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DIFFERENTIAL CALCULUS ON ALMOST COMMUTATIVE ALGEBRAS AND APPLICATIONS TO THE QUANTUM HYPERPLANE

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Abstract. In this paper we introduce a new class of differential graded algebras named DG ρ-algebras and present Lie operations on this kind of algebras. We give two examples: the algebra of forms and the algebra of noncommutative differential forms of a ρ-algebra. Then we introduce linear connections on a ρ-bimodule M over a ρ-algebra A and extend these connections to the space of forms from A to M. We apply these notions to the quantum hyperplane.

1. Introduction

Let N be a $\mathbb{C}^\infty$ manifold and $\mathbb{C}^\infty(N)$ the algebra of $\mathbb{C}^\infty$ functions on N. The basic properties of N are determinated by the purely algebraic properties of $\mathbb{C}^\infty(N)$: the derivations of $\mathbb{C}^\infty(N)$ are vector fields on N and they form a $\mathbb{C}^\infty(N)$ module in which forms and the tensor fields together with usual operations can be obtained within the framework of multilinear algebra of $\mathbb{C}^\infty(N)$ modules. The algebraic skeleton of differential geometry suggests a piece of mathematics that can stand on its own and that be associated with an arbitrary commutative algebra. The next step is to generalize it to the case of noncommutative algebra and to obtain in this way the noncommutative version of differential geometry. Next we present the case when the algebra A is from a class of noncommutative algebras which are almost commutative algebras (ρ-commutative algebras).

In this paper we introduce the notion of differential graded ρ-algebra (DG ρ-algebra) which is a generalization of DG-algebra and of the DG-superalgebras. We define Lie operations on a DG ρ-algebra which are the generalizations of Lie operations on a DG-algebra.

We give two examples of DG ρ-algebras: the algebra of forms $\Omega(A)$ of an almost commutative algebra and the algebra of noncommutative differential forms $\Omega_\alpha A$ of a ρ-algebra A. We also present linear connections on a ρ-bimodule M over an

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almost commutative algebra A, then we extend these connections to the space of forms $\Omega(A, M)$ of A with values in the ρ-bimodule M and we introduce the classical differential operators associated to a such connection. This differential calculus generalizes the fermionic differential calculus from [9] and extends the calculus over an almost commutative algebra from [1]. We also introduce distributions on a ρ-bimodule M over a ρ-algebra A, we present parallel and globally integrable distribution with respect to a M connection. We apply these notions to the quantum hyperplane.

The paper is organized as follows: In the second section we review the basic notions concerning almost commutative algebras and the derivations on this kind of algebras (see [1]). In the third section we define the DG ρ-algebras, the differential calculus over an almost commutative algebra and present two examples: the algebra of forms over an almost commutative algebra A denoted by $\Omega(A)$ and the algebra of noncommutative differential forms $\Omega_\alpha A$ of A. Then we introduce the Lie operation of a ρ-Lie algebra in a DG ρ-algebra and, as examples, we present Lie operations on $\Omega(A)$ and $\Omega_\alpha A$. We also introduce the algebraic Frölicher-Nijenhuis bracket on the almost commutative algebra A.

In the fourth section we present linear connections on a ρ-bimodule M over a ρ-algebra A and extend these connections to the algebra $\Omega(A)$, we give the classical differential operators associated to a such connection and prove some relations between them. In the last section we apply these notions to the quantum hyperplane.

2. Almost commutative algebras

In this section we present shortly a class of noncommutative algebras which are almost commutative algebras, for more details see [1].

Let G be an abelian group, additively written, and let A be a G-graded algebra. This means that, as a vector space, A has a G-grading $A = \oplus_{a \in G} A_a$. Moreover, $A_a A_b \subset A_{a+b}$ $(a, b \in G)$. The G-degree of a (nonzero) homogeneous element f of A is denoted as $|f|$. Furthermore, let $\rho : G \times G \to k$ be a map which satisfies

$$\rho(a, b) = \rho(b, a)^{-1}\,, \qquad a, b \in G\,, \tag{1}$$

$$\rho(a + b, c) = \rho(a, c)\rho(b, c)\,, \qquad a, b, c \in G\,. \tag{2}$$

This implies $\rho(a, b) \neq 0$, $\rho(0, b) = 1$, and $\rho(c, c) = \pm 1$, for all $a, b, c \in G$, $c \neq 0$. We define the *ρ-commutator* $[f, g]_\rho = fg - \rho(|f|, |g|)gf$, where f and g are homogeneous elements in A.

This expression as it is makes sense only for homogeneous elements f and g, but it can be extended linearly to general elements. The ρ-commutator has the following properties:

$$[A_a, A_b] \subset A_{a+b}\,, \qquad a, b \in G\,, \tag{3}$$

$$[f, g]_\rho = -\rho(|f|\,|g|)[g, f]_\rho\,, \tag{4}$$

$$
\begin{aligned}
(5) \qquad 0 = \rho(|f|, |h|)^{-1}[f, [g, h]_\rho]_\rho + \rho(|g|, |f|)^{-1}[g, [h, f]_\rho]_\rho \\
+ \rho(|h|, |g|)^{-1}[h, [f, g]_\rho]_\rho
\end{aligned}
$$

for any homogeneous elements $f, g, h \in A$.

Eq. (4) may be called *ρ-antisymmetry* and Eq. (5) is the *ρ-Jacobi identity.* A G-graded algebra A with a given cocycle ρ is called *ρ-algebra.* The ρ-algebra A is *ρ-commutative* if $fg = \rho(|f|, |g|)gf$ for all homogeneous elements f and g in A. A ρ-commutative algebra is also called *almost commutative algebra.*

Let α be an element of the group G. A *ρ-derivation* X of A, *of degree* α, is a linear map $X : A \to A$, of G-degree $|X|$ i.e. $X : A_* \to A_{*+|X|}$, such that has for all elements $f \in A_{|f|}$ and $g \in A$

$$
X(fg) = (Xf)g + \rho(\alpha, |f|)f(Xg)\,. \tag{6}
$$

Without any difficulties it can be obtained that if the algebra A is ρ-commutative, $f \in A_{|f|}$ and X is a ρ-derivation of degree α, then fX is a ρ-derivation of degree $|f| + \alpha$ and the G-degree $|f| + |X|$ i.e.

$$
(fX)(gh) = ((fX)g)h + \rho(|f| + \alpha, |g|)g(fX)h
$$

and $fX : A_* \to A_{*+|f|+|X|}$.

We say that $X : A \to A$ is a *ρ-derivation* if it has the degree $|X|$ and the G-degree $|X|$ i.e. $X : A_* \to A_{*+|X|}$ and $X(fg) = (Xf)g + \rho(|X|, |f|)f(Xg)$ for any $f \in A_{|f|}$ and $g \in A$.

It is known that the ρ-commutator of two ρ-derivations is again a ρ-derivation and the linear space of all ρ-derivations is a ρ-Lie algebra, denoted by ρ-Der A.

One verifies immediately that for an almost commutative algebra A, ρ-Der A is not only a ρ-Lie algebra but also a left A-module with the action of A on ρ-Der A defined by

$$
(fX)g = f(Xg) \qquad f, g \in A, \quad X \in \rho\text{- Der } A\,. \tag{7}
$$

Let M be a G-graded left module over a ρ-commutative algebra A, with the usual properties, in particular $|f\psi| = |f| + |\psi|$ for $f \in A$, $\psi \in M$. Then M is also a right A-module with the right action on M defined by $\psi f = \rho(|\psi|, |f|)f\psi$. In fact M is a bimodule over A, i.e. $f(\psi g) = (f\psi)g$, for any $f, g \in A$, $\psi \in M$.

