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PROLONGATION OF PAIRS OF CONNECTIONS INTO CONNECTIONS ON VERTICAL BUNDLES

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Abstract. Let F be a natural bundle. We introduce the geometrical construction transforming two general connections into a general connection on the F-vertical bundle. Then we determine all natural operators of this type and we generalize the result by I. Kolář and the second author on the prolongation of connections to F-vertical bundles. We also present some examples and applications.

Introduction

Let $\mathcal{M}f_m$ be the category of m-dimensional manifolds and local diffeomorphisms, $\mathcal{FM}$ be the category of fibered manifolds and fiber respecting mappings and $\mathcal{FM}_{m,n}$ be the category of fibered manifolds with m-dimensional bases and n-dimensional fibers and locally invertible fiber respecting mappings.

Consider an arbitrary bundle functor F on the category $\mathcal{M}f_n$ and denote by V^F its vertical modification. Our starting point is the paper [9] by I. Kolář and the second author, who studied the prolongation of a connection Γ on an arbitrary fibered manifold $Y \to M$ with respect to an F-vertical functor V^F. In particular, they have introduced an F-vertical prolongation $\mathcal{V}^F\Gamma$ of a connection Γ and have proved that $\mathcal{V}^F$ is the only natural operator of finite order transforming connections on $Y \to M$ into connections on $V^F Y \to M$. They have also described some conditions under which every natural operator of such a type has finite order. Further, in the case of the vertical Weil functor V^A they have proved that the operator transforming a connection Γ on $Y \to M$ into its vertical prolongation $\mathcal{V}^A\Gamma$ is the only natural one.

The aim of this paper is to study the prolongation of a pair of connections Γ_1 and Γ_2 on $Y \to M$ into a connection on $V^F Y \to M$. Our main result is Theorem 1 which describes all such natural operators. As a direct consequence we prove the generalization of a result by I. Kolář and the second author. In particular, we show that $\mathcal{V}^F$ is the only natural operator transforming connections on $Y \to M$ into connections on $V^F Y \to M$ (without any additional assumption

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on the finite order). In Section 1 we discuss the prolongation of connections on $Y \to M$ into connections on $GY \to M$, where G is a bundle functor on $\mathcal{FM}_{m,n}$. Section 2 is devoted to the construction of a connection on $V^F Y \to M$ by means of a pair Γ_1, Γ_2 of connections on $Y \to M$. This geometrical construction will be based on linear natural operators transforming vector fields on n-manifolds N into vector fields on FN. In Section 3 we introduce some examples and applications. We also show, that in the case of a vertical Weil functor V^A the connection on $V^A Y \to M$ depending on a pair Γ_1, Γ_2 can be constructed by means of the vertical prolongation of the deviation $\delta(\Gamma_1, \Gamma_2)$ of Γ_1 and Γ_2. Finally, the whole Section 4 is devoted to the proof of Theorem 1.

In what follows $Y \to M$ stands for $\mathcal{FM}_{m,n}$-objects and N stands for $\mathcal{M}f_n$-objects. All manifolds and maps are assumed to be of the class C^∞. Unless otherwise specified, we use the terminology and notation from the book [7].

1. Prolongation of connections to $GY \to M$

Recently it has been clarified that the order of bundle functors on $\mathcal{FM}$ is characterized by three integers (r, s, q), $s \geq r \leq q$ and is based on the concept of (r, s, q)-jet, [7]. Consider two fibered manifolds $p : Y \to M$ and $\overline{p} : \overline{Y} \to \overline{M}$ and let r, $s \geq r$, $q \geq r$ be integers. We recall that two $\mathcal{FM}$-morphisms $f, g : Y \to \overline{Y}$ with the base maps $\underline{f}, \underline{g} : M \to \overline{M}$ determine the same (r, s, q)-jet $j^{r,s,q}_y f = j^{r,s,q}_y g$ at $y \in Y$, $p(y) = x$, if

$$j^r_y f = j^r_y g,\ j^s_y(f|Y_x) = j^s_y(g|Y_x),\ j^q_x \underline{f} = j^q_x \underline{g}\,.$$

The space of all such (r, s, q)-jets will be denoted by $J^{r,s,q}(Y, \overline{Y})$. By 12.19 in [7], the composition of $\mathcal{FM}$-morphisms induces the composition of (r, s, q)-jets.

