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DISCRETE SINGULAR FUNCTIONALS

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Abstract. In the paper the discrete version of the Morse's singularity condition is established. This condition ensures that the discrete functional over the unbounded interval is positive semidefinite on the class of the admissible functions. Two types of admissibility are considered.

1. Introduction

The relationship between the extremal values of the quadratic functional and the properties of the corresponding Euler-Lagrange differential equation is well known in both continuous and discrete case. One of the classical results of the calculus states that the quadratic functional is positive (semi-)definite on the class of admissible functions with zero boundary conditions if and only if the corresponding Euler-Lagrange differential equation is disconjugate, see [2] and [3] for details. This property has been extended in many directions which include among others the general boundary conditions, the vector case and the singular functionals. Řehák [9] studied for $p > 1$, $r_k \in \mathbb{R} \setminus \{0\}$ and $c_k \in \mathbb{R}$ the discrete p-degree functional

$$J(x; 0, n) = \sum_{k=0}^{n} r_k |\Delta x_k|^p - c_k |x_{k+1}|^p \,. \tag{1}$$

which can be viewed as a generalization of the discrete quadratic functional. The corresponding Euler-Lagrange equation for functional (1) is the equation

$$L_k[y] = \Delta(r_k \Delta\Phi(y_k)) + c_k \Phi(y_{k+1}) = 0\,, \tag{2}$$

where $\Phi(y) = |y|^{p-2} y$ is a generalized power function. For $p = 2$ equation (2) is a linear equation, however in the general case $p \neq 2$ the additivity of the set of the solutions is lost and only homogeneity remains. From this reason equation (2) is usually referred as a half-linear equation. In the sequel we introduce the

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concept of oscillation which is connected with equation (2). Remark that under the interval I we actually mean the discrete set $I \cap \mathbb{N}$ throughout the paper.

Definition 1.1 (generalized zero)**.** An interval $(m, m+1]$ is said to contain a *generalized zero* of a solution $y = (y_k)$ of equation (2) if $y_m \neq 0$ and $r_m y_m y_{m+1} \leq 0$.

Definition 1.2 (disconjugacy)**.** Equation (2) is said to be *disconjugate* on the interval $[m, n]$ if every solution of (2) has at most one generalized zero on the interval $(m, n+1]$ and the solution satisfying $y_m = 0$ has no generalized zero on the interval $(m, n+1]$. Equation (2) is said to be disconjugate on the interval $[m, \infty)$ if it is disconjugate on the interval $[m, n]$ for every n, $n > m$.

Řehák [9] studied the positive definiteness of the functional (3) and proved the following theorem.

Theorem 1.1 (Řehák, [9])**.** *The functional* (1) *satisfies the conditions*

(i) $J(x; 0, n) \geq 0$ *and*
(ii) $J(x; 0, n) = 0$ *if and only if* $x = 0$

for every sequence $x = (x_k)_{k=0}^{n+1}$ *with boundary conditions* $x_0 = 0 = x_{n+1}$ *if and only if equation* (2) *is disconjugate on* $[0, n]$*.*

The aim of this paper is to extend Theorem 1.1 to the case of the singular functional

$$\liminf_{n \to \infty} J(x; 0, n) \tag{3}$$

where $J(x; 0, n)$ is defined by (1). The functional is studied on the set of the real sequences $x = (x_k)_{k=0}^{\infty}$ which are admissible in the sense of the following definition.

Definition 1.3 (admissibility)**.** The sequence $x = (x_k)_{k=0}^{\infty}$ of the real numbers is said to be an *admissible sequence* if

$$x_0 = 0 = \lim_{k \to \infty} x_k$$

holds.

We will focus our attention on the necessary and sufficient condition for the positive semidefiniteness (rather than the positive definiteness as in [9]) of the singular functional (3).

Definition 1.4 (positive semidefiniteness)**.** The functional (3) is said to be *positive semidefinite* on the class of the admissible sequences if

$$\liminf_{n \to \infty} J(x; 0, n) \geq 0 \tag{4}$$

for every admissible sequence x.

