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FIXED POINTS AND BEST APPROXIMATION IN MENGER CONVEX METRIC SPACES

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Abstract. We obtain necessary conditions for the existence of fixed point and approximate fixed point of nonexpansive and quasi nonexpansive maps defined on a compact convex subset of a uniformly convex complete metric space. We obtain results on best approximation as a fixed point in a strictly convex metric space.

1. Introduction

Nonexpansive mappings have been studied extensively in recent years by many authors.The first fixed point theorem of a general nature for nonlinear nonexpansive mappings in noncompact setting were proved independently by Browder [8] and Gohde [12]. Later on, Kirk [17] proved the same results under slightly weaker assumptions. A fundamental problem in fixed point theory of nonexpansive mappings is to find conditions under which a set has the fixed point property for these mappings. It is intimately connected with differential equations and with the geometry of the Banach spaces.The interplay between the geometry of Banach spaces and fixed point theory has been very strong and fruitful. In particular, geometrical properties play key role in metric fixed point problems, see, for example, [2–5, 7, 19, 20] and references mentioned therein. These results use mainly convexity hypothesis and geometric properties of Banach spaces. These results were starting point for a new mathematical feild : the application of the geometric theory of Banach spaces to fixed point theory. The problem of proximity of subsets in normed spaces has been studied extensily, see, for example, [13, 15, 19, 20]. Aronszajn and Panitchapakdi [2] and Menger [18] defined the convexity structure on metric spaces through closed ball and studied their properties. Khalil [16] further studied existence of fixed points and best approximation in these convex metric spaces. This paper addresses the convexity structure of a metric space, using geometric

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properties of this space; we study fixed points, approximate fixed points and structure of the set of fixed points. We establish results on invariant approximation for the mapping defined on a class of non-convex sets in a convex metric space.

2. Preliminaries

We gather here some basic definitions and set out our terminology needed in the sequel. We also present some previously known results about fixed points.

Definition 1 (Menger [18])**.** Let (X,d) be a metric space. It is said to be (*Menger*) *convex metric space* if for every x, y in X, $x \neq y$, $0 \leq r \leq d(x,y)$,

$$B[x,r] \cap B[y,d(x,y)-r] \neq \emptyset\,,$$

where $B[x,r] = \{y \in X : d(x,y) \leq r\}$.

Some of the properties of these spaces were introduced by Blumenthal [7]. Further results can be found in [6, 15]. A subset E of a convex metric space is called convex if for all x, y in E,

$$B[x,r] \cap B[y,d(x,y)-r] \subseteq E\,, \quad 0 \leq r \leq d(x,y)\,.$$

We note that in general in a convex metric space, the set $B[x,r]$ is not convex.

Definition 2. A convex metric space X is said to have a *property* (A), if for every $x, y \in X$, the set

$$B[x,(1-t)d(x,y)] \cap B[y,td(x,y)]$$

is a singleton set for $t \in [0,1]$.

We denote this singleton set by $m(x,y,t)$. When $x = y$ then obviously $m(x,y,t) = x$. In a convex metric space having property (A), $B[x,r]$ is a convex set.

Definition 3. A convex metric space X is said to have *property* (B), if

$$d(m(x,y,t),m(z,y,t)) \leq td(x,z)\,,$$

for all $x,y,z \in X$ and $t \in (0,1)$.

Definition 4. Let X be a convex metric space. It is said to be *uniformly convex* if it has property (A) and for every given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $r > 0$, and $x,y,z \in X$ with $d(z,x) \leq r$, $d(z,y) \leq r$ and $d(x,y) \geq r\varepsilon$, imply that $d\big(z,m\big(x,y,\frac{1}{2}\big)\big) \leq r\big(1-\delta(\varepsilon)\big) < r$.

Definition 5 (Khalil [16])**.** A convex metric space X is said to be *strictly convex*, if $x,y \in B[z,r]$ with $x \neq y$, then $B[x,(1-t)d(x,y)] \cap B[y,td(x,y)] \subseteq B(z,r)$, for all $t \in (0,1)$, and all $z \in X$, $r > 0$. Where, $B(z,r) = \{x \in X : d(x,z) < r\}$.