Definition 1 ([9]). A bundle functor G on $\mathcal{FM}_{m,n}$ is said to be of order (r, s, q), if $j^{r,s,q}_y f = j^{r,s,q}_y g$ implies $Gf|G_y Y = Gg|G_y Y$.

Then the integer q is called the base order, s is called the fiber order and r is called the total order of G.

If $X : N \to TN$ is a vector field and F is a bundle functor on $\mathcal{M}f_n$, then we can define the flow prolongation $\mathcal{F}X : FN \to TFN$ of X with respect to F by

$$\mathcal{F}X = \frac{\partial}{\partial t}\Big|_0 F(\exp tX) \tag{1}$$

where $\exp tX$ denotes the flow of X, [7]. Quite analogously, a projectable vector field on a fibered manifold $Y \to M$ is an $\mathcal{FM}$-morphism $Z : Y \to TY$ over the underlying vector field $M \to TM$, and its flow $\exp tZ$ is formed by local $\mathcal{FM}_{m,n}$-morphisms. Further, if G is a bundle functor on $\mathcal{FM}_{m,n}$, the flow prolongation of Z with respect to G is defined by

$$\mathcal{G}Z = \frac{\partial}{\partial t}\Big|_0 G(\exp tZ)\,.$$

By [9], this map is $\mathbf{R}$-linear and preserves bracket.

Proposition 1 ([9]). *If G is of order (r,s,q), then the value of $\mathcal{G}Z$ at each point of G_yY depends on $j_y^{r,s,q}Z$ only.*

Thus the construction of the flow prolongation of projectable vector fields can be interpreted as a map

$$\mathcal{G}_Y : GY \times_Y J^{r,s,q}TY \to TGY\,,$$

where $J^{r,s,q}TY$ denotes the space of all (r,s,q)-jets of projectable vector fields on Y. Since the flow prolongation is $\mathbf{R}$-linear, $\mathcal{G}_Y$ is linear in the second factor.

Now let $\Gamma : Y \to J^1Y$ be a general connection on $p : Y \to M$. In [7] and [9] it is clarified, that if the functor G on $\mathcal{FM}_{m,n}$ has the base order q, then one can construct the induced connection $\mathcal{G}(\Gamma,\Delta)$ on $GY \to M$ by means of an auxiliary linear q-th order connection Δ on the base manifold M. The geometrical construction of the connection $\mathcal{G}(\Gamma,\Delta)$ is the following. Let X be a vector field on M with the coordinate components $X^i(x)$ and let

$$dy^p = \Gamma_i^p(x,y)\,dx^i$$

be the coordinate expression of Γ. Then the Γ-lift of X is a vector field ΓX on Y, whose coordinate form is

$$X^i(x)\frac{\partial}{\partial x^i} + \Gamma_i^p(x,y)X^i(x)\frac{\partial}{\partial y^p}\,.$$

By Proposition 1, the flow prolongation $\mathcal{G}(\Gamma X)$ depends on the q-jets of X only. So we obtain a map

$$\mathcal{G}\Gamma : GY \times_M J^qTM \to TGY\,, \tag{2}$$

which is linear in the second factor. Further, let $\Delta : TM \to J^qTM$ be a linear q-th order connection on M. By linearity, the composition

$$\mathcal{G}(\Gamma,\Delta) := \mathcal{G}\Gamma \circ (\mathrm{id}_{GY} \times_{\mathrm{id}_M} \Delta) : GY \times_M TM \to TGY \tag{3}$$

