}_{n=0}^{\infty}$ *of* (2) *satisfies*

$$z_n \in K(\lambda_0) \quad for \quad n = 0, 1, \dots, N+1\,, \tag{15}$$

$$z_0 \in \hat{K}(\gamma_0)\,, \tag{16}$$

then there is an $n_1 \in \mathbb{N}$ *such that*

$$z_{n_1} \in K(\beta), \quad z_n \in K(\delta,\nu) \setminus K(\beta) \quad for \quad n = 0, 1, \dots, n_1 - 1\,.$$

Proof. Clearly $z_0 \in K(\delta,\nu) \setminus K(\beta)$ holds. Suppose that $z_k \in K(\delta,\nu)$, where $k \in \{0, 1, \dots, N\}$. Then

$$\Delta V(z_k) \le (\alpha_k - 1)V(z_k)$$

and

$$V(z_{k+1}) \le \alpha_k V(z_k) \le \alpha_k \alpha_{k-1} V(z_{k-1}) \le \dots \le \Big(\prod_{j=0}^{k} \alpha_j\Big) V(z_0) \le \gamma_0 < \nu\,.$$

This implies that

$$z_{k+1} \in K(\nu)\,. \tag{17}$$

If $z_k \notin K(\beta)$ for $k = 0, 1, \dots, N+1$ then, in view of previous consideration, $z_k \in K(\delta,\nu) \setminus K(\beta)$ for $k = 0, 1, \dots, N+1$ and

$$\beta \le V(z_{N+1}) \le \Big(\prod_{j=0}^{N} \alpha_j\Big) V(z_0) < \gamma_0^{-1}\beta\gamma_0 = \beta\,,$$

which is a contradiction. Therefore there exists an $n_1 \in \{1, 2, \dots, N+1\}$ such that $z_{n_1} \in K(\beta)$ and $z_n \in K(\delta,\nu) \setminus K(\beta)$ for $n = 0, 1, \dots, n_1 - 1$. □

Remark 2. If $V(z) \ge \nu$ for $z \in \Omega \setminus K(\lambda_0)$, then the condition (15) can be omitted.

Theorem 4. *Suppose* $0 \le \delta < \gamma_0 < \beta \le \nu \le \lambda_0$, $N \in \mathbb{N}$. *Assume that there is a sequence* $\{\alpha_n\}_{n=0}^{\infty}$ *such that* $\alpha_n \ge 1$ *for* $n \in \mathbb{N}_0$,

$$\prod_{k=0}^{N} \alpha_k > \gamma_0^{-1}\beta\,, \tag{18}$$

$$\Big|1 + \frac{f(n,z)}{z}\Big|\Big|\exp\int_z^{z+f(n,z)} r(\vartheta)d\vartheta\Big| \ge \alpha_n \tag{19}$$

for $n = 0, 1, \dots, N$, $z \in K(\delta, \nu)$. *If a solution* $\{z_n\}_{n=0}^{\infty}$ *of* (2) *satisfies*

$$z_n \in K(\lambda_0) \quad \textit{for} \quad n \in \mathbb{N}_0\,, \tag{20}$$

$$z_0 \in \hat{K}(\gamma_0)\,, \tag{21}$$

then there is an $n_1 \in \mathbb{N}$ *such that*

$$z_{n_1} \in K(\beta, \lambda_0), \quad z_n \in K(\delta, \nu) \cap \operatorname{Cl} K(\beta) \quad \textit{for} \quad n = 0, 1, \dots, n_1 - 1\,.$$

Proof. It holds that $z_0 \in K(\delta, \nu) \cap \operatorname{Cl} K(\beta)$. Suppose $z_k \in K(\delta, \nu)$, where $k \in \{0, 1, \dots, N\}$. Now

$$\Delta V(z_k) \geq (\alpha_k - 1) V(z_k)$$

and

$$V(z_{k+1}) \geq \alpha_k V(z_k) \geq \alpha_k \alpha_{k-1} V(z_{k-1}) \geq \dots \geq \Big(\prod_{j=0}^{k} \alpha_j \Big) V(z_0) \geq \gamma_0 > \delta\,.$$