It follows from above definition that uniformly convex metric space is strictly convex but converse in general does not hold.

Definition 6. A subset F of a convex metric space X is called T-*regular* set if and only if $T : F \longrightarrow X$ and $m(x,T(x),\frac{1}{2}) \in F$, for each $x \in F$.

Remark 7. Every convex set invariant under a map T in a convex metric space is a T- regular set. But a T-regular set need not be a convex set.

Definition 8. Let X be a metric space and $T : X \longrightarrow X$, a point $x \in X$ is called

(a) *fixed point* of T, if it is the solution of functional equation $T(x) = x$.

(b) ε-*fixed point* of T if $d(x, T(x)) < \varepsilon$ for given $\varepsilon > 0$.

Obviously fixed point of T is ε-fixed point for every given $\varepsilon > 0$ but converse is not true in general. Existence of ε-fixed point of T for each $\varepsilon > 0$ sometimes guarantees the existence of fixed point of T, for example, if a continuous map T defined on a closed subset F of metric space X with $T(F)$ contained in some compact subset of X, has ε-fixed point for each $\varepsilon > 0$, then T has a fixed point.

Definition 9. Let (X, d) be a metric space and $T : X \longrightarrow X$. The mapping T is called *nonexpansive* if for all $x, y \in X$, $d\big(T(x), T(y)\big) \le d(x, y)$.

Definition 10. Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is said to be *quasi-nonexpansive* provided T has a fixed point in X and if x_0 is a fixed point of T, then $d(T(x), x_0) \le d(x, x_0)$, for all x in X.

Remark 11. The class of nonexpansive mappings is strictly contained in the class of quasi-nonexpansive mappings.

We present the following example which supports the Remark 11.

Example 12. Define the self map T on Euclidean space $\mathbf{R}$ as $T(x) = x \sin\left(\frac{1}{x}\right)$ when $x \neq 0$ and $T(0) = 0$. Obviously 0 is the only fixed point of T and T is quasi-nonexpansive. Take $x = \frac{2}{\pi}$, $y = \frac{2}{3\pi}$, then,

$$|T(x) - T(y)| = \frac{8}{3\pi} > \frac{4}{3\pi} = |x - y| ,$$

gives that T is not nonexpansive.

Definition 13. Let (X, d) be a metric space and $M \subseteq X$. For $x \in X$, we define

$$P_M(x) = \{z \in M : d(x, z) = d(x, M)\} ,$$

where $d(x, M) = \inf\{d(x, y), y \in M\}$. Any $z \in P_M(x)$ is called point of *best approximation* for x from M.

The set $P_M(x) \neq \phi$, for any compact subset M of a metric space X.If $P_M(x)$ is singleton for each x in X then we may define the nearest point projection $P : X \longrightarrow M$ by assigning the point of best approximation for x from M to each x in X. It has been proved [16] that $P_M(x)$ is convex for a closed convex set M in a convex metric space X having property (A).

Definition 14. Let K be a subset of a complete convex metric space and $\{f_\alpha : \alpha \in K\}$ be family of maps from $[0, 1]$ into K, having property that for each $\alpha \in K$ we have $f_\alpha(1) = \alpha$. Such a family is said to be

(**a**) *contractive* provided there exist a function $\varphi : (0, 1) \longrightarrow (0, 1)$ such that for all α, $\beta \in K$ and for all $t \in (0, 1)$, we have $d(f_\alpha(t), f_\beta(t)) \le \varphi(t)d(\alpha, \beta)$;

(**b**) *jointly continuous* if $t \to t_0$ in $[0, 1]$ and $\alpha \to \alpha_0$ in K imply $f_\alpha(t) \to f_{\alpha_0}(t_0)$.