The interest in the study of the (continuous) singular quadratic functional

$$\liminf_{b \to \infty} \int_a^b \left[r(t)|\eta'(t)|^2 - c(t)|\eta(t)|^2 \right] \mathrm{d}t \tag{5}$$

has been initiated by Leighton and Morse in the paper [6] and continued by papers [1, 4, 5]. (In [6] is, in fact, the singular point $t = 0$ considered, instead of the singular point $t = \infty$ in (5). Nevertheless the transformation $t \to \frac{1}{t}$ transforms the singular point 0 into ∞, as it is considered e.g. in [1].) In [6] it is showed that the disconjugacy of the Euler-Lagrange equation is no more sufficient for the positive semidefiniteness of the singular functional. To establish a necessary and sufficient condition for the positive semidefiniteness of (5) a concept of singularity condition is introduced. This condition together with disconjugacy presents the desired necessary and sufficient condition for positive semidefiniteness of (5) on the class of the admissible functions with zero boundary conditions.

The aim of this paper is to extend the Leighton-Morse's concept of singularity condition for the case of the discrete p-degree functional (3).

2. Main results

First let us present the discrete singularity condition.

Definition 2.1 (singularity condition)**.** Let $y = (y_k)$ be a solution of (2) given by the initial conditions

$$y_0 = 0 \quad \text{and} \quad y_1 = 1. \tag{6}$$

The functional (3) is said to satisfy the *singularity condition* if

$$\liminf_{n\to\infty} |x_n|^p r_n \frac{\Phi(\Delta y_n)}{\Phi(y_n)} \geq 0 \tag{7}$$

for every admissible sequence x along which the functional (3) is finite.

Lemma 2.1 (Picone identity, [9])**.** *Let $x = (x_k)_{k=0}^{n+1}$, $y = (y_k)_{k=0}^{n+2}$ be real sequences, $L_k[y] = 0$ for $k \in [0, n]$ and $y_k \neq 0$ for $k \in [1, n+1]$ ($k \in [0, n+1]$). Then for $k \in [1, n]$ ($k \in [0, n]$)*

$$\Delta\left\{-|x_k|^p r_k \frac{\Phi(\Delta y_k)}{\Phi(y_k)}\right\} = c_k|x_{k+1}|^p - r_k|\Delta x_k|^p + \frac{r_k y_k}{y_{k+1}} G_k(x, y) \tag{8}$$

where

$$G_k(x, y) = \frac{y_{k+1}}{y_k}|\Delta x_k|^p - \frac{y_{k+1}\Phi(\Delta y_k)}{y_k \Phi(y_{k+1})}|x_{k+1}|^p + \frac{y_{k+1}\Phi(\Delta y_k)}{y_k \Phi(y_k)}|x_k|^p\,. \tag{9}$$

The function G satisfies

$$G_k(x, y) \geq 0 \tag{10}$$

with equality if and only if $\Delta x_k = x_k \frac{\Delta y_k}{y_k}$, i.e. if and only if $x_{k+1} = x_k \frac{y_{k+1}}{y_k}$.

Lemma 2.2. *If $x_0 = 0$ and equation (2) is disconjugate on the interval $[0, n]$, then*

$$J(x; 0, n) = |x_{n+1}|^p r_{n+1} \frac{\Phi(\Delta y_{n+1})}{\Phi(y_{n+1})} + \sum_{k=1}^{n} \frac{r_k y_k}{y_{k+1}} G_k(x, y)\,, \tag{11}$$

where $y = (y_k)$ is a solution of (2) which satisfies the initial conditions (6).