This together with $z_{k+1} \in K(\lambda_0)$ yields

$$z_{k+1} \in K(\delta, \lambda_0)\,. \tag{22}$$

If $z_k \notin K(\beta, \lambda_0)$ for $k = 0, 1, \dots, N+1$ then, in view of the previous consideration, $z_k \in K(\delta, \nu) \cap \operatorname{Cl} K(\beta)$ for $k = 0, 1, \dots, N+1$ and

$$\beta \geq V(z_{N+1}) \geq \Big(\prod_{j=0}^{N} \alpha_j \Big) V(z_0) > \gamma_0^{-1} \beta \gamma_0 = \beta\,,$$

a contradiction. Therefore there exists an $n_1 \in \{1, 2, \dots, N+1\}$ such that $V(z_{n_1}) \in K(\beta, \lambda_0)$, $z_n \in K(\delta, \nu) \cap \operatorname{Cl} K(\beta)$ for $n = 0, 1, \dots, n_1 - 1$. ☐

Theorem 5. *Suppose* $0 < \nu \leq \lambda_0$, $f(n, 0) = 0$ *for* $n \in \mathbb{N}_0$. *Assume that there is a sequence* $\{\alpha_n\}_{n=0}^{\infty}$ *such that* $0 \leq \alpha_n \leq 1$ *for* $n \in \mathbb{N}_0$,

$$\lim_{n \to \infty} \prod_{k=0}^{n} \alpha_k = 0\,, \tag{23}$$

$$\Big| 1 + \frac{f(n,z)}{z} \Big| \Big| \exp \int_{z}^{z + f(n,z)} r(\vartheta) d\vartheta \Big| \leq \alpha_n \tag{24}$$

for $n \in \mathbb{N}_0$, $z \in K(0, \nu)$. *If a solution* $\{z_n\}_{n=0}^{\infty}$ *of* (2) *satisfies*

$$z_n \in K(\nu) \quad \textit{for} \quad n \in \mathbb{N}_0\,,$$

then

$$\lim_{n \to \infty} z_n = 0\,.$$

Proof. In view of the condition $f(n, 0) = 0$ it can be assumed without loss of generality that $z_n \neq 0$ for $n \in \mathbb{N}_0$. Put $\gamma_0 = V(z_0)$. Similarly as before we have

$$V(z_{n+1}) \leq \Big(\prod_{k=0}^{n} \alpha_k \Big) V(z_0) = \Big(\prod_{k=0}^{n} \alpha_k \Big) \gamma_0\,.$$

Using (23) we get $\lim_{n\to\infty} V(z_{n+1}) = 0$. Since V is positive definite, we have $\lim_{n\to\infty} z_{n+1} = 0$. □

Remark 3. If $V(z) \geq \nu$ for $z \in \Omega \setminus K(\nu)$, then the condition $z_n \in K(\nu)$ for $n \in \mathbb{N}_0$ can be replaced by $z_0 \in K(\nu)$.

Theorem 6. *Suppose* $0 \leq \delta < \lambda_0$. *Assume there is a sequence* $\{\alpha_n\}_{n=0}^{\infty}$ *such that* $\alpha_n \geq 0$ *for* $n \in \mathbb{N}_0$*,*

$$\liminf_{n\to\infty} \prod_{k=0}^{n} \alpha_k = \varkappa > 1\,, \tag{25}$$

$$\Big|1 + \frac{f(n,z)}{z}\Big|\Big|\exp \int_z^{z+f(n,z)} r(\vartheta)d\vartheta\Big| \geq \alpha_n \tag{26}$$

for $n \in \mathbb{N}_0$*,* $z \in K(\delta, \lambda_0)$. *If a solution* $\{z_n\}_{n=0}^{\infty}$ *of* (2) *satisfies*

$$z_n \in K(\delta, \lambda_0) \quad \textit{for} \quad n \in \mathbb{N}_0\,,$$

and $z_0 \in \hat{K}(\gamma_0)$*, where* $\gamma_0 \varkappa \geq \lambda_0$ *then*

$$\lim_{n\to\infty} V(z_n) = \lambda_0\,.$$

Proof. Similarly as before we have

$$V(z_{n+1}) \geq \Big(\prod_{k=0}^{n} \alpha_k\Big) V(z_0) = \Big(\prod_{k=0}^{n} \alpha_k\Big)\gamma_0 \geq \Big(\prod_{k=0}^{n} \alpha_k\Big)\varkappa^{-1}\lambda_0\,.$$