3. Differential graded ρ-algebras

Next we introduce the notion of differential graded ρ-algebras which is a generalization of the usual DG-algebras and the DG-superalgebras. Let G be a group as in the previous section, $\rho : G \times G \to k$ a cocycle which satisfies the relations (1) and (2). We denote by $G' = \mathbb{Z} \times G$ and we define the map $\rho' : G' \times G' \to k$ by

$$
\rho'\left((n, \alpha), (m, \beta)\right) = (-1)^{nm} \rho\left(\alpha, \beta\right)\,. \tag{8}
$$

It is obvious that ρ' satisfies the properties (1) and (2).

Definition 1. We say that $\Omega = \underset{(n,\alpha)\in G'}{\oplus} \Omega^n_\alpha$ is a *differential graded ρ-algebra* (*DG ρ-algebra*) if there is an element $\alpha \in G$, a differential d of degree $(1, \alpha)$ and the

$G'-$degree $|d|' = (1, 0)$, i.e. there are the following properties:

$$d^2 = 0 \tag{9}$$

and

$$d(\omega\theta) = (d\omega)\,\theta + (-1)^n\,\rho\,(\alpha, |\omega|)\,\omega d\theta \tag{10}$$

for any $\omega \in \Omega^n_{|\omega|}$ and $\theta \in \Omega$.

Remark that if we denote by $|\omega|' = (n, |\omega|)$ the G'-degree of $\omega \in \Omega^n_{|\omega|}$, the equality (10) may be written equivalently in the following manner:

$$d(\omega\theta) = (d\omega)\theta + \rho'(|d|', |\omega|')\omega d\theta\,.$$

Without any difficulties it can be shown that Ω is a ρ'-algebra.

Example 2. In the case when the group G is trivial then the map ρ is identity and Ω is a DG ρ-algebra then Ω is the usual DG-algebra.

Example 3. When the group G is $\mathbb{Z}_2$ and the map $\rho(a, b)$ is $(-1)^{ab}$ we obtain that Ω is DG-superalgebra.

Definition 4. Let A be a ρ-algebra. We say that $\left(\Omega\,(A) = \underset{(n,\alpha)\in G'}{\oplus} \Omega^n_\alpha\,(A)\,, d\right)$ is a *ρ-differential calculus* on A if $\Omega\,(A)$ is a differential graded ρ-algebra, $\Omega\,(A)$ is an A-bimodule and $\Omega^0\,(A) = A$.

In the following subsections we give other examples of ρ-differential calculus over a given ρ-algebra A.

3.1. **The algebra of forms of a ρ-algebra.** In this subsection we construct the algebra of forms $\Omega(A)$ of an almost commutative algebra A (see [1]).

The algebra of forms of an the ρ-algebra A is given in the classical manner: $\Omega^0(A) := A$, and $\Omega^p(A)$ for $p = 1, 2, \ldots,$ as the G-graded space of p-linear maps $\alpha_p : \times^p \rho\text{-Der } A \to A$, p-linear in sense of left A-modules

$$\alpha_p(fX_1, \ldots, X_p) = f\alpha_p(X_1, \ldots, X_p)\,, \tag{11}$$

$$\alpha_p(X_1, \ldots, X_j f, X_{j+1}, \ldots, X_p) = \alpha_p(X_1, \ldots, X_j, fX_{j+1}, \ldots X_p) \tag{12}$$

and ρ-alternating

$$\begin{aligned}\alpha_p(X_1, \ldots, , X_j, X_{j+1}, \ldots, X_p)&\\ = -\rho(|X_j|\,, |X_{j+1}|)&\alpha_p(X_1, \ldots, X_{j+1}, X_j, \ldots, X_p)\end{aligned} \tag{13}$$

for $j = 1, \ldots, p-1$; $X_k \in \rho$-Der (A), $k = 1, \ldots, p$; $f \in A$ and Xf is the right A-action on ρ-Der A. $\Omega^p(A)$ is in natural way a G-graded right A-module with

$$|\alpha_p| = |\alpha_p(X_1, \ldots, X_p)| - (|X_1| + \cdots + |X_p|) \tag{14}$$

and with the right action of A defined as

$$(\alpha_p f)(X_1, \ldots, X_p) = \alpha_p(X_1, \ldots, X_p)f\,. \tag{15}$$

From the previous considerations, it follows that $\Omega(A) = \oplus_{p=0}^{\infty}\Omega^p(A)$ is again a G-graded A-bimodule.

One defines exterior differentiation as a linear map $d : \Omega^p(A) \to \Omega^{p+1}(A)$, for all $p \geq 0$, as

$$df(X) = X(f),$$

and for $p = 1, 2, \dots,$

$$\begin{aligned} d\alpha_p(X_1, \dots, X_{p+1}) :&= \sum_{j=1}^{p+1}(-1)^{j-1}\rho(\sum_{i=1}^{j-1}|X_i|, |X_j|)X_j\alpha_p(X_1, \dots, \widehat{X}_j, \dots, X_{p+1}) \\ &+ \sum_{1\leq j<k\leq p+1}(-1)^{j+k}\rho(\sum_{i=1}^{j-1}|X_i|, |X_j|)\rho(\sum_{i=1}^{j-1}|X_i|, |X_k|) \\ &\times \rho(\sum_{i=j+1}^{k-1}|X_i|, |X_k|)\alpha_p([X_j, X_k]_\rho, \dots, X_1, \dots, \widehat{X}_j, \\ &\dots, \widehat{X}_k, \dots X_{p+1}) \,. \end{aligned}$$

One can show that d has degree 0, and that $d^2 = 0$.

There is an exterior product $\Omega^p(A) \times \Omega^q(A) \to \Omega^{p+q}(A)$, $(\alpha_p, \beta_q) \mapsto \alpha_p \wedge \beta_q$, defined by the ρ-antisymmetrization formula:

$$\begin{aligned} &\alpha_p \wedge \beta_q (X_1, \dots, X_{p+q}) \\ &\quad = \sum_\sigma \text{sign}(\sigma)\,(\rho\text{-factor})\,\alpha_p\left(X_{\sigma(1)}, \dots, X_{\sigma(p)}\right)\beta_q\left(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)}\right) \,. \end{aligned}$$

The sum is over all permutations σ of the cyclic group S_{p+q} such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$. The ρ-factor is the product of all $\rho(\left|X_{\sigma(j)}\right|, |\alpha_p|)$ for $p+1 \leq j \leq p+q$ and all $\rho(\left|X_{\sigma(j)}\right|, \left|X_{\sigma(k)}\right|)^{-1}$ for $j < k$ and $\sigma(j) > \sigma(k)$.

$\Omega(A)$ is a $G'-$graded algebra with the group $G' = \mathbb{Z} \times G$. Denote the G' degree of α_p as $|\alpha_p|' = (p, |\alpha_p|)$. It is easy to see that the map $\rho' : G' \times G' \to k$ defined by $\rho'((p,a),(q,b)) = (-1)^{pq}\rho(a,b)$ is a cocycle and that $\Omega(A)$ is a ρ'-commutative algebra. Moreover, the map d is a ρ'-derivation of $\Omega(A)$ with G'-degree $|d|' = (+1, 0)$.