is the lifting map of a connection on $GY \to M$. Clearly, if the base order of G is zero, then (2) is a connection on $GY \to M$ and we need no auxiliary linear connection Δ. This is the case of a vertical functor V^F, which is defined as follows. Let F be a bundle functor on $\mathcal{M}f_n$ of order s. Its vertical modification V^F is a bundle functor on $\mathcal{FM}_{m,n}$ defined by

$$V^FY = \bigcup_{x\in M} F(Y_x)\,, \quad V^Ff = \bigcup_{x\in M} F(f_x)\,,$$

where f_x is the restriction and corestriction of $f : Y \to \overline{Y}$ over $\underline{f} : M \to \overline{M}$ to the fibers Y_x and $\overline{Y}_{\underline{f}(x)}$, [9]. Obviously, the order of the functor V^F is $(0,s,0)$. Since the base order of V^F is zero, the map (2) defines a connection $\mathcal{V}^F\Gamma$ for every connection Γ on $Y \to M$.

Definition 2 ([9])**.** The connection $\mathcal{V}^F\Gamma$ is called the F-vertical prolongation of Γ.

If $F = T^A$ is a Weil functor, then V^{T^A} is the vertical Weil functor on $\mathcal{FM}_{m,n}$, which will be denoted by V^A. This functor induces the vertical A-prolongation $\mathcal{V}^A\Gamma$. In particular, for $F = T$ we obtain the classical vertical bundle, which will be denoted by V instead of V^T and the corresponding vertical prolongation of Γ will be denoted by $\mathcal{V}\Gamma$. I. Kolář [5] has proved that $\mathcal{V}\Gamma$ is the only natural operator transforming connections on $Y \to M$ into connections on $VY \to M$, see also [7], p. 255. Moreover, the following naturality property of the F-vertical prolongation $\mathcal{V}^F\Gamma$ is an interesting generalization of the well known result concerning the classical vertical prolongation $\mathcal{V}\Gamma$ to an arbitrary bundle functor F on $\mathcal{M}f_n$.

Proposition 2 ([9])**.** *$\mathcal{V}^F$ is the only natural operator of finite order transforming connections on $Y \to M$ into connections on $V^FY \to M$.*

Propositon 3 ([9])**.** *If the standard fiber $F_0(\mathbf{R}^n)$ of F is compact or if $F_0(\mathbf{R}^n)$ contains a point z_0 such that $F(b\mathrm{id}_{\mathbf{R}^n})(z) \to z_0$ if $b \to 0$ for any $z \in F_0(\mathbf{R}^n)$, then every natural operator D transforming connections on $Y \to M$ into connections on $V^FY \to M$ has finite order.*

For example, the assumption of Proposition 3 is satisfied in the case F is a Weil functor T^A. On the other hand, this assumption is not satisfied in the case F is a cotangent bundle functor T^*.

Remark 1. It is well known, that there is no natural operator transforming connections on $Y \to M$ into connections on $J^1Y \to M$, see [5] and [7]. Quite analogously, I. Kolář and the first author have proved that there is no first order natural operator transforming connections on $Y \to M$ into connections on $TY \to M$, [2]. The second author has recently proved the following general result, [13]: If G is a bundle functor on $\mathcal{FM}_{m,n}$ such that $G^1 : \mathcal{M}f_m \to \mathcal{FM}$, $G^1M = G(M \times \mathbf{R}^n)$, $G^1(\varphi) = G(\varphi \times \mathrm{id}_{\mathbf{R}^n})$ is not of order zero, then there is no natural operator transforming connections on $Y \to M$ into connections on $GY \to M$. This means that in this case, the use of an auxiliary linear connection Δ on the base manifold M in the construction (3) is unavoidable. We remark that all natural operators transforming a connection Γ on $Y \to M$ and a linear connection $\Delta : TM \to J^1TM$ into a connection on $J^1Y \to M$ are determined in [5].