Proof. The sum of the Picone identity (8) over the interval $[1,n]$ gives

$$J(x;1,n)=|x_{n+1}|^p r_{n+1}\frac{\Phi(\Delta y_{n+1})}{\Phi(y_{n+1})}-|x_1|^p r_1\frac{\Phi(\Delta y_1)}{\Phi(y_1)}+\sum_{k=1}^{n}\frac{r_k y_k}{y_{k+1}}G_k(x,y)\,.$$

From (2) follows

$$r_1\Phi(\Delta y_1)-r_0\Phi(\Delta y_0)+c_0\Phi(y_1)=0\,.$$

Since $\Delta y_0=y_1$ we have from here

$$\frac{r_1\Phi(\Delta y_1)}{\Phi(y_1)}-r_0+c_0=0$$

and hence

$$J(x;1,n)=|x_{n+1}|^p r_{n+1}\frac{\Phi(\Delta y_{n+1})}{\Phi(y_{n+1})}-r_0|x_1|^p+c_0|x_1|^p+\sum_{k=1}^{n}\frac{r_k y_k}{y_{k+1}}G_k(x,y)\,.$$

The last relation with the fact that $x_1=\Delta x_0$ imply

$$\begin{aligned}J(x;0,n)&=J(x;1,n)+r_0|x_1|^p-c_0|x_1|^p\\&=|x_{n+1}|^p r_{n+1}\frac{\Phi(\Delta y_{n+1})}{\Phi(y_{n+1})}+\sum_{k=1}^{n}\frac{r_k y_k}{y_{k+1}}G_k(x,y)\,.\end{aligned}$$

The proof is complete. □

Theorem 2.1. *The functional* (3) *is positive semidefinite on the class of the admissible sequences if and only if equation* (2) *is disconjugate on the interval* $[0,\infty)$ *and the functional* (3) *satisfies singularity condition* (7)*.*

Proof. *Sufficient condition.* Suppose that equation (2) is disconjugate and the singularity condition (7) is satisfied. Let x be an admissible sequence. If $\liminf_{n\to\infty} J(x;0,n)=\infty$, then $\liminf_{n\to\infty} J(x;0,n)\geq 0$. Suppose that $\liminf J_{n\to\infty}(x;0,n)<\infty$. Then (11) holds for every $n>0$. Taking limes inferior of (11) and using inequality (10) we get

$$\begin{aligned}\liminf_{n\to\infty} J(x;0,n)&=\liminf_{n\to\infty}\Big(|x_{n+1}|^p r_{n+1}\frac{\Phi(\Delta y_{n+1})}{\Phi(y_{n+1})}+\sum_{k=1}^{n}\frac{r_k y_k}{y_{k+1}}G_k(x,y)\Big)\\&\geq\liminf_{n\to\infty}|x_{n+1}|^p r_{n+1}\frac{\Phi(\Delta y_{n+1})}{\Phi(y_{n+1})}\,.\end{aligned}$$

Hence (7) implies (4).

Necessary condition. Suppose that the functional is positive semidefinite for every admissible sequence $x=(x_k)_{k=0}^{\infty}$. We will continue in two steps: first we will show that (2) is disconjugate and the solution given by the initial conditions (6) has no generalized zero on $(0,\infty)$, and then we prove the validity of the singularity condition.

If the functional (3) is positive semidefinite and m is an arbitrary integer, then $J(x;0,m)$ is positive semidefinite on the class of the sequences satisfying zero boundary conditions $x_0 = 0 = x_{m+1}$. Really, if we define $x_k = 0$ for $k > m+1$, then

$$J(x;0,m) = \liminf_{n\to\infty} J(x;0,n) \geq 0\,. \tag{12}$$

Suppose, by contradiction, that the solution $y = (y_k)$ of IVP (2),(6) has a generalized zero on $(0,\infty)$. Then there exists $n > 0$ such that

$$\begin{aligned} &r_k y_k y_{k+1} > 0 \qquad k \in (0, n-1] \\ &r_n y_n y_{n+1} \leq 0\,. \end{aligned}$$

Let us consider the cases $y_{n+1} \neq 0$ and $y_{n+1} = 0$ separately. In each case we will show that the assumptions contradict (12).