3.2. **The algebra of noncommutative differential forms of a ρ-algebra.** Next we present our construction of the algebra of noncommutative differential forms $\Omega_\alpha A$ of the ρ-algebra A. This is a generalization of the algebra of noncommutative differential forms of an associative algebra and also is a generalization of the algebra of noncommutative differential forms of a superalgebra ([10]).

Let α be an arbitrary element of G. By definition the *algebra of noncommutative differential forms of the ρ-algebra A* is the algebra $\Omega_\alpha A$ generated by the algebra A and the symbols da, $a \in A$ which satisfy the following relations:

1. da is linear in a.
2. the ρ-Leibniz rule: $d(ab) = d(a)b + \rho(\alpha, |a|)adb$.

3. $d(1) = 0$.

Let $\Omega_\alpha^n A$ the space of n-forms $a_0 da_1 \dots da_n$, $a_i \in A$ for any $0 \leq i \leq n$. $\Omega_\alpha^n A$ is an A-bimodule with the left multiplication

$$a(a_0 da_1 \dots da_n) = aa_0 da_1 \dots da_n \,, \tag{16}$$

and with the right multiplication:

$$\begin{aligned}(a_0 da_1 \dots da_n) a_{n+1} = \sum_{i=1}^{n} (-1)^{n-i} \rho(\alpha, \sum_{j=i+1}^{n} |a_j|)(a_0 da_1 \dots d(a_i a_{i+1}) \dots da_{n+1}) \\ + (-1)^n \rho(\alpha, \sum_{i=1}^{n} |a_j|) a_0 a_1 da_2 \dots da_{n+1} \,.\end{aligned} \tag{17}$$

$\Omega_\alpha A = \underset{n \in \mathbb{Z}}{\oplus} \Omega_\alpha^n A$ is $\mathbb{Z}$-graded algebra with the multiplication

$$\Omega_\alpha^n A \cdot \Omega_\alpha^m A \subset \Omega_\alpha^{n+m} A$$

given by:

$$(a_0 da_1 \dots da_n)(a_{n+1} da_{n+2} \dots da_{m+n}) = ((a_0 da_1 \dots da_n) a_{n+1}) da_{n+2} \dots da_{m+n}) \,,$$

for any $a_i \in A$, $0 \leq i \leq n+m$, $n, m \in \mathbb{N}$.

We define the G-degree of the n-form $a_0 da_1 \dots da_n$ by

$$|a_0 da_1 \dots da_n| = \sum_{i=0}^{n} |a_i| \;.$$

It is obvious that $|\omega_n \cdot \omega_m| = |\omega_n| + |\omega_m|$ for any homogeneous forms $\omega_n \in \Omega_\alpha^n A$ and $\omega_m \in \Omega_\alpha^m A$.

Remark 5. $\Omega_\alpha A$ is a $G' = \mathbb{Z} \times G$-graded algebra with the G' degree of the n-form $a_0 da_1 \dots da_n$ as follows $|a_0 da_1 \dots da_n|' = (n, \sum\limits_{i=0}^{n} |a_i|)$.

We define the cocycle $\rho' : G' \times G' \to k$ on the algebra $\Omega_\alpha A$ in the following way:

$$\rho'(|a_0 da_1 \dots da_n|', |b_0 db_1 \dots db_m|') = (-1)^{nm} \rho\Big(\sum_{i=0}^{n} |a_i|, \sum_{i=0}^{m} |b_i|\Big) \tag{18}$$

or, equivalently,

$$\rho'(|\omega_n|', |\omega_m|') = (-1)^{nm} \rho(|\omega_n|, |\omega_m|) \tag{19}$$

for any $\omega_n \in \Omega_\alpha^n A$, $\omega_m \in \Omega_\alpha^m A$. It is obvious that $\Omega_\alpha A$ is a ρ'-algebra. We obtain that the G'-degree of the map d is $(1,0)$ i.e. $d : \Omega_{|\omega|}^n \to \Omega_{|\omega|}^{n+1}$ and the G'-degree of an element $x \in A$ is $|x|' = (0, |x|)$.

Theorem 6 ([4])**.** *$d : \Omega_\alpha^* A \to \Omega_\alpha^{*+1} A$ satisfies:*

$$d(\omega\theta) = (d\omega)\theta + (-1)^n \rho(\alpha, |\omega|) \omega d\theta$$

for any $\omega \in \Omega_\alpha^n A$, $\theta \in \Omega_\alpha^m A$.

If we use the notation $|d|' = (1, \alpha)$ we get that d is a graded derivation of degree $|d|'$ and the G'-degree $(1,0)$, i.e. $d : \Omega^*_\alpha A \to \Omega^{*+1}_\alpha A$ and $d(\omega\theta) = (d\omega)\theta + \rho'(|d|', |\omega|')\omega d\theta$.

Concluding we have the following result:

Theorem 7. *$(\Omega_\alpha A, d)$ is a ρ-differential calculus over A.*

$\Omega_\alpha A$ is called *the algebra of noncommutative differential forms* of the ρ-algebra A.

Remark 8. The algebra $\Omega_\alpha A$ depends by the election of the element $\alpha \in G$.

Example 9. In the case when the group G is trivial, A is an associative algebra and $\Omega_\alpha A$ is the algebra of noncommutative differential forms of A from [6].

Example 10. If the group G is $\mathbb{Z}_2$ and the cocycle is $\rho(a,b) = (-1)^{ab}$ then A is a superalgebra and in this case $\Omega_\alpha A$ is the superalgebra of noncommutative differential forms of A from [10].

3.3. **Operations of a ρ-Lie algebra in a DG ρ-algebra.** In this section we generalize the Lie operation of a Lie algebra into a DG-algebra from ([7]) and present our definition of an operation of a ρ-Lie algebra into a DG ρ-algebra.

Let L be a ρ-Lie algebra, Ω a DG ρ-algebra both over the group G and the cocycle $\rho : G \times G \to k$, $G' = G \times \mathbb{Z}$ and $\rho' : G' \times G' \to k$ be the cocycle from (8).

Definition 11. An *operation* of the ρ-Lie algebra L in the DG ρ-algebra Ω is a linear map $X \mapsto i_X$ of L into the spaces of ρ-derivations of G'-degree $(-1, |X|)$ of Ω such that one has for any $X, Y \in L$:

$$i_X i_Y + \rho(|X|, |Y|) i_Y i_X = 0 \quad \text{i.e.} \quad [i_X, i_Y]_{\rho'} = 0\,.$$

The *Lie-derivative* $L_X = [d, i_X]_{\rho'}$ is a ρ'-derivation of G'-degree $(0, |X|)$ and satisfies the following relations $[L_X, i_Y]_{\rho'} = i_{[X,Y]_\rho}$ and $[L_X, L_Y]_{\rho'} = L_{[X,Y]_\rho}$. This means that $X \mapsto L_X$ is a ρ-Lie algebra homomorphism of L into the ρ-Lie algebra of ρ'-derivations of Ω.

Next we present some examples of operations of a ρ-Lie algebra in a DG ρ-algebra.