Moreover $V(z_n) \leq \lambda_0$ in view of $z_n \in K(\delta, \lambda_0)$. With respect to (25) we obtain

$$\lim_{n\to\infty} V(z_n) = \liminf_{n\to\infty} V(z_n) = \lambda_0\,.$$

□

3. Applications

Consider a Riccati difference equation

$$\Delta z_n = \frac{(z_n - a)(b_n - z_n)}{z_n - q_n}\,, \tag{27}$$

where $a \in \mathbb{C}$, $b_n, q_n \in \mathbb{C}$ for $n \in \mathbb{N}_0$. The equation (27) can be written in the form

$$z_{n+1} = \frac{[(a + b_n) - q_n]z_n - ab_n}{z_n - q_n}\,.$$

A substitution $w = z - a$ transfers (27) to

$$\Delta w_n = \frac{w_n(b_n - a - w_n)}{w_n + a - q_n}\,.$$

Writing z_n instead of w_n in the last equation, we have

$$\Delta z_n = \frac{z_n(b_n - a - z_n)}{z_n + a - q_n}\,. \tag{28}$$

Notice that the function $f(n,z)=z(b_n-a-z_n)/(z-a-q_n)$ satisfies the condition $f(n,0)=0$ which is required in Theorem 1 and Theorem 5. This is a reason for our supposition that a is constant in (27) and not a sequence.

The right-hand side of (28) can be written as

$$\frac{1}{|z_n+a-q_n|^2}\, z_n\,(b_n-a-z_n)\overline{(z_n+a-q_n)}\,. \tag{29}$$

If we replace z_n by z and suppose that b_n is sufficiently close to $b\in\mathbb{C}\backslash\{a\}$, then neglecting the real factor $1/|z_n+a-q_n|^2$ and a nonholomorphic factor $\overline{(z_n+a-q_n)}$, we can try to suppose that the function (29) is close to a holomorphic function

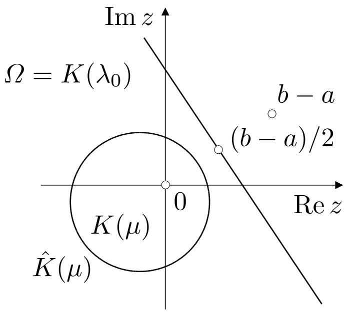
$$h(z)=z(b-a-z)\,. \tag{30}$$

Putting $\Omega=\{z\in\mathbb{C}:2\operatorname{Re}[(\bar b-\bar a)z]<|a-b|^2\}$, we observe $h'(z)=b-a-2z$, $h'(0)=b-a\neq 0$, $r(z)=1/(b-a-z)$, $\lambda_0=|a-b|$,

$$v(z)=(a-b)z/(z-b+a)\,,$$

$$V(z)=|a-b|\frac{|z|}{|z-b+a|}\,. \tag{31}$$

Then we have $K(\mu)=\{z\in\mathbb{C}:|a-b||z|<\mu|z-b+a|\}$ for $0<\mu\le|a-b|$ and $K(\lambda_0)=K(|a-b|)=\Omega$. If $0<\mu<|a-b|$, then $\hat K(\mu)=\{z\in\mathbb{C}:|a-b||z|=\mu|z-b+a|\}$ and $\hat K(\mu)$ are circles (see figure).



Suppose $0<\nu\le|a-b|$ and put $f(n,z)=z(b_n-a-z)/(z+a-q_n)$. It holds that

$$\begin{aligned}
&\Big|1+\frac{f(n,z)}{z}\Big|\Big|\exp\int_z^{z+f(n,z)} r(\zeta)d\zeta\Big|\\
&=\Big|1+\frac{f(n,z)}{z}\Big|\Big|\frac{z+a-b}{z+f(n,z)+a-b}\Big|\\
&=\Big|1+\frac{b_n-a-z}{z+a-q_n}\Big|\left|\frac{z+a-b}{z+\frac{z(b_n-a-z)}{z+a-q_n}+a-b}\right|\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left|1+\frac{f(n,z)}{z}\right|\left|\exp\int_z^{z+f(n,z)} r(\zeta)d\zeta\right| \\
&= \left|1+\frac{b_n-a-z}{z+a-q_n}\right|\left|1+\frac{z(b_n-a-z)}{(z+a-q_n)(z+a-b)}\right|^{-1} \\
&= |b_n-q_n|\left|z+a-q_n+\frac{z(b_n-a-z)}{z-b+a}\right|^{-1} \\
&= \frac{|b_n-q_n||z-b+a|}{|(a+b_n-b-q_n)(z+a)-ab_n+bq_n|} \\
&= \frac{|b_n-q_n|}{\left|a+b_n-b-q_n+\frac{(b_n-b)(b-a)}{z+a-b}\right|}
\end{aligned}$$