2. Prolongation of pairs of connections into connections on vertical bundles

Let $F : \mathcal{M}f_n \to \mathcal{FM}$ be a natural bundle of order s and $V^F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be the corresponding vertical modification. Suppose we have a natural linear operator

$$L : T \rightsquigarrow TF$$

transforming vector fields on N into vector fields on FN. Let $\Gamma_1, \Gamma_2 : Y \times_M TM \to TY$ be connections on an $\mathcal{FM}_{m,n}$-object $Y \to M$. We are going to construct a connection $\mathcal{V}^{F,L}(\Gamma_1,\Gamma_2)$ on $V^FY \to M$ depending canonically on Γ_1 and Γ_2. Clearly, such a connection can be written in the form

$$\mathcal{V}^{F,L}(\Gamma_1,\Gamma_2) : V^FY \times_M TM \to TV^FY\,.$$

Firstly, we define a fiber linear map

$$(\Gamma_1,\Gamma_2)^{F,L} : V^FY \times_M TM \to V(V^FY) \tag{4}$$

covering the identity on V^FY as follows. Let $(u,v) \in (V^FY \times_M TM)_x$, $x \in M$ and let $v^{\Gamma_1}, v^{\Gamma_2}$ (defined on Y_x) be the horizontal lifts of v with respect to Γ_1 and Γ_2 respectively. The difference $v^{\Gamma_1,\Gamma_2} := (v^{\Gamma_1} - v^{\Gamma_2})$ is vertical, so it can be considered as the vector field on Y_x, $v^{\Gamma_1,\Gamma_2} : Y_x \to T(Y_x) = (VY)_x$. Using the linear operator L, we have the vector field

$$L(v^{\Gamma_1,\Gamma_2}) : F(Y_x) = (V^FY)_x \to T\big((V^FY)_x\big) = \big(V(V^FY)\big)_x$$

which can be considered as (defined on $(V^FY)_x$) vertical vector field $L(v^{\Gamma_1,\Gamma_2}) : V^FY \to V(V^FY)$. We put

$$(\Gamma_1,\Gamma_2)^{F,L}(u,v) = L(v^{\Gamma_1,\Gamma_2})(u)\,.$$

Since L is a linear operator, the map $(\Gamma_1,\Gamma_2)^{F,L}$ is linear in the second factor. Further,

$$\mathcal{V}^{F,L}(\Gamma_1,\Gamma_2) := \mathcal{V}^F\Gamma_1 + (\Gamma_1,\Gamma_2)^{F,L} : V^FY \times_M TM \to TV^FY$$

is a connection on $V^FY \to M$ canonically dependent on Γ_1 and Γ_2.

Definition 3. The connection $\mathcal{V}^{F,L}(\Gamma_1,\Gamma_2)$ is called the (F,L)-vertical prolongation of (Γ_1,Γ_2).

From the geometrical construction of $(\Gamma_1,\Gamma_2)^{F,L}$ it follows directly

Lemma 1. *We have*

(i) $(\Gamma_1,\Gamma_2)^{F,L} = -(\Gamma_2,\Gamma_1)^{F,L}$,
(ii) $(\Gamma_1,\Gamma_2)^{F,c_1L_1+c_2L_2} = c_1(\Gamma_1,\Gamma_2)^{F,L_1} + c_2(\Gamma_1,\Gamma_2)^{F,L_2}$, $c_1, c_2 \in \mathbf{R}$,
(iii) $\mathcal{V}^{F,L}(\Gamma,\Gamma) = \mathcal{V}^F\Gamma$.

The main result of the present paper is the following classification theorem.

Theorem 1. $\mathcal{V}^{F,L}$ *are the only natural operators transforming pairs of connections on* $Y \to M$ *into connections on* $V^FY \to M$.

We have the following corollary of Theorem 1.

Corollary 1. $\tilde{\mathcal{V}}^F(\Gamma_1,\Gamma_2) := \frac{1}{2}(\mathcal{V}^F\Gamma_1+\mathcal{V}^F\Gamma_2)$ *is the only natural symmetric operator transforming pairs of connections on* $Y \to M$ *into connections on* $V^FY \to M$.