3.3.1. *Lie operations of the ρ-Lie algebra ρ-*Der A *into the DG ρ-algebra $\Omega(A)$.* There is a contraction $i_X : \Omega(A) \to \Omega(A)$ of G'-degree $(-1, |X|)$ defined in the following way:

$$i_X \alpha_p(X_1, \ldots, X_{p-1}) := \rho\Big(\sum_{i=1}^{p-1} |X_i|, |X|\Big)\alpha_p(X, X_1, \ldots, X_{p-1}) \tag{20}$$

and $i_X \alpha_0 = 0$, $\alpha_0 \in \Omega^0(A)$. The *Lie derivation* $L_X : \Omega(A) \to \Omega(A)$ of G'-degree $(0, |X|)$ is given by

$$\begin{aligned} L_X \alpha_p(X_1, \ldots, X_p) : &= \rho\Big(\sum_{i=1}^{p} |X_i|, |X|\Big) X\big(\alpha_p(X, X_1, \ldots, X_{p-1})\big) \\ &- \sum_{i=1}^{p} \rho\Big(\sum_{i=1}^{p} |X_i|, |X|\Big)\alpha_p(X_1, \ldots, [X, X_i]_\rho, \ldots, X_p)\,, \end{aligned} \tag{21}$$

The relations between i_X, L_X and d are:

$$[d,d]_{\rho'} = 0, \qquad [d,i_X]_{\rho'} = L_X, \qquad [d,L_X]_{\rho'} = 0\,,$$
$$[i_X,i_Y]_{\rho'} = 0, \qquad [i_X,L_Y]_{\rho'} = i_{[X,Y]_\rho}\,,$$
$$[L_X,L_Y]_{\rho'} = L_{[X,Y]_\rho}\,.$$

So the map $X \to i_X$ is a Lie operation on the ρ-Lie algebra ρ-Der A into the DG ρ-algebra $\Omega(A)$.

In the second example we continue the noncommutative calculus on a ρ-algebra from the subsection 3.2.

3.3.2. *The Frölicher -Nijenhuis bracket of a ρ-algebra.* Next we study the case when the element $\alpha \in G$ is zero and A is a ρ-commutative algebra. In this case $\Omega_0 A$ is denoted by ΩA.

First we describe the derivations of the algebra ΩA. Denote by $\mathrm{Der}^{\,\mathrm{alg}}\,\Omega A$ the submodule of ΩA of all algebraic derivations, i.e. the ρ-derivations X of ΩA such that $X\,|_{\Omega^0 A}= 0$.

Since any algebraic derivation is determined by its values on $\Omega^1 A$ we get the isomorphism:

$$\mathrm{Der}^{\,\mathrm{alg}}_{\,(k,\alpha)}\,\Omega A \backsimeq \mathrm{Hom}_{\,\alpha}(\Omega^1 A, \Omega^{k+1} A) \backsimeq \mathrm{Der}_{\,\alpha}(\Omega^{k+1} A) \tag{22}$$

where $\mathrm{Der}^{\,\mathrm{alg}}_{\,(k,\alpha)}\,\Omega A$ is the space of all algebraic ρ'-derivations of G'-degree (k,α), $\mathrm{Hom}_{\,\alpha}(\Omega^1 A, \Omega^{k+1} A)$ is the space of all morphisms from $\Omega^1 A$ to $\Omega^{k+1} A$ of G'-degree $(0,\alpha)$ and $\mathrm{Der}_{\,\alpha}(\Omega^{k+1} A)$ is the space of ρ'-derivations of G'-degree $(0,\alpha)$.

For any derivation $X \in \mathrm{Der}_{\,\alpha}(\Omega^{k+1} A)$ we will denote by i_X the corresponding algebraic (*inner*) derivation of ΩA.

In other words the operator may be defined as a ρ'-derivation in ΩA such that:

1) $i_X : \Omega^n_\beta A \to \Omega^{n+k}_{\alpha+\beta} A$,
2) $i_X(\omega\theta) = i_X(\omega)\theta + (-1)^{jk}\rho(\alpha,\beta)\omega i_X(\theta)$,
3) $i_X(a) = 0$, $i_X(da) = X(a)$,

where $j,k \in \mathbb{N}$, $\alpha,\beta \in G$, $\omega \in \Omega^j_\alpha A$ and $a \in A$.

The module of ρ'-derivations is closed with respect to the ρ'-commutator of derivations. Therefore we get a ρ'-Lie algebra structure on

$$\mathcal{N}ij(A) = \sum_{k\in\mathbf{Z},\ \alpha\in G} \mathrm{Hom}_{\,\alpha}(\Omega^1 A, \Omega^{k+1} A)$$

which it will be called the *Nijenhuis algebra* of the ρ-commutative algebra A and the bracket will be called a *ρ-algebraic Nijenhuis bracket.*

By definition the Nijenhuis bracket of the elements $X \in \mathrm{Hom}_{\,\alpha}(\Omega^1 A, \Omega^{k+1} A)$ and $Y \in \mathrm{Hom}_{\,\beta}(\Omega^1 A, \Omega^{l+1} A)$ is given by formula:

$$[X,Y]\,(\omega) = i_X(Y(\omega)) - (-1)^{kl}\rho(\alpha,\beta) i_Y\big(X(\omega)\big)\,,$$

for all $\omega \in \Omega^1 A$.

Any ρ-derivation $X \in \rho$-Der A determines an inner derivation $i_X \in \rho$-Der A of G' degree (-1,$|X|$) and a *Lie derivation*: $L_X = [i_X, d]$.

For Lie derivations we have the same properties as for the usual ones.

Theorem 12. 1) *L_X is a ρ-derivation of G' degree $(k,|X|)$ of the algebra $\Omega^* A$:*

$$L_X(\omega_1\omega_2) = L_X(\omega_1)\omega_2 + (-1)^k \rho(|X|\,, |\omega_1|)\omega_1 L_X(\omega_2).$$

2) *The bracket $[L_X, L_Y]$ is a Lie derivation L_Z for some element $Z = [X, Y]$, and is called the Frölicher-Nijenhuis bracket.*

3) *The Frölicher-Nijenhuis bracket*

$$\mathrm{Hom}_{\,\alpha}(\Omega^1 A, \Omega^{k+1} A) \times \mathrm{Hom}_{\,\beta}(\Omega^1 A, \Omega^{l+1} A) \to \mathrm{Hom}_{\,\alpha\beta}(\Omega^1 A, \Omega^{k+l+2} A)$$

determines a G'-graded ρ-Lie algebra in the Nijenhuis algebra.

4. Linear connections on a differential calculus of a ρ-algebra

Let A be a ρ-algebra, $\Omega\,(A) = \underset{n\in\mathbb{Z}}{\oplus} \Omega^n\,(A)$ a differential calculus over the ρ-algebra A and M a ρ-bimodule over A. A *linear connection ∇ on the A-bimodule M over the ρ-differential calculus* $\Omega\,(A)$ is the linear map $\nabla : M \otimes \Omega^n\,(A) \to M \otimes \Omega^{n+1}\,(A)$ such that

$$\nabla(m\omega) = \nabla\,(m)\,\omega + (-1)^n\, m \otimes d\omega$$

for any $m \in M$ and $\omega \in \Omega^n\,(A)$. If the A-bimodule M is $\Omega^1\,(A)$ then ∇ is a *linear connection on the differential calculus* $\Omega\,(A)$.