for $z \in K(0,\nu)$. Clearly $z \in K(0,\nu)$ if and only if $0 < |a-b||z|/|z-b+a| < \nu$. Putting $w = (a-b)z/(z-b+a)$, we obtain $(a-b)z = (z-b+a)w$, $z = (b-a)w/(w-a+b)$, $z-b+a = -(b-a)^2/(w-a+b)$, $0 < |w| < \nu$. Hence

$$\begin{aligned}
&\left|1+\frac{f(n,z)}{z}\right|\left|\exp\int_z^{z+f(n,z)} r(\zeta)d\zeta\right| \\
&= \frac{|b_n-q_n|}{\left|b_n+a-b-q_n+\frac{(w-a+b)(b_n-b)}{a-b}\right|} \\
&= \frac{|b_n-q_n||a-b|}{|b_n-b|\left|\frac{(b_n+a-b-q_n)(a-b)}{b_n-b}+w-a+b\right|} \\
&= \frac{|b_n-q_n||a-b|}{|b_n-b|\left|\frac{(a-q_n)(a-b)}{b_n-b}+w\right|} \\
&\le \frac{|b_n-q_n||a-b|}{|b_n-b|\left|\frac{(a-q_n)(a-b)}{b_n-b}+\nu\frac{(q_n-a)(a-b)}{b_n-b}\frac{|b_n-b|}{|q_n-a||a-b|}\right|} \\
&\le \frac{|b_n-q_n||a-b|}{|q_n-a||a-b|-\nu|b_n-b|}\,,
\end{aligned}$$

if we assume $|b_n-b| < \nu^{-1}|a-b||q_n-a|$. Similarly we obtain the inequality

$$\left|1+\frac{f(n,z)}{z}\right|\left|\exp\int_z^{z+f(n,z)} r(\zeta)d\zeta\right| \ge \frac{|b_n-q_n||a-b|}{|q_n-a||a-b|+\nu|b_n-b|}\,.$$

Remark 4. The condition $h(z)=0 \iff z=0$ is satisfied on Ω. However this condition is not true on $\mathbb{C}$. Nevertheles the functions h, v and V are defined not only on Ω but even on $\mathbb{C}$. The function v is meromorphic on $\mathbb{C}$ with a pole at the point $b-a$. It can be easily seen that, on the assumption $|b_n-b| < \nu^{-1}|a-b||q_n-a|$,

$$V(z_{k+1}) \leq \frac{|b_k - q_k||a-b|}{|q_k - a||a-b| - \nu|b_k - b|} V(z_k)$$

holds for any solution $\{z_n\}_{n=0}^{\infty}$ of (28) and any $k \in \mathbb{N}_0$ such that $z_k \in K(\nu)$ without the supposition $z_{k+1} \in \Omega$. Obviously Remarks 1–3 remain true if we replace Ω by $\mathbb{C}$.

Theorem 7. *Let $a, b \in \mathbb{C}$, $a \neq b$, $b_n, q_n \in \mathbb{C}$ for $n \in \mathbb{N}_0$, $0 < \vartheta \leq 1$ and*

$$|b_n - b| < \vartheta^{-1}\,|q_n - a| \quad \textit{for} \quad n \in \mathbb{N}_0\,.$$

Suppose there is a sequence $\{\alpha_n\}_{n=0}^{\infty}$ such that $\alpha_n \geq 0$ for $n \in \mathbb{N}_0$,

$$\sup_{n\in\mathbb{N}_0} \prod_{k=0}^{n} \alpha_k = \varkappa < \infty$$

and

$$\frac{|b_n - q_n|}{|q_n - a| - \vartheta|b_n - b|} \leq \alpha_n$$

for $n \in \mathbb{N}_0$. If a solution $\{z_n\}_{n=0}^{\infty}$ of (27) satisfies $|z_0 - a| \leq \delta_0|z_0 - b|$, where $0 < \delta_0 \max(1, \varkappa) < \vartheta$, then

$$|z_n - a| \leq \delta_0 \varkappa |z_n - b|$$

for $n \in \mathbb{N}$.