Proof of Corollary 1. Let D be such an operator. By Theorem 1, $D(\Gamma_1,\Gamma_2) = \mathcal{V}^F\Gamma_1 + (\Gamma_1,\Gamma_2)^{F,L}$. By the symmetry of D we get $\mathcal{V}^F\Gamma_1 + (\Gamma_1,\Gamma_2)^{F,L} = \mathcal{V}^F\Gamma_2 - (\Gamma_1,\Gamma_2)^{F,L}$ because $(\Gamma_2,\Gamma_1)^{F,L} = -(\Gamma_1,\Gamma_2)^{F,L}$. Then $(\Gamma_1,\Gamma_2)^{F,L} = \frac{1}{2}(\mathcal{V}^F\Gamma_2 - \mathcal{V}^F\Gamma_1)$ and $D(\Gamma_1,\Gamma_2) = \frac{1}{2}(\mathcal{V}^F\Gamma_1 + \mathcal{V}^F\Gamma_2)$ as well. □

Now we show that one can omit the finite order assumption in Proposition 2. In this way we obtain the following generalization of this result:

Proposition 2'. $\mathcal{V}^F$ *is the only natural operator transforming connections on* $Y \to M$ *into connections on* $V^FY \to M$.

Proof. Write $\Gamma_1 = \Gamma_2 = \Gamma$ in Corollary 1. Then we obtain $\widetilde{\mathcal{V}}^F(\Gamma,\Gamma) = \mathcal{V}^F\Gamma$. □

Remark 2. The (F,L)-prolongation is a geometrical construction, which transforms two connections Γ_1 and Γ_2 on $Y \to M$ into a connection $\mathcal{V}^{F,L}(\Gamma_1,\Gamma_2)$ on $V^FY \to M$. Another example of a geometrical construction defined on pairs of connections is the mixed curvature, which is defined as the Frölicher-Nijenhuis bracket $[\Gamma_1,\Gamma_2]$. We remark that the mixed curvature is a section $Y \to VY \otimes \otimes^2T^*M$, see 27.4 in [7].

By Theorem 1, natural operators transforming pairs of connections on $Y \to M$ into a connection on $V^FY \to M$ depend on linear natural operators $L : T \rightsquigarrow TF$ on vector fields. Now we show that it suffices to find the basis of such linear operators.

Proposition 4. *Let* $L_1,\dots,L_k$ *be the basis of linear natural operators* $T \rightsquigarrow TF$ *transforming vector fields on* n*-manifolds* N *into vector fields on* FN. *Then all natural operators transforming pairs of connections on* $Y \to M$ *into a connection on* $V^FY \to M$ *are of the form*

$$(\Gamma_1,\Gamma_2) \mapsto \mathcal{V}^F\Gamma_1 + c_1(\Gamma_1,\Gamma_2)^{F,L_1} + \dots + c_k(\Gamma_1,\Gamma_2)^{F,L_k}, \quad , c_i \in \mathbf{R}\,.$$

Proof. An arbitrary linear operator $L : T \rightsquigarrow TF$ is of the form $L = c_1L_1 + \dots + c_kL_k$, $c_i \in \mathbf{R}$. Then the assertion follows from Theorem 1 and from Lemma 1. □

3. Applications

Clearly, the flow prolongation (1) is a natural linear operator $T \rightsquigarrow TF$. So for an arbitrary natural bundle F on $\mathcal{M}f_n$ there exists a natural operator transforming pairs of connections Γ_1,Γ_2 on $Y \to M$ into a connection $\mathcal{V}^{F,\mathcal{F}}(\Gamma_1,\Gamma_2)$ on $V^FY \to M$. Now let $F = T^A$ be a Weil functor determined by a Weil algebra A. By [7], all product preserving functors on $\mathcal{M}f$ are of this type. We have the following action