3. Fixed points and approximate fixed points

In this section we study fixed point and ε-fixed point of nonexpansive and quasi nonexpansive mappings defined on a compact convex subset of a metric space.

Theorem 15. *Let F be a nonempty compact convex subset of a convex complete metric space X having properties* (A) *and* (B)*, then any nonexpansive self map T on F has a fixed point.*

Proof. Fix $y \in F$, consider the mapping $T_n : F \longrightarrow F$ defined by $T_n(x) = m\big(T(x), y, \frac{1}{n}\big)$.
Then

$$d\big(T_n(x), T_n(z)\big) = d\Big(m\Big(T(x), y, \frac{1}{n}\Big), m\Big(T(z), y, \frac{1}{n}\Big)\Big) \leq \frac{1}{n} d(x, z)\,.$$

It implies that T_n is a contraction for each positive integer n, Banach contraction principle ([12]) further implies that for each n there exists $x_n \in F$ for which $T(x_n) = x_n$. Now the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_j}\}$ converging to x_0. It follows by the continuity of T that $T(x_{n_j}) \to T(x_0)$. Now

$$d\big(x_{n_j}, T(x_0)\big) = d\big(T_n(x_{n_j}), T(x_0)\big) = d\Big(m\Big(T(x_{n_j}), y, \frac{1}{n}\Big), T(x_0)\Big)\,.$$

When $n \to \infty$, $\{x_{n_j}\} \to T(x_0)$. Now the result follows by the uniqueness of the limit. $\square$

Remark 16. Theorem 15, generalizes [11, Theorem 3.1].

The special feature of our next theorem is that in this theorem we relaxe the condition of convexity on the domain of definition of the mapping. It not only improves Theorem 15 but also contains the generalization of Theorem 1 [9].

Theorem 17. *Let F be a compact subset of a complete convex metric space X. Suppose there exists a contractive, jointly continuous family of maps associated with F, then any nonexpansive mapping T of F into itself has a fixed point in F.*

Proof. For each $n = 1, 2, 3, \ldots.$ Let $k_n = \frac{n}{n+1} \in [0, 1]$. Define a mapping T_n from F into itself by $T_n(x) = f_{T(x)}(k_n)$ for all $x \in F$. Since $T(F) \subset F$ and $k_n < 1$, each T_n is well defined map from F into itself. Moreover, for each n and all $x, y \in F$, we have

$$\begin{aligned} d\big(T_n(x), T_n(y)\big) &= d\big(f_{T(x)}(k_n), f_{T(y)}(k_n)\big) \\ &\leq \phi(k_n) d\big(T(x), T(y)\big) \\ &\leq \phi(k_n) d(x, y)\,. \end{aligned}$$

So for each n, T_n is a contraction mapping on F. Banach contraction principle [12] further implies that each T_n has unique fixed point $x_n \in F$. Now compactness of F gives a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to x_0$ in F. Since $T_{n_j}(x_{n_j}) = x_{n_j}$, we have $T_{n_j}(x_{n_j}) \to x_0$. By the continuity of T, $T(x_{n_j}) \to T(x_0)$, also by the joint continuity of the family, we have

$$T_{n_j}(x_{n_j}) = f_{T(x_{n_j})}(k_{n_j}) \to f_{T(x_0)}(1) = T(x_0)\,.$$

It gives that x_0 is the fixed point of T. $\square$

Now we present the theorem which will drop the condition of compactness on the domain of the function and thus further improves the Theorem 15. Moreover, it extends well known Schauder's fixed point theorem for nonexpansive mapping to convex metric spaces.