Proof. Put $\vartheta = \nu|a-b|^{-1}$, $\delta_0 = \gamma_0|a-b|^{-1}$ and define h and V by (30) and (31), respectively. Applying Theorem 1 to the equation (28) and transferring the variable z back to that of the equation (27), we obtain the given result. Notice that $V(z) \geq \lambda_0$ for $z \in \mathbb{C} \setminus K(\lambda_0)$ and Remark 1 together with Remark 4 can be used. ☐

Theorem 8. *Let $a, b \in \mathbb{C}$, $a \neq b$, $b_n, q_n \in \mathbb{C}$ for $n \in \mathbb{N}_0$, $0 \leq \theta < \delta_0 < \vartheta \leq 1$, $N \in \mathbb{N}$,*

$$|b_n - b| < \vartheta^{-1}\,|q_n - a| \quad \textit{for} \quad n \in \mathbb{N}_0\,.$$

Assume that there is a sequence $\{\alpha_n\}_{n=0}^{\infty}$ such that $0 \leq \alpha_n \leq 1$ for $n \in \mathbb{N}_0$,

$$\prod_{k=0}^{N} \alpha_k < \delta_0^{-1}\theta$$

and

$$\frac{|b_n - q_n|}{|q_n - a| - \vartheta|b_n - b|} \leq \alpha_n$$

for $n = 0, 1, \ldots, N$. If a solution $\{z_n\}_{n=0}^{\infty}$ of (27) satisfies $|z_0 - a| = \delta_0|z_0 - b|$, then there is an $n_1 \in \mathbb{N}$ such that

$$\begin{gathered} |z_{n_1} - a| < \theta|z_{n_1} - b|\,, \\ \theta|z_n - b| \leq |z_n - a| < \vartheta|z_n - b| \quad \textit{for} \quad n = 0, 1, \ldots, n_1 - 1\,. \end{gathered} \tag{32}$$

Proof. Since $0 \le \alpha_n \le 1$, the assumptions of Theorem 7 are fulfilled with $\delta = 0$, $\vartheta = \nu|a-b|^{-1}$, $\delta_0 = \gamma_0|a-b|^{-1}$, $\theta = \beta|a-b|^{-1}$, $\varkappa = 1$. Hence $|z_n - a| \le \delta_0|z_n - b|$ for $n \in \mathbb{N}$. From Theorem 3 and from Remarks 2,4 it follows that there is an $n_1 \in \mathbb{N}$ such that (32) holds true. ☐

Theorem 9. *Let $a, b \in \mathbb{C}$, $a \ne b$, $b_n, q_n \in \mathbb{C}$ for $n \in \mathbb{N}_0$, $0 < \vartheta \le 1$ and*

$$|b_n - b| < \vartheta^{-1}\,|q_n - a| \quad \text{for} \quad n \in \mathbb{N}_0\,.$$

Assume that there is a sequence $\{\alpha_n\}_{n=0}^{\infty}$ such that $0 \le \alpha_n \le 1$ for $n \in \mathbb{N}_0$,

$$\lim_{n\to\infty} \prod_{k=0}^{n} \alpha_k = 0$$

and

$$\frac{|b_n - q_n|}{|q_n - a| - \vartheta|b_n - b|} \le \alpha_n$$

for $n \in \mathbb{N}_0$. If a solution $\{z_n\}_{n=0}^{\infty}$ of (27) satisfies $|z_0 - a| < \vartheta|z_0 - b|$, then

$$\lim_{n\to\infty} z_n = a\,. \tag{33}$$

Proof. Since $0 \le \alpha_n \le 1$, the assumptions of Theorem 7 are satisfied with $\varkappa = 1$. Hence $|z_n - a| < \vartheta|z - b|$ for $n \in \mathbb{N}$. Using Theorem 5 and Remarks 3,4 with $\vartheta = \nu|a-b|^{-1}$, we obtain (33). ☐