4.1. Linear connections on a ρ-bimodule over an almost commutative algebra.

In this subsection we present linear connections on a ρ-bimodule M over an almost commutative algebra A.

Definition 13. A *linear connection* on M is a linear map of ρ-Der A into the linear endomorphisms of M

$$\nabla : \rho\text{- Der } A \to \mathrm{End}\,(M)\,,$$

such that one has:

$$\nabla_X : M_p \to M_{p+|X|}\,, \tag{23}$$

$$\nabla_{aX}(m) = a\nabla_X(m)\,, \tag{24}$$

and

$$\nabla_X(am) = \rho(|X|\,, |m|)X(a)m + a\nabla_X(m)\,, \tag{25}$$

for all $p \in G$, $a \in A$, and homogeneous elements $X \in \rho$-Der A, and $m \in M$.

Let ∇ be a connection as above. Its *curvature* R is the map

$$R : (\rho\text{- Der } A) \times (\rho\text{- Der } A) \to \mathrm{End}\,(M)$$
$$(X, Y) \longmapsto R_{X,Y}$$

defined by:

$$R_{X,Y}(m) = [\nabla_X, \nabla_Y]_{\rho'}(m) - \nabla_{[X,Y]_\rho}(m) \tag{26}$$

for any $X, Y \in \rho$-Der A, and $m \in M$, where the brackets are:

$$[\nabla_X, \nabla_Y]_{\rho'} = \nabla_X\nabla_Y - \rho(|X|\,, |Y|)\nabla_Y\nabla_X$$

and

$$[X,Y]_\rho = X \circ Y - \rho(|X|, |Y|) Y \circ X\,.$$

Theorem 14. *The curvature of any connection ∇ has the following properties:*

1) *A-linearity:*

$$R_{aX,Y}(m) = aR_{X,Y}\,, \tag{27}$$

2) *$R_{X,Y}$ is right A-linear:*

$$R_{X,Y}(ma) = R_{X,Y}(m)a\,, \tag{28}$$

3) *$R_{X,Y}$ is left A-linear:*

$$R_{X,Y}(am) = \rho(|X| + |Y|, |a|) R_{X,Y}(m)\,, \tag{29}$$

4) *R is a ρ-symmetric map:*

$$R_{X,Y} = -\rho(|X|, |Y|) R_{Y,X} \tag{30}$$

*for any $a \in A_{|a|}$, $m \in M$, $X, Y \in \rho$-*Der *A.*

In the case when the ρ-bimodule M is ρ-Der A then we may introduce the *torsion* of the connection ∇ the following linear map

$$T_\nabla : (\rho\text{- Der } A) \times (\rho\text{- Der } A) \to \rho\text{- Der } A$$

defined by

$$T_\nabla(X,Y) = [\nabla_X Y, \nabla_Y X]_\rho - [X,Y]_\rho\,,$$

for any $X, Y \in \rho$-Der A.

4.2. **Distributions.** Here we introduce the notion of distribution on a ρ-bimodule M over the almost commutative algebra A.

Definition 15. A *distribution* on the ρ-bimodule M over A is a simply a ρ-subbimodule N of M.

Let ∇ be a linear connection on M, we say that the distribution N is *parallel* with respect to the connection ∇ if $\nabla_X m \in N$ for any $X \in \rho$-Der A and for any $m \in N$.

A *distribution* $\mathcal{D}$ on a given differential calculus $\Omega A = \underset{n \in \mathbb{Z}}{\oplus} \Omega^n A$ over A, is a subbimodule of $\Omega^1 A$. The distribution $\mathcal{D}$ is *totally integrable* if there is a subalgebra B of the algebra A such that $\mathcal{D}$ is generated by $A(dB)$.

4.3. **Classical differential operators on $\Omega(A, M)$ associated to the connection ∇.** In this section A is an almost commutative algebra, M is a ρ-bimodule over A and ∇ is a connection on M. Let $\Omega\,(A, M) = \underset{n \in \mathbb{Z}}{\oplus} \Omega^n\,(A, M)$ with $\Omega^n\,(A, M)$ the space of n- forms of A with values in the ρ-bimodule M (is defined in the same way like $\Omega(A)$ just that the values are in the ρ-bimodule M).

Next we define the classical differential operators on $\Omega(A, M)$ associated to the linear connection ∇ and we give some relations between them.

The classical operators attached to the connection ∇ are defined on the space $\Omega(A,M)$ in the following way:

$$\nabla\wedge : \Omega^p(A,M) \to \Omega^{p+1}(A,M) \tag{31}$$

$$(\nabla\wedge\alpha_p)(X_1,\ldots,X_{p+1}) := \sum_{j=1}^{p+1}(-1)^{j-1}\rho\Big(\sum_{l=1}^{j-1}|X_l|\,,|X_j|\Big) \times \nabla_{X_j}\alpha_p(X_1,\ldots,\widehat{X}_j,\ldots,X_{p+1})\,,$$

for any $X_1,\ldots,X_{p+1}\in\rho$-$\mathrm{Der}(A)$. The *exterior differential attached to the connection* ∇ is:

$$d_\nabla = d_0 + \nabla\wedge\,, \tag{32}$$

where $d_0 : \Omega^p(A,M)\to\Omega^{p+1}(A,M)$ is defined by

$$d_0\alpha_p(X_1,\ldots,X_{p+1}) = \sum_{1\le j<k\le p+1}(-1)^{j+k}\rho\Big(\sum_{i=1}^{j-1}|X_i|\,,|X_j|+|X_k|\Big) \times \rho\Big(\sum_{i=j+1}^{k-1}|X_i|\,,|X_k|\Big)\alpha_p([X_j,X_k]_\rho,\ldots,X_1,\ldots,\widehat{X}_j,\ldots,\widehat{X}_k,\ldots X_{p+1})\,,$$

where $\widehat{x}$ means omission of the element x.

Let $X\in\rho$-Der (A), we define the operator:

$$L^0_X : \Omega^p(A,M)\to\Omega^p(A,M) \tag{33}$$

$$(L^0_X\alpha_p)(X_1,\ldots,X_p) = -\sum_{i=1}^{p}\rho\Big(\sum_{l=j}^{p}|X_i|\,,|X|\Big)\alpha_p(X_1,\ldots,[X,X_j]_\rho,\ldots,X_p)\,,$$

The *extension of the connection* ∇ to the space $\Omega(A,M)$ is:

$$\nabla_X : \Omega^p(A,M)\to\Omega^p(A,M) \tag{34}$$

$$(\nabla_X\alpha_p)(X_1,\ldots,X_p) = \rho\Big(\sum_{l=1}^{p}|X_l|\,,|X|\Big)\nabla_X\big(\alpha_p(X_1,\ldots,X_p)\big)\,.$$

The *Lie derivative associated to the connection* ∇ is defined in the following way:

$$L^\nabla_X = L^0_X + \nabla_X \tag{35}$$

and the inner derivation $i_X : \Omega^{p-1}(A,M)\to\Omega^p(A,M)$ is from formula (20).

The relations between the operators i_X, L^0_X and d_0 are given in the following theorem.