Case I: $y_{n+1} \neq 0$. Let us consider the sequence $x = (x_k)$ defined with

$$x_k = \begin{cases} y_k & k \leq n \\ 0 & k = n+1\,. \end{cases}$$

Summation by parts, the definition of the sequence x (namely the relations $x_{n+1} = x_0 = 0$, $L_k[x] = 0$ for $k \leq n-2$, $\Delta x_{n-1} = \Delta y_{n-1}$, $x_n = y_n$ and $\Delta x_n = -y_n$) and the condition $y_{n+1} \neq 0$ give

$$\begin{aligned} J(x;0,n) &= \sum_{k=0}^{n} r_k|\Delta x_k|^p - c_k|x_{k+1}|^p \\ &= [x_k r_k \Phi(\Delta x_k)]_{k=0}^{n+1} - \sum_{k=0}^{n} x_{k+1} L_k[x] \\ &= -\sum_{k=0}^{n} x_{k+1} L_k[x] = -x_n L_{n-1}[x] \\ &= -x_n\Big[\Delta(r_{n-1}\Phi(\Delta x_{n-1})) + c_{n-1}\Phi(x_n)\Big] \\ &= -y_n\Big[r_n\Phi(\Delta x_n) - r_{n-1}\Phi(\Delta y_{n-1}) + c_{n-1}\Phi(y_n)\Big] \\ &= -y_n\Big[r_n\Phi(\Delta x_n) - r_n\Phi(\Delta y_n)\Big] \\ &= -y_n r_n \Phi(-y_n) + y_n r_n \Phi(\Delta y_n) \\ &= \frac{r_n y_n y_{n+1}}{y_{n+1}^2}|y_n|^p\Big[\frac{y_{n+1}}{y_n} + \frac{y_{n+1}}{y_n}\Phi\Big(\frac{y_{n+1}}{y_n} - 1\Big)\Big]\,. \end{aligned}$$

In view of the fact that $\alpha + \alpha\Phi(\alpha - 1) > 0$ for $\alpha \neq 0$ and $y_{n+1} \neq 0$, the expression in brackets is positive and the term $r_n y_n y_{n+1}$ is negative according to the assumptions. Hence it follows $J(x;0,n) < 0$, a contradiction to (12).

Case II: $y_{n+1} = 0$. The summation by parts proceeded in the same way as in the Case I shows

$$J(y;0,n) = -\sum_{k=0}^{n} y_{k+1} L_k[y] = 0\,.$$

Now let us consider the functional $J(x;0,n+1)$ and the sequence $x=(x_k)$ defined

$$x_k = \begin{cases} y_k & k \le n \\ \lambda & k = n+1 \\ 0 & k = n+2\,. \end{cases}$$

We will show that there exists $\lambda \in \mathbb{R}$ such that $J(x;0,n+1) < 0$. The sequences x and y differ on the interval $[0,n+1]$ only in the terms x_{n+1} and y_{n+1}. Hence

$$\begin{aligned} J(x;0,n+1) &= J(x;0,n) + r_{n+1}|\Delta x_{n+1}|^p - c_{n+1}|x_{n+2}|^p \\ &= J(y;0,n) - r_n|\Delta y_n|^p + c_n|y_{n+1}|^p \\ &\quad + r_n|\Delta x_n|^p - c_n|x_{n+1}|^p + r_{n+1}|\Delta x_{n+1}|^p - c_{n+1}|x_{n+2}|^p \\ &= 0 - r_n|y_n|^p + 0 + r_n|\Delta x_n|^p - c_n|x_{n+1}|^p + r_{n+1}|\Delta x_{n+1}|^p - 0\,. \end{aligned}$$

The definition of the sequence x gives

$$J(x;0,n+1) = -r_n|y_n|^p + r_n|\lambda - y_n|^p + c_n|\lambda|^p + r_{n+1}|\lambda|^p =: F(\lambda)\,.$$

The function $F(\lambda)$ satisfies $F(0)=0$ and

$$F'(0) = p\Big[r_n\Phi(\lambda - y_n) + c_n\Phi(\lambda) + r_{n+1}\Phi(\lambda)\Big]\Big|_{\lambda=0} = -pr_n\Phi(y_n) \neq 0$$

Hence, depending on the sign of the product $r_n\Phi(y_n)$, there exists either $\lambda_0 > 0$ or $\lambda_0 < 0$ such that

$$J(x;0,n+1) = F(\lambda_0) < 0$$

which contradicts (12).