$$A \times TT^AN \to TT^AN \tag{5}$$

of the elements of A on the tangent vectors on T^AN. Indeed, the multiplication of the tangent vectors of N by reals is a map $m : \mathbf{R} \times TN \to TN$. Applying the functor T^A and using the fact that $T^A\mathbf{R} = A$ we obtain a map $T^A m : A \times T^A TN \to T^A TN$. Finally, the canonical identification $T^A TN \cong TT^A N$ yields the action (5). So for an arbitrary $a \in A$ we have a natural affinor on $T^A N$ of the form

$$\mathrm{af}(a)_N : TT^A N \to TT^A N .$$

By [7], all natural linear operators $T \rightsquigarrow TT^A$ transforming vector fields on N into vector fields on $T^A N$ are of the form

$$\mathrm{af}\,(a) \circ \mathcal{T}^A$$

for all $a \in A$, where $\mathcal{T}^A$ means the flow operator. Thus, we have

Proposition 5. *All natural operators transforming pairs of connections on $Y \to M$ into a connection on $V^A Y \to M$ are of the form*

$$(\Gamma_1, \Gamma_2) \mapsto \mathcal{V}^{T^A,\, \mathrm{af}(a)\circ \mathcal{T}^A}(\Gamma_1, \Gamma_2)$$

for all $a \in A$.

It is well known that $J^1 Y \to Y$ is an affine bundle with the associated vector bundle $VY \otimes T^*M$. So the difference of two connections $\Gamma_1, \Gamma_2 : Y \to J^1 Y$ is a map $\delta(\Gamma_1, \Gamma_2) : Y \to VY \otimes T^*M$, which is called the deviation of Γ_1 and Γ_2. Clearly, this map can be written as

$$\delta(\Gamma_1, \Gamma_2) : Y \times_M TM \to VY . \tag{6}$$

A. Cabras and I. Kolář [1] have constructed the vertical A-prolongation of (6) with respect to the first factor

$$\mathcal{V}_1^A \delta(\Gamma_1, \Gamma_2) : V^A Y \times_M TM \to VV^A Y \tag{7}$$

fiberwise in the following way. Denoting by $q : TM \to M$ the bundle projection, we can write $\delta_z : Y_x \to (VY)_x$ for the map $y \mapsto \delta(\Gamma_1, \Gamma_2)(y, z)$, $y \in Y$, $z \in TM$, $q(z) = x$. Applying T^A we obtain a map

$$(V_1^A \delta)_z := T^A(\delta_z) : T^A(Y_x) = (V^A Y)_x \to T^A((VY)_x) = (V^A VY)_x$$

which yields a map $V_1^A \delta : V^A Y \times_M TM \to V^A VY$. Further, the canonical exchange diffeomorphism of Weil functors $i_N^{B,A} : T^B(T^A N) \to T^A(T^B N)$ from [7] induces the exchange diffeomorphism $i_Y : V^A VY \to VV^A Y$, [1]. Then the map (7) can be defined by

$$\mathcal{V}_1^A \delta(\Gamma_1, \Gamma_2) = i_Y \circ V_1^A \delta . \tag{8}$$

On the other hand, we can construct the vertical A-prolongations $\mathcal{V}^A \Gamma_1, \mathcal{V}^A \Gamma_2 : V^A Y \times_M TM \to TV^A Y$ of Γ_1 and Γ_2. The deviation of the connections $\mathcal{V}^A \Gamma_1$ and $\mathcal{V}^A \Gamma_2$ is a map

$$\delta(\mathcal{V}^A \Gamma_1, \mathcal{V}^A \Gamma_2) : V^A Y \times_M TM \to V(V^A Y) . \tag{9}$$

A. Cabras and I. Kolář have proved the formula

$$\delta(\mathcal{V}^A \Gamma_1, \mathcal{V}^A \Gamma_2) = \mathcal{V}_1^A \delta(\Gamma_1, \Gamma_2) . \tag{10}$$