Theorem 16. 1) $\big[i_X, L^0_Y\big]_{\rho'} = i_{[X,Y]_\rho}$,

2) $[i_X, L_Y]_{\rho'} = i_{[X,Y]_\rho}$,

3) $[i_X, i_Y]_{\rho'} = 0$,

4) $\big[L^0_X, L^0_Y\big]_{\rho'} = L^0_{[X,Y]_\rho}$,

5) $[d_0, i_X]_{\rho'} = L^0_X$,

6) $[d_0, L_X]_{\rho'} = 0$,

7) $d_0^2 = 0$.

There are the following relations between the extension of the connection ∇, the exterior differential associated to ∇ and the Lie derivative associated to ∇.

Theorem 17. 1) $[\nabla\wedge, i_X]_{\rho'} = \nabla_X$,
2) $[d_\nabla, i_X]_{\rho'} = L_X^\nabla$,
3) $[i_X, \nabla_Y]_{\rho'} = 0$,
4) $\left[i_X, L_Y^\nabla\right]_{\rho'} = i_{[X,Y]_\rho}$.

Proof. 1) First remark that

$$[\nabla\wedge, i_X]_{\rho'} = \nabla \wedge i_X + i_X \nabla \wedge\ .$$

$$\begin{aligned}
&[\nabla\wedge, i_X]_{\rho'}(\alpha_p)(X_1,\dots,X_p)\\
&\qquad = \nabla \wedge i_X(\alpha_p)(X_1,\dots,X_p) + i_X\nabla\wedge(\alpha_p)(X_1,\dots,X_p)\\
&\qquad = \sum_{j=1}^{p}(-1)^{j-1}\rho\Big(\sum_{l=1}^{j-1}|X_l|\,,|X_j|\Big)\nabla_{X_j}(i_X\alpha_p)\big(X_1,\dots,\widehat{X}_j,\dots,X_p)\big)\\
&\qquad\quad - \sum_{j=1}^{p}\rho(|X_j|\,,|X|)\big(\nabla\wedge(\alpha_p)\big)(X,X_1,\dots,X_p)\\
&\qquad = \sum_{j=1}^{p}(-1)^{j-1}\rho\Big(\sum_{l=1}^{j-1}|X_l|\,,|X_j|\Big)\sum_{i=1}^{p}\rho(|X_i|\,,|X|)\\
&\qquad\quad \times\nabla_{X_j}\big(\alpha_p(X,X_1,\dots,\widehat{X}_j,\dots,X_p))\big)\\
&\qquad\quad + \sum_{j=1}^{p}\rho(|X_j|\,,|X|)\Big\{\nabla_X\big(\alpha_p(X_1,\dots,X_p)\big)\\
&\qquad\quad + \sum_{j=1}^{p}(-1)^j\rho\Big(\,|X| + \sum_{l=1}^{j-1}|X_l|\,,|X_j|\Big)\nabla_{X_j}\big(\alpha_p(X,X_1,\dots,\widehat{X}_j,\dots,X_p)\big)\Big\}\\
&\qquad = \sum_{j=1}^{p}\rho(|X_j|\,,|X|)\nabla_X\big(\alpha_p(X_1,\dots,X_p)\big) = \nabla_X(\alpha_p)(X_1,\dots,X_p)\,.
\end{aligned}$$

2)

$$\begin{aligned}
[d_\nabla, i_X]_{\rho'} &= [d^0 + \nabla\wedge, i_X]_{\rho'}\\
&= [d^0, i_X]_{\rho'} + [\nabla\wedge, i_X]_{\rho'} = L_X^0 + \nabla_X\,.
\end{aligned}$$

3)

$$
\begin{aligned}
&[i_X, \nabla_Y]_{\rho'}(\alpha_p)(X_1, \ldots, X_p) \\
&\quad = i_X \nabla_Y(\alpha_p)(X_1, \ldots, X_p) - \rho(|X|, |Y|)\nabla_Y i_X(\alpha_p)(X_1, \ldots, X_p) \\
&\quad = \sum_{j=1}^{p} \rho(|X_j|, |X|)\nabla_Y(\alpha_p)(X, X_1, \ldots, X_p) \\
&\qquad - \rho(|X|, |Y|)\sum_{j=1}^{p} \rho(|X_j|, |Y|)\nabla_Y\big(i_X\alpha_p(X_1, \ldots, X_p)\big) \\
&\quad = \sum_{j=1}^{p} \rho(|X_j|, |X|)\sum_{j=1}^{p} \rho(|X| + |X_j|, |Y|)\nabla_Y\big(\alpha_p(X, X_1, \ldots, X_p)\big) \\
&\qquad - \rho(|X|, |Y|)\sum_{j=1}^{p} \rho(|X_j|, |Y|)\sum_{i=1}^{p} \rho(|X_i|, |Y|) \\
&\qquad \times \nabla_Y\big(\alpha_p(X, X_1, \ldots, X_p)\big) = 0\,.
\end{aligned}
$$

4)

$$
\big[i_X, L_Y^{\nabla}\big]_{\rho'} = \big[i_X, L_Y^{0} + \nabla_Y\big]_{\rho'} = i_{[X,Y]_\rho}\,. \qquad \square
$$

Next we define the extension of the curvature R to $\Omega(A, M)$ in the following way:

(36)
$$
R_{X,Y}^{\nabla} : \Omega^p(A, M) \to \Omega^p(A, M)\,.
$$

It has the G-degree $|X| + |Y|$ and is:

(37)
$$
R_{X,Y}^{\nabla}(\alpha_p)(X_1, \ldots, X_p) = \sum_{i=1}^{p}\rho(|X_i|, |X| + |Y|)R_{X,Y}^{\nabla}\big(\alpha_p(X_1, \ldots, X_p)\big)\,.
$$

The relation between $R_{X,Y}^{\nabla}$, ∇_X and ∇_Y is given in the next theorem:

Theorem 18. $R_{X,Y}^{\nabla} = [\nabla_X, \nabla_Y]_{\rho'} - \nabla_{[X,Y]_\rho}$.

Proof.

$$
\begin{aligned}
&([\nabla_X, \nabla_Y]_{\rho'} - \nabla_{[X,Y]_\rho})(\alpha_p)(X_1, \ldots, X_p) \\
&\quad = \big\{\nabla_X\nabla_Y - \rho(|X|, |Y|)\nabla_Y\nabla_X - \nabla_{[X,Y]_\rho}\big\}(\alpha_p)(X_1, \ldots, X_p) \\
&\quad = \sum_{i=1}^{p}\rho(|X_i|, |X| + |Y|)\nabla_X\big(\nabla_Y(\alpha_p(X_1, \ldots, X_p))\big) \\
&\qquad - \rho(|X|, |Y|)\sum_{i=1}^{p}\rho(|X_i|, |Y| + |X|)\nabla_Y\big(\nabla_X(\alpha_p(X_1, \ldots, X_p))\big)
\end{aligned}
$$

$$- \sum_{i=1}^{p} \rho(|X_i|, |Y| + |X|) \nabla_{[X,Y]_\rho} \big(\alpha_p(X_1, \ldots, X_p)\big)$$
$$= R^{\nabla}_{X,Y}(\alpha_p)(X_1, \ldots, X_p) . \qquad \square$$

Remark 19. $R^{\nabla}_{X,Y} = [L^{\nabla}_X, L^{\nabla}_Y]_{\rho'} - L^{\nabla}_{[X,Y]_\rho}$.