Hence neither Case I, nor Case II, can occur and the solution of the initial problem (2)–(6) has no generalized zero on $(0,\infty)$. By the Sturm–type separation theorem for the solutions of equation (2) (see [9] for details) every other linearly independent solution of (2) has at most one generalized zero on $(0,\infty)$.

The second step is to show that the singularity condition holds. Suppose that x is an admissible sequence for which (3) is finite. Let $(n_t)_{t=0}^{\infty}$ be an increasing unbounded sequence of the integers such that

$$\liminf_{n\to\infty} J(x;0,n) = \lim_{t\to\infty} J(x;0,n_t)\,. \tag{13}$$

We state that

$$\liminf_{t\to\infty} |x_{n_t+1}|^p r_{n_t+1}\frac{\Phi(\Delta y_{n_t+1})}{\Phi(y_{n_t+1})} \ge 0\,. \tag{14}$$

To prove this let us consider the one-parametric family $x^{(t)} = (x_k^{(t)})_{k=0}^{\infty}$ of sequences defined

$$x_k^{(t)} = \begin{cases} y_k x_{n_t+1}/y_{n_t+1} & \text{for} \quad k \le n_t \\ x_k & \text{for} \quad k > n_t\,. \end{cases}$$

The sequence $x^{(t)}$ is admissible for every t and equal to x for large k. Hence

$$0 \le \liminf_{n\to\infty} J(x^{(t)};0,n) = \lim_{s\to\infty} J(x^{(t)};0,n_s) < \infty$$

for every t and clearly

$$\liminf_{t\to\infty} \lim_{s\to\infty} J(x^{(t)};0,n_s) \geq 0\,. \tag{15}$$

Further

$$\begin{aligned}
&\liminf_{t\to\infty} \lim_{s\to\infty} J(x^{(t)};0,n_s)\\
&\qquad = \lim_{t\to\infty}\lim_{s\to\infty} J(x^{(t)};n_t+1,n_s) + \liminf_{t\to\infty} J(x^{(t)};0,n_t)\\
&\qquad = \lim_{t\to\infty}\lim_{s\to\infty} J(x;n_t+1,n_s) + \liminf_{t\to\infty} \frac{|x_{n_t+1}|^p}{|y_{n_t}+1|^p} J(y;0,n_t)\\
&\qquad = 0 + \liminf_{t\to\infty} \frac{|x_{n_t+1}|^p}{|y_{n_t}+1|^p} |y_{n_t}+1|^p r_{n_t+1} \frac{\Phi(\Delta y_{n_t+1})}{\Phi(y_{n_t+1})}\\
&\qquad = \liminf_{t\to\infty} |x_{n_t+1}|^p r_{n_t+1} \frac{\Phi(\Delta y_{n_t+1})}{\Phi(y_{n_t+1})}\,.
\end{aligned}$$

and (14) is a consequence of (15). From (11) we have the following relation

$$\lim_{t\to\infty} J(x;0,n_t) = \lim_{t\to\infty} |x_{n_t+1}|^p r_{n_t+1} \frac{\Phi(\Delta y_{n_t+1})}{\Phi(y_{n_t+1})} + \lim_{t\to\infty} \sum_{k=1}^{n_t} \frac{r_k y_k}{y_{k+1}} G_k(x,y)\,. \tag{16}$$

Note that

$$0 \leq \lim_{t\to\infty} J(x;0,n_t) < \infty\,,$$

(14) holds and all terms inside the sum are nonnegative. Hence both limits in the right-hand side of (16) really exist, are nonnegative and finite. We proved that (14) holds even with "lim" instead of "lim inf" and the limit

$$\lim_{t\to\infty} \sum_{k=1}^{n_t} \frac{r_k y_k}{y_{k+1}} G_k(x,y)$$

exists as a finite number. This fact and the fact that all terms in the sum are nonnegative imply that the following limit exists as well

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{r_k y_k}{y_{k+1}} G_k(x,y) = \lim_{t\to\infty} \sum_{k=1}^{n_t} \frac{r_k y_k}{y_{k+1}} G_k(x,y) < +\infty\,. \tag{17}$$