Theorem 18. *Let F be a nonempty closed convex subset of a convex metric space having properties* (A) *and* (B). *Let $T : F \longrightarrow F$ be a nonexpansive and $T(F)$ a subset of a compact subset of F, then T has a fixed point.*

Proof. Let $x_0 \in F$, for $n = 2, 3, 4, \ldots$, define $T_n(x) = m\big(x_0, T(x), \frac{1}{n}\big)$. Convexity of F implies that T_n is a self map on F for each n. Now

$$\begin{aligned} d\big(T_n(x), T_n(y)\big) &= d\Big(m\Big(x_0, T(x), \frac{1}{n}\Big), m\Big(x_0, T(y), \frac{1}{n}\Big)\Big) \\ &\le \frac{1}{n} d\big(T(x), T(y)\big) \le \frac{1}{n} d(x, y)\,. \end{aligned}$$

Banach contraction principle [12] in turns implies T_n has a unique fixed point x_n in F for each n. Since $T(F)$ lies in a compact subset of F. The sequence $\{x_n\}$ in F has a subsequence $\{x_{n_j}\}$ such that $T(x_{n_j}) \to x_0$, where $x_0 \in F$. Now

$$x_{n_j} = T_n(x_{n_j}) = m\Big(x_0, T(x_{n_j}), \frac{1}{n}\Big) \to x_0\,, \quad \text{as} \quad n \to \infty\,.$$

By the continuity of T, the result follows immediately. $\square$

Now we prove the theorem that gives necessary and sufficient conditions for the convergence of iterative sequence (Banach type) to the fixed point of quasi-nonexpansive mapping in a convex metric space setting.

Theorem 19. *Let F be a closed subset of a convex complete metric space X and $T : F \to X$ be a quasi-nonexpansive mapping. Suppose there exists a point $x_0 \in F$ with $x_n = T^n(x_0) \in F$. Then the sequence $\{x_n\}$ converges to the fixed point of T if and only if $\lim_{n\to\infty} \big(x_n, \mathrm{Fix}\,(T)\big) = 0$, where $\mathrm{Fix}\,(T)$ stands for the set of fixed points of F.*

Proof. As $d\big(x_n, \mathrm{Fix}\,(T)\big) = \sup_{x\in \mathrm{Fix}\,(T)} d(x_n, x)$. Thus $\lim_{n\to\infty} \big(x_n, \mathrm{Fix}\,(T)\big) = 0$, immediately follows by the convergence of sequence $\{x_n\}$ to the fixed point of T. For the converse, let $\varepsilon > 0$, then there exists n_0 such that for all $n \ge n_0$, we have $d\big(x_n, \mathrm{Fix}\,(T)\big) < \frac{\varepsilon}{2}$.

Take the point p in $\mathrm{Fix}\,(T)$ and $m, n \ge n_0$, we have

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, p) + d(p, x_m) \\ &= d\big(T^n(x_0), p\big) + d\big(p, T^m(x_0)\big) \\ &\le 2d\big(T^{n_0}(x_0), p\big) \le 2d(x_{n_0}, p) \\ &\le 2d\big(x_{n_0}, \mathrm{Fix}\,(T)\big)\,, \end{aligned}$$

give $\{x_n\}$ is a Cauchy sequence in F. As F is a closed subset of a complete metric space. So there exists a point u in F such that $x_n \to u$. Hence $d\big(u, \mathrm{Fix}\,(T)\big) = 0$, $\mathrm{Fix}\,(T)$ is closed due to quasi-nonexpansiveness of T, we have $u \in \mathrm{Fix}\,(T)$. $\square$

Theorem 20. *Let F be a closed subset of a convex metric space X Then a compact continuous map $T : F \to X$, has a fixed point if and only if T has ε-fixed point for each $\varepsilon > 0$.*

Proof. Assume that T has ε-fixed point for each $\varepsilon > 0$. Now for each positive integer n, let x_n be $\frac{1}{n}$-fixed point, that is $d\big(x_n, T(x_n)\big) < \frac{1}{n}$. Since T is a compact, $T(F)$ is contained in a compact subset D of X. For a sequence $\{x_n\}$ in F, we have a subsequence $\{x_{n_j}\}$ and a point u in F such that $T(x_{n_j}) \to u$ as $n_j \to \infty$ gives $x_{n_j} \to u$, as $n_j \to \infty$. □

Now closedness of F and continuity of T give $T(x_{n_j}) \to T(u)$, $u \in F$, and this together with argument used in Theorem 18 imply the result.