We define the operator

$$R^{\nabla} \wedge : \Omega^p(A, M) \to \Omega^{p+2}(A, M)$$

of G-degree 0 thus:

$$R^{\nabla} \wedge \big(\alpha_p(X_1, \ldots, X_{p+2})\big) = - \sum_{1 \le j < k \le p+2} \alpha_{j,k} \rho\Big(|X_j| + |X_k| , \sum_{i=1}^{p+2} |X_i| \Big) \times R^{\nabla}_{X_j, X_k}(\alpha_p)(X_1, \ldots, \widehat{X}_j, \ldots, \widehat{X}_k, \ldots X_{p+2})$$

where

$$\alpha_{i,j} = (-1)^{j+k} \rho\Big(\sum_{i=1}^{j-1} |X_i| , |X_j| + |X_k| \Big) \rho\Big(\sum_{i=j+1}^{k-1} |X_i| , |X_k| \Big) .$$

Finally we get the following theorem.

Theorem 20. $R^{\nabla} \wedge = d^2_{\nabla}$.

5. Linear connections on N-dimensional quantum hyperplane

In this section we present linear connections on the N-dimensional quantum hyperplane S^q_N. First we present shortly the N-dimensional quantum hyperplane from [1] and then we present the linear connections on S^q_N.

5.1. **N-dimensional quantum hyperplane.** The N-dimensional quantum hyperplane is the algebra S^q_N generated by the unit element and N linearly independent elements $x_1, \ldots x_N$ satisfying the relations:

$$x_i x_j = q x_j x_i , \qquad i < j$$

for some fixed $q \in k$, $q \neq 0$.

S^q_N is the $\mathbb{Z}^N$-graded algebra

$$S^q_N = \overset{\infty}{\underset{n_1, \ldots n_N}{\oplus}} (S^q_N)_{n_1 \ldots n_N} ,$$

with $(S^q_N)_{n_1 \ldots n_N}$ the one-dimensional subspace spanned by products $x^{n_1} \cdots x^{n_N}$. The $\mathbb{Z}^N$-degree of the element $x^{n_1} \cdots x^{n_N}$ is $n = (n_1, \ldots n_N)$. The cocycle ρ is $\rho : \mathbb{Z}^N \times \mathbb{Z}^N \to k$

$$\rho(n, n') = q^{\sum_{j,k=1}^{N} n_j n'_k \alpha_{jk}} , \tag{38}$$

with $\alpha_{jk} = 1$ for $j < k$, 0 for $j = k$ and -1 for $j > k$. The N-dimensional quantum hyperplane S_N^q is an almost commutative algebra with the cocycle ρ from (38).

We are in a special case when we have coordinates vector fields, the ρ-derivations $\partial/\partial x_i,\ j = 1, \dots N$, of $\mathbb{Z}^N$-degree $|\partial/\partial x_i|$, with $|\partial/\partial x_i| = -|x_i|$ defined by $\partial/\partial x_i(x_j) = \delta_{ij}$. One has

$$\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k} = q\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}\,, \qquad \text{for } j < k \tag{39}$$

and

$$\frac{\partial}{\partial x_j}(x_1^{n_1} \dots\ x_N^{n_N}) = n_j q^{(n_1+\cdots+n_j)}(x_1^{n_1} \dots\ x_j^{n_{j-1}} \dots\ x_N^{n_N})\,. \tag{40}$$

The space of ρ-derivations ρ-Der ${}_*(S_N^q)$ of S_N^q is a free S_N^q-module of rank N with the basis $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_N}$. An arbitrary ρ-derivation X can be written as

$$X = \sum_{i=1}^{N} X_i \frac{\partial}{\partial x_i}\,, \tag{41}$$

with $X_i = X(x_i) \in S_N^q$, for i=1,…,N.

The space of one-forms $\Omega^1(S_N^q)$ of S_N^q is a ρ-S_N^q-module and is also free of rank N. The coordinate of one-forms $dx_1, \dots, dx_N$, are defined by

$$dx_i(X) = X(x_i) \quad \text{or} \quad dx_i(\frac{\partial}{\partial x_j}) = \delta_{ij} \tag{42}$$

and there form a basis in $\Omega^1(S_N^q)$, dual to the basis $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_N}$ in ρ-Der ${}_*(S_N^q)$. Note that $|dx_i| = |x_i|$ and for any $f \in S_N^q$ there is the following relation:

$$df = \sum_{i=1}^{N}(dx_i)\frac{\partial}{\partial x_i} f\,. \tag{43}$$

An arbitrary one-form can be written as

$$\alpha_1 = \sum_{i=1}^{N}(dx_i)A_i\,, \quad \text{with} \quad A_i = \alpha_1(\frac{\partial}{\partial x_i}) \in S_N^q\,. \tag{44}$$

5.2. **Linear connections on N-dimensional quantum hyperplane.** In this subsection we introduce linear connections on S_N^q.

We note that any linear connection along the field $X = \sum_{i=1}^{N} X_i \frac{\partial}{\partial x_i}$ is a linear map $\nabla_X : \rho\text{-Der}_*\, S_N^q \to \rho\text{-Der}_{*+|X|}\, S_N^q$ and we have $\bigtriangledown_X = \sum_{i=1}^{N} X_i \bigtriangledown_{\frac{\partial}{\partial x_i}}$. Next we will use the following notations:

$$\bigtriangledown_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} = \Gamma_{i,j}^k \frac{\partial}{\partial x_k}\,, \qquad \text{for any} \quad i, j = 1, \dots, N\,, \tag{45}$$

where $\Gamma^k_{i,j} \in S^q_N$ are the *connection coefficients*. If we take $X = \sum_{i=1}^{N} X_i \frac{\partial}{\partial x_i}$ and $Y = \sum_{j=1}^{N} Y_j \frac{\partial}{\partial x_j}$, we obtain the following relations:

$$\begin{aligned}\nabla_X Y &= \sum_{i=1}^{N} X_i \nabla_{\frac{\partial}{\partial x_i}} \Big(\sum_{j=1}^{N} Y_j \frac{\partial}{\partial x_j} \Big) \\ &= \sum_{i=1}^{N} X_i \sum_{j=1}^{N} \frac{\partial Y_j}{\partial x_i} \frac{\partial}{\partial x_j} - \rho(|x_i|, |Y_j|\, \Gamma^k_{i,j} \frac{\partial}{\partial x_k}) \,.\end{aligned}$$

The curvature R of the linear connection $\bigtriangledown$ is well defined on the basis $\left(\frac{\partial}{\partial x_i}\right)$, $i = 1, \ldots, N$ and is given by the *curvature coefficients*: $R^l_{i,j,k} \in S^q_N$ such that:

$$R_{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_k} = R^l_{i,j,k} \frac{\partial}{\partial x_l} \qquad \text{for any} \quad i, j, k = 1, \ldots, N \,.$$

Using the Theorem 4.1. we obtain that $R^l_{i,j,k} = q^{\alpha_{ij}} R^l_{j,i,k}$, for any $i, j, k, l \in 1, \ldots, N$. The relations between the curvature coefficients and the connection coefficients are:

$$\begin{aligned} R^l_{i,j,k} =& \frac{\partial \Gamma^l_{j,k}}{\partial x_i} - \rho(|x_i|, \left|\Gamma^l_{j,k}\right|) \Gamma^p_{j,k} \Gamma^l_{i,p} \\ & - q^{\alpha_{i,j}} (\frac{\partial \Gamma^l_{i,k}}{\partial x_j} - \rho(|x_j|, \left|\Gamma^p_{i,k}\right|) \Gamma^p_{i,k} \Gamma^l_{j,p}) \end{aligned} \tag{46}$$

for any $i, j, k, l = 1, \ldots, N$.