Taking limit of (11) we obtain

$$\liminf_{n\to\infty} J(x;0,n) = \liminf_{n\to\infty} |x_{n+1}|^p r_{n+1} \frac{\Phi(\Delta y_{n+1})}{\Phi(y_{n+1})} + \lim_{n\to\infty} \sum_{k=1}^{n} \frac{r_k y_k}{y_{k+1}} G_k(x,y) \tag{18}$$

By (13) the left-hand sides of equalities (16) and (18) are both finite and equal and by (17) the second terms on the right-hand sides are finite and equal as well. We conclude from (16) and (18) that

$$\liminf_{n\to\infty} |x_{n+1}|^p r_{n+1} \frac{\Phi(\Delta y_{n+1})}{\Phi(y_{n+1})} = \lim_{t\to\infty} |x_{n_t+1}|^p r_{n_t+1} \frac{\Phi(\Delta y_{n_t+1})}{\Phi(y_{n_t+1})}$$

holds and the validity of the singularity condition follows from (14). The proof is complete. □

3. Singular functional with free end point

In the last section we will use the approach from the preceding section to the case of the functional with free end point. The functional

$$S(x;0,n)=\alpha|x_0|^p+J(x;0,n),\qquad \alpha\in\mathbb{R} \tag{19}$$

defined on the class of real sequences $x=(x_k)_{k=0}^{n+1}$ with a boundary condition $x_{n+1}=0$ is studied in [8] as a modification of the functional (1). Now let us study the singular version of this functional, namely the singular functional with the free end point

$$\liminf_{n\to\infty} S(x;0,n) \tag{20}$$

on the class of the real sequences $x=(x_k)_{k=0}^{\infty}$ satisfying the boundary condition $\lim_{k\to\infty} x_k=0$. Hence the class of admissible sequences is more comprehensive than the class of admissible sequences for the functional (3).

Definition 3.1 (admissibility for the functional with free end point)**.** The sequence $x=(x_k)_{k=0}^{\infty}$ of the real numbers is said to be an *admissible sequence* for the functional (20) if

$$\lim_{k\to\infty} x_k=0$$

holds.

The following variant of Lemma 2.2 holds.

Lemma 3.1. *Suppose that the solution $y=(y_k)$ of (2) given by the initial conditions*

$$y_0=1,\qquad y_1=1+\Phi^{-1}\Bigl(\frac{\alpha}{r_0}\Bigr), \tag{21}$$

where Φ^{-1} is the inverse function to the function Φ, has no generalized zero on $[0,n+1]$. Then for every sequence $x=(x_k)$ we have

$$S(x;0,n)=|x_{n+1}|^p r_{n+1}\frac{\Phi(\Delta y_{n+1})}{\Phi(y_{n+1})}+\sum_{k=0}^{n}\frac{r_k y_k}{y_{k+1}}G_k(x,y)\,, \tag{22}$$

where $G_k(x,y)$ is defined by (9).

Proof. The initial conditions (21) ensure that the sequence y satisfies

$$r_0\Phi\Bigl(\frac{\Delta y_0}{y_0}\Bigr)=\alpha\,.$$

This fact and the summation of the Picone identity (8) over the interval $[0,n]$ imply (22). □

In the case of the functional with free end point the following modification of the singularity condition is necessary.

Definition 3.2 (singularity condition for functional with free end point)**.** Let $y = (y_k)$ be the solution of (2) given by the initial conditions (21). The functional (20) is said to satisfy the *singularity condition* if (7) holds for every admissible sequence along which the functional (20) is finite.

The difference between the Definitions 2.1 and 3.2 lies in another specification of the sequence y. The necessary and sufficient condition for positive semidefiniteness of the functional (20) is introduced in the following theorem.

Theorem 3.1. *The functional* (20) *is positive semidefinite on the class of the admissible sequences if and only if the solution* y *of equation* (2) *given by the initial conditions* (21) *has no generalized zero on the interval* $[0,\infty)$ *and functional* (20) *satisfies singularity condition.*

Proof. The proof is almost the same as the proof of Theorem 2.1 and is omitted here. □

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