Consider now a linear map (4), where we put $F = T^A$ and $L = \mathcal{T}^A$, $(\Gamma_1, \Gamma_2)^{T^A, \mathcal{T}^A} : V^A Y \times_M TM \to V(V^A Y)$. We have

Proposition 6. *Let $\mathcal{T}^A$ be the flow operator. Then we have*

$$(\Gamma_1, \Gamma_2)^{\mathcal{T}^A, \mathcal{T}^A} = \mathcal{V}_1^A \delta(\Gamma_1, \Gamma_2)\,. \tag{11}$$

Proof. Denoting by $\delta := \delta(\Gamma_1, \Gamma_2)(y, -) : (TM)_x \to (VY)_x$, we have $\delta(v) = \Gamma_1 v - \Gamma_2 v$ for $v \in (TM)_x$. Since $\delta(v)$ is vertical, it can be considered as a vector field $Y_x \to T(Y_x)$. Applying the flow operator $\mathcal{T}^A$ we obtain a vector field $\mathcal{T}^A\delta(v) : T^A(Y_x) = (V^AY)_x \to T((V^AY)_x) = (V(V^AY))_x$, which can be considered as a vertical vector field on V^AY. This defines the map
(12)

$$(\Gamma_1, \Gamma_2)^{\mathcal{T}^A, \mathcal{T}^A} : V^AY \times_M TM \to V(V^AY)\,, \quad (\Gamma_1, \Gamma_2)^{\mathcal{T}^A, \mathcal{T}^A}(u, v) = \mathcal{T}^A\delta(v)(u)\,.$$

In general, given a vector field $\xi : N \to TN$, the flow prolongation $\mathcal{T}^A\xi$ can be also constructed as the composition $\mathcal{T}^A\xi = \mathrm{i}_N^{A,\mathbb{D}} \circ T^A\xi$, where $\mathrm{i}_N^{A,\mathbb{D}} : T^ATN \to TT^AN$ is the canonical exchange diffeomorphism and $\mathbb{D}$ is the Weil algebra of dual numbers corresponding to the tangent bundle T. By (8) and (12) we have $\mathcal{T}^A\delta = \mathcal{V}_1^A\delta$. □

Remark 3. It is interesting to pose a question whether the formulas (10) and (11) can be generalized for an arbitrary natural bundle F on $\mathcal{M}f_n$. Given any connections Γ_1 and Γ_2 on $Y \to M$, one can construct their F-vertical prolongations $\mathcal{V}^F\Gamma_1, \mathcal{V}^F\Gamma_2 : V^FY \times_M TM \to T(V^FY)$ and then the deviation

$$\delta(\mathcal{V}^F\Gamma_1, \mathcal{V}^F\Gamma_2) : V^FY \times_M TM \to V(V^FY)\,. \tag{13}$$

Further, for any linear natural operator $L : T \rightsquigarrow TF$ we have the map (4). From Theorem 1 it follows that

$$\delta(\mathcal{V}^F\Gamma_1, \mathcal{V}^F\Gamma_2) = (\Gamma_1, \Gamma_2)^{F,L}$$

for some linear natural operator L. By (10) and (11), if $F = T^A$, then $L = \mathcal{T}^A$. From the proof of Theorem 1 (see the construction (14) of L^D) it follows that even in the general case of an arbitrary natural bundle F we have $L = \mathcal{F}$, where $\mathcal{F}$ is the flow operator (1). We remark that the construction of the vertical prolongation (7) and the proof of (11) essentially depend on the existence of the exchange diffeomorphism $\mathrm{i}_Y : V^AVY \to VV^AY$. We recall that the bundle functor F is said to have the point property, if $F(\mathrm{pt}) = \mathrm{pt}$, where pt denote the one-point manifold. From Theorem 39.2 in [7] it follows directly that if F has the point property, then there exists a natural equivalence $\mathrm{i}_Y^F : V^FVY \to VV^FY$ if and only if F is a Weil functor T^A. In this case, i_Y^F coincides with i_Y.