The torsion T of any linear connection is given by the *torsion coefficients* $T^k_{i,j} \in S^q_N$ from the following relations:

$$T_{\bigtriangledown}(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = T^k_{i,j} \frac{\partial}{\partial x_k} \tag{47}$$

for any $i, j, k = 1, \ldots, N$.

Evidently, the relations between the torsion coefficients and the connection coefficients are:

$$T^k_{i,j} = \Gamma^k_{i,j} - q^{\alpha_{i,j}} \Gamma^k_{j,i} \,.$$

5.3. **Linear connections on quantum hyperplane over the bimodule $\Omega^1(S^q_N)$.** We present linear connections over ρ-bimodule $\Omega^1(S^q_N)$ over the quantum hyperplane. Without any confusion we use the notation

$$\bigtriangledown_{\frac{\partial}{\partial x_i}} dx_j = \Gamma^k_{i,j} dx_k \,, \qquad \text{for} \quad i, j = 1, \ldots, N \,, \tag{48}$$

where $\Gamma^k_{i,j} \in S^q_N$, $i,j,k = 1,\dots,N$ are again denoted *connection coefficients* over the ρ-bimodule $\Omega^1(S^q_N)$. We obtain that

$$\begin{aligned}\bigtriangledown_{\frac{\partial}{\partial x_i}}(x_1^{n_1}\dots x_N^{n_N}dx_k) =& -n_i q^{(n_1+\dots+n_i)} q^{\alpha_{ik}}(x_1^{n_1}\dots x_i^{n_i-1}\dots x_N^{n_N})dx_k \\ &+ x_1^{n_1}\dots x_N^{n_N}\Gamma^l_{ik}dx_l\,.\end{aligned}$$

The curvature R of the linear connection $\bigtriangledown$ is given by the *curvature coefficients*: $R^l_{i,j,k} \in S^q_N$ defined by:

$$R_{\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}}dx_k = R^l_{i,j,k}dx_l \qquad \text{for } i,j,k = 1,\dots,N.$$

We get that $R^l_{i,j,k} = q^{\alpha_{ij}}R^l_{j,i,k}$, for any $i,j,k,l \in 1,\dots,N$, and the relation between the curvature coefficients and the connection coefficients is the same like (46).

5.4. **The algebra of noncommutative differential forms on $N-$dimensional quantum hyperplane.** Next we will apply the construction of the algebra of the noncommutative differential forms of a ρ-algebra from the section 4 to the ρ-algebra S^q_N and, thus, we will give a new differential calculus on the N-dimensional quantum hyperplane denoted by $\Omega_n S^q_N$.

Let $n = (n_1,\dots,n_N)$ be an arbitrary element from $\mathbb{Z}^N$. We define $\Omega_n S^q_N$ the algebra generated by $a \in S^q_N$ and the symbols da ,which satisfy the following relations:

1. da is linear in a.
2. the ρ-Leibniz rule: $d(ab) = (da)b + \rho(n,|a|)adb$.
3. $d(1) = 0$.

Next we present the structure of the algebra $\Omega_n S^q_N$.

We use the following notations $y_i = dx_i$, for any $i \in \{1,\dots,N\}$. By an easy computation we get the following lemmas:

Lemma 21. $y_i x_j = \rho(n+|x_i|,|x_j|)x_j y_i$, *for any* $i,j \in \{1,\dots,N\}$.

Lemma 22. $y_j y_i = \rho(n,|x_i|)\rho(n+|x_i|,|x_j|)y_j y_i$, *for any* $i,j \in \{1,\dots,N\}$.

Lemma 23. $d(x_i^m) = m\rho^{m-1}(n,|x_i|)x_i^{m-1}y_i$, *for any* $m \in \mathbb{N}$ *and* $i \in \{1,\dots,N\}$.

Putting together the previous lemmas we obtain the following theorem which gives the structures of the algebra $\Omega_n S^q_N$:

Theorem 24. $\Omega_n S^q_N$ *is the algebra spanned by the elements* x_i *and* y_i *with* $i \in \{1,\dots,N\}$ *which satisfies the following relations:*

1) $x_i x_j = \rho(|x_i|,|x_j|)x_j x_i$,
2) $y_i x_j = \rho(n+|x_i|,|x_j|)x_j y_i$,
3) $y_j y_i = \rho(n,|x_i|)\rho(n+|x_i|,|x_j|)y_j y_i$, *for any* $i,j \in \{1,\dots,N\}$.

Definition 25. The algebra $\Omega_n S^q_N$ is the *algebra of noncommutative differential forms* of the quantum hyperplane.

Remark that the algebra $\Omega_n S^q_N$ depends by the election of the element $n \in \mathbb{Z}^N$ and is not an almost commutative algebra.

Remark 26. The algebra $\Omega_0 S_N^q$, where $0 = (0, \ldots, 0) \in \mathbb{Z}^n$, is the algebra of noncommutative differential forms of quantum hyperplane from the paper [11].

Using the Definition 13 in the case of ρ-S_N^q-bimodule $\Omega_n^1 S_N^q$ we obtain the following definition of linear connections on $\Omega_n^1 S_N^q$.

Definition 27. A *linear connection* on $\Omega_n^1 S_N^q$ along the field X is a linear morphism

$$\nabla_X : \Omega_n^1 S_N^q \to \Omega_n^1 S_N^q\,,$$

such that

$$\nabla_{\alpha X}(adb) = \alpha \nabla_X(adb)$$

and

$$\nabla_X(adb) = \rho(|X|, |a|)X(a)db + a\nabla_X(db)$$

for any homogeneous elements α, $a, b \in S_N^q$.

Remark that any linear connection ∇ on $\Omega_n^1 S_N^q$ is well defined by *connections coefficients* $\Gamma_{i,j}^k \in S_N^q$ given by the following equations:

$$\nabla_{\frac{\partial}{\partial x_i}} y_j = \Gamma_{i,j}^k y_k\,, \quad \text{for any} \quad i, j \in \{1, \ldots, N\}\,. \tag{49}$$

Next we introduce distributions on $\Omega_n S_N^q$ and we give a characterized globally integrable and parallel distributions with respect to a linear connection.

A *distribution* $\mathcal{D}$ on $\Omega_n S_N^q$ is a subbimodule of $\Omega_n^1 S_N^q$. The distribution $\mathcal{D}$ of dimension p is *globally integrable* if there is a subset of p elements, denoted by I of $\{1, \ldots, N\}$, such that $\mathcal{D}$ is generated by $x_j y_i$ for any $j \in \{1, \ldots, N\}$ and $i \in I$. In this situation we say that the distribution $\mathcal{D}$ has the dimension p.

Theorem 28. *Any globally integrable and parallel distributions $\mathcal{D}$ of dimension p with respect to a linear connection ∇ is well defined by the following equations:*

$$\Gamma_{i,j}^k = 0 \tag{50}$$

for one subset I of $\{1, \ldots, N\}$ with p elements, $i \in \{1, \ldots, N\}$, $j \in I$, $k \in \{1, \ldots, N\} \setminus I$.

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