4. Best approximation

In this section we present application of fixed point theorems to approximation theory in a convex metric space setting.

Theorem 21. *Let M be a closed convex subset of a uniformly convex complete metric space X. If $P_M(x)$ is singleton for each $x \in X$, then the nearest point projection $P : X \to M$ is continuous.*

Proof. Let the sequence $\{x_n\}$ converges to x in X and $P_{\mathrm{M}}(x) = \{z\}$. For the sake of simplicity, denote $P(x_n)$ by u_n. Now $\{u_n\}$ is a Cauchy sequence in M, if not, then there are positive real numbers ε and subsequences $\{u_{n_k}\}$ and $\{u_{m_k}\}$ such that $m_k > n_k$ and $d(u_{n_k}, u_{m_k}) \geq \varepsilon$ for all k. Put $a_k = u_{n_k}$, $b_k = u_{m_k}$ and $M_k = \max\{d(x, a_k), d(x, b_k)\}$.

Note that $M_k \to d(x, M)$ as $k \to \infty$. Now $d(x, a_k) \leq M_k, d(x, b_k) \leq M_k$ and $d(a_k, b_k) \geq \big(\frac{\varepsilon}{M_k}\big)M_k$. It gives

$$d\Big(x, m\Big(a_k, b_k, \frac{1}{2}\Big)\Big) \leq M_k\Big(1 - \delta\Big(\frac{\varepsilon}{M_k}\Big)\Big) \leq M_k\Big(1 - \delta\Big(\frac{d(a_k, b_k)}{M_k}\Big)\Big) .$$

Also $\delta\big(\frac{\varepsilon}{M_k}\big) \leq 1 - \frac{d(x,M)}{M_k}$, letting $k \to \infty$, $\delta\big(\frac{\varepsilon}{M_k}\big) \to 0$ and ε can not be positive. Thus $\{P(x_n)\}$ is a Cauchy sequence in M and therefore converges to a point z in M, as $d(x, z) = d(x, M)$, and $z = P(x)$. □

Theorem 22. *Let F be a bounded T-regular subset of a uniformly convex complete metric space X. Then either $T(x) = x$ for all x in F or there exists a point x_0 in F such that $d(x_0, F) < \operatorname{diam}(F)$.*

Proof. Suppose for some $x \in F$, $x \neq T(x)$, let $d\big(x, T(x)\big) = \varepsilon$. Now for any y in F, $d\big(y, T(x)\big) \leq \operatorname{diam}(F)$ and $d(y, x) \leq \operatorname{diam}(F)$. As F is a T-regular set, so $m\big(x, T(x), \frac{1}{2}\big) \in F$. Now by uniform convexity of X, there exists positive real number $\delta(\varepsilon)$ with

$$d\Big(y, m\Big(x, T(x), \frac{1}{2}\Big)\Big) \leq \big(1 - \delta(\varepsilon)\big) \operatorname{diam}(F) ,$$

which in turn implies

$$d\Big(F, m\Big(x, T(x), \frac{1}{2}\Big)\Big) \leq \big(1 - \delta(\varepsilon)\big) \operatorname{diam}(F) ,$$

and the result follows. □

We prove the following pair of propositions needed for Theorem 25.

Proposition 23. *Let X be a strictly convex metric space, $u \in X$ and M a subset of X. If $x,\ y \in P_M(u)$ with $x \neq y$. Then $m(x,y,t) \notin M$, where $t \in (0,1)$.*

Proof. If $m(x,y,t) \in M$, then $x,\ y \in P_M(u)$ gives $d(x,u) \leq d\big(u,\ m(x,y,t)\big)$, and $d(y,u) \leq d\big(u,\ m(x,y,t)\big)$. Since X is strictly convex metric space, so we arrive at a contradiction. Hence $m(x,y,t) \notin M$, $0 < t < 1$. □