Theorem 10. *Let $a, b \in \mathbb{C}$, $a \ne b$, $b_n, q_n \in \mathbb{C}$ for $n \in \mathbb{N}_0$, $0 < \vartheta \le 1$ and $|b_n - b| < \nu^{-1}|q_n - a|$ for $n \in \mathbb{N}_0$. Assume there is a sequence $\{\alpha_n\}_{n=0}^{\infty}$ such that $\alpha_n \ge 0$ for $n \in \mathbb{N}_0$,*

$$\inf_{n\in\mathbb{N}_0} \prod_{k=0}^{n} \alpha_k = \varkappa > 0\,,$$

and

$$\frac{|b_n - q_n|}{|q_n - a| + \vartheta|b_n - b|} \ge \alpha_n$$

for $n \in \mathbb{N}_0$. If a solution $\{z_n\}_{n=0}^{\infty}$ of (27) satisfies conditions $0 < |z_n - a| < \vartheta|z_n - b|$ for $n \in \mathbb{N}_0$, $|z_0 - a| = \delta_0|z_0 - b|$, where $0 < \delta_0 \max(1, \varkappa) < \vartheta$, then

$$|z_n - a| \ge \delta_0\varkappa|z_n - b|$$

for $n \in \mathbb{N}$.

Proof. Putting $\delta = 0$, $\vartheta = \nu|a-b|^{-1}$, $\delta_0 = \gamma_0|a-b|^{-1}$ and applying Theorem 2, we obtain the statement of Theorem 10. ☐

Theorem 11. *Let $a, b \in \mathbb{C}$, $a \ne b$, $b_n, q_n \in \mathbb{C}$ for $n \in \mathbb{N}_0$, $0 < \delta_0 < \theta < \vartheta \le 1$, $N \in \mathbb{N}$ and $|b_n - b| < \nu^{-1}|q_n - a|$ for $n \in \mathbb{N}_0$. Assume that there is a sequence $\{\alpha_n\}_{n=0}^{\infty}$ such that $\alpha_n \ge 1$ for $n \in \mathbb{N}_0$,*

$$\prod_{k=0}^{N} \alpha_k > \delta_0^{-1}\theta$$

and

$$\frac{|b_n - q_n|}{|q_n - a| + \vartheta |b_n - b|} \geq \alpha_n$$

for $n = 0, 1, \ldots, N$. *If a solution* $\{z_n\}_{n=0}^{\infty}$ *of* (27) *satisfies conditions* $|z_n - a| < |z_n - b|$ *for* $n \in \mathbb{N}_0$, $|z_0 - a| = \delta_0 |z_0 - b|$, *then there is an* $n_1 \in \mathbb{N}$ *such that*

$$\theta |z_{n_1} - b| < |z_{n_1} - a| < |z_{n_1} - b|, \quad 0 < |z_n - a| \leq \theta |z_n - b| \quad \text{for} \quad n = 0, 1, \ldots, n_1 - 1\,.$$

Proof. The result follows from Theorem 4, if we put $\delta = 0$, $\vartheta = \nu |a - b|^{-1}$, $\delta_0 = \gamma_0 |a - b|^{-1}$, $\theta = \beta |a - b|^{-1}$. □

Theorem 12. *Let* $a, b \in \mathbb{C}$, $a \neq b$, $b_n, q_n \in \mathbb{C}$ *for* $n \in \mathbb{N}_0$ *and* $|b_n - b| < |q_n - a|$ *for* $n \in \mathbb{N}_0$. *Assume there is a sequence* $\{\alpha_n\}_{n=0}^{\infty}$ *such that* $\alpha_n \geq 0$ *for* $n \in \mathbb{N}_0$ *and*

$$\liminf_{n \to \infty} \prod_{k=0}^{n} \alpha_k = \varkappa > 1\,,$$

$$\frac{|b_n - q_n|}{|q_n - a| + |b_n - b|} \geq \alpha_n$$

for $n \in \mathbb{N}_0$. *If a solution* $\{z_n\}_{n=0}^{\infty}$ *of* (27) *satisfies* $0 < |z_n - a| < |z_n - b|$ *for* $n \in \mathbb{N}_0$ *and* $|z_0 - a| = \delta_0 |z_0 - b|$, *where* $\delta_0 \varkappa \geq 1$, $\delta_0 < 1$, *then*

$$\lim_{n \to \infty} \frac{|z_n - a|}{|z_n - b|} = 1\,.$$

Proof. The result follows from Theorem 6, if we put $\delta = 0$, $\delta_0 = \gamma_0 |a - b|^{-1}$. □

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Department of Mathematical Analysis
Faculty of Science, Masaryk University
Janáčkovo nám. 2a, 602 00 Brno, Czech Republic
E-mail: kalas@math.muni.cz