Proposition 24. *Let M be any subset of a strictly convex metric space X and $T : M \to M$. If $P_M(u)$ is a nonempty T-regular set for any $u \in X$, then each point of $P_M(u)$ is a fixed point of T.*

Proof. Suppose for some x in $P_M(u)$, we have $T(x) \neq x$. By the Proposition 23, $m\big(x, T(x), \frac{1}{2}\big) \notin M$. Hence $m\big(x, T(x), \frac{1}{2}\big) \notin P_M(u)$. Since $P_M(u)$ is a T-regular set, therefore $x = T(x)$ must hold. Thus each best M-approximation of u is a fixed point of T. □

Theorem 25. *Let M be a nonempty closed and T-regular subset of a strictly convex metric space X, where T is a compact mapping and u be a point in M. Suppose that $d\big(T(x), u\big) \leq d(x,u)$ for all x in M. Then each x in M, which is best approximation to u, is a fixed point of T.*

Proof. Let $r = d(u, M)$, then there is a minimizing sequence $\{y_n\}$ in M such that $\lim_{n\to\infty} d(u, y_n) = r$. Clearly $\{y_n\}$ is a bounded sequence. Since T is a compact, $\operatorname{cl}(\{T(y_n)\})$ is a compact subset of M and so $\{T(y_n)\}$ has a convergent sequence $\{T(y_{n_k})\}$ with $\lim_{n\to\infty} T(y_{n_k}) = x$ in M. Now

$$r \leq d(u,x) = \lim_{n_k} d\big(u, T(y_{n_k})\big) \leq \lim_{n_k} d(u, y_{n_k}) = \lim d(u, y_n) = r\,.$$

Thus $x \in P_M(u)$. Also, if $y \in P_M(u)$ and $r \leq d\big(T(y), u\big) \leq d(y,u) = r$ imply that $T(y) \in P_M(u)$. Therefore $y \in P_M(u)$ gives $d\big(T(y), u\big) = r$. Now strict convexity of X implies $r \leq d\big(m\big(y, T(y), \frac{1}{2}\big), u\big) < r$. Thus $m\big(y, T(y), \frac{1}{2}\big) \in P_M(u)$.

The result now follows by using Proposition 23. □

Theorem 26. *Let M be a nonempty closed and T-regular subset of strictly convex metric space X, where $T : X \longrightarrow X$ is a compact mapping. If u is a fixed point of T in $X \setminus M$, and*

$$\begin{aligned} d\big(T(x), T(y)\big) \leq \alpha d(x,y) &+ \beta\big(d\big(x, T(x)\big) + d\big(y, T(y)\big)\big) \\ &+ \gamma\big(d\big(x, T(y)\big) + d\big(y, T(x)\big)\big)\,, \end{aligned}$$

for all $x, y \in X$, where α, β and γ are real numbers with $\alpha + 2\beta + \gamma \leq 1$. Then each best approximation in M to u is a fixed point of T.

Proof. For $x \in X$, consider

$$\begin{aligned}\big(T(x), T(u)\big) &\leq \alpha d(x,u) + \beta\big(d\big(x, T(x)\big) + d(u, T(u))\big) + \gamma\big(d\big(x, T(u) + d\big(u, T(x)\big)\big)\\ &= \alpha d(x,u) + \beta(d\big(x, T(x)\big)\big) + \gamma\big(d(x,u) + d\big(T(u), T(x)\big)\big)\\ &\leq \alpha d(x,u) + \beta\big(d(x,u) + d(T(u), T(x))\big) + \gamma\big(d(x,u) + d(T(u), T(x))\big)\\ &= (\alpha+\beta+\gamma) d(x,u) + (\beta+\gamma) d\big(T(u), T(x)\big)\,.\end{aligned}$$

Thus $d\big(T(x), T(u)\big) \leq \big(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\big) d(x,u)$, and so $d\big(T(x), T(u)\big) \leq d(x,u)$.

The result follows by Theorem 25. $\square$

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