Let $T^{r*}N = J^r(N, \mathbf{R})_0$ be the space of all r-jets from an n-manifold N into reals with target 0. Since $\mathbf{R}$ is a vector space, $T^{r*}N$ has a canonical structure of the vector bundle over N. $T^{r*}N$ is called the r-th order cotangent bundle and the dual vector bundle

$$T^{(r)}N = (T^{r*}N)^*$$

is called the r-th order tangent bundle. For every map $f : N \to N_1$ the jet composition $A \mapsto A \circ (j_x^r f)$, $x \in N$, $A \in (T^{r*}N_1)_{f(x)}$ defines a linear map

$(T^{r*}N_1)_{f(x)} \to (T^{r*}N)_x$. The dual map $T_x^{(r)}f : (T^{(r)}N)_x \to (T^{(r)}N_1)_{f(x)}$ is called the r-th order tangent map of f at x. This defines a vector bundle functor $T^{(r)}$, which is defined on the whole category $\mathcal{M}f$ of all smooth manifolds and all smooth maps. Clearly, for $r = 1$ we obtain the classical tangent functor T and for $r > 1$ the functor $T^{(r)}$ does not preserve products. Obviously, we have the canonical inclusion $TN \subset T^{(r)}N$. Using fiber translations on $T^{(r)}N$, we can extend every section $X : N \to TN$ into a vector field $V(X)$ on $T^{(r)}N$. This defines a linear natural operator $V : T \rightsquigarrow TT^{(r)}$. The second author has in [10] determined all natural operators $T \rightsquigarrow TT^{(r)}$. From this result we obtain directly that all linear natural operators $T \rightsquigarrow TT^{(r)}$ transforming vector fields on N into vector fields on $T^{(r)}N$ are of the form $c_1\mathcal{T}^{(r)} + c_2V$, $c_i \in \mathbf{R}$. Using Proposition 4 we have

Proposition 7. *All natural operators transforming pairs of connections on $Y \to M$ into a connection on $V^{T^{(r)}}Y \to M$ are of the form*

$$(\Gamma_1, \Gamma_2) \mapsto \mathcal{V}^{T^{(r)}}\Gamma_1 + c_1(\Gamma_1, \Gamma_2)^{T^{(r)}, \mathcal{T}^{(r)}} + c_2(\Gamma_1, \Gamma_2)^{T^{(r)}, V}, \quad c_i \in \mathbf{R}.$$

By Corollary 4.1 in [11], all linear natural operators $T \rightsquigarrow TT^*$ are linear combinations (with real coefficients) of the flow operator $\mathcal{T}^*$ and the operator V defined by $V(X)_\omega = \langle \omega, X_x \rangle \cdot C_\omega$, where C is the Liouville vector field of the cotangent bundle and $X \in \mathcal{X}(N)$, $\omega \in T_x^*N$, $x \in N$. Thus, we have

Proposition 8. *All natural operators transforming pairs of connections on $Y \to M$ into a connection on $V^{T^*}Y \to M$ are of the form*

$$(\Gamma_1, \Gamma_2) \mapsto \mathcal{V}^{T^*}\Gamma_1 + c_1(\Gamma_1, \Gamma_2)^{T^*, \mathcal{T}^*} + c_2(\Gamma_1, \Gamma_2)^{T^*, V}, \quad c_i \in \mathbf{R}.$$

Using [11], we can generalize this result in the following way. First, we have r linear natural operators $E_1, \dots, E_r : T \rightsquigarrow TT^{r*}$ defined by

$$E_k(X)(j_x^r\gamma) = \big\langle X(x), j_x^1\gamma \big\rangle \cdot \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 (j_x^r\gamma + tj_x^r(\gamma)^k), \quad k = 1, \dots, r$$

where $X \in \mathcal{X}(N)$ is a vector field on N, $j_x^r\gamma \in T_x^{r*}N$ and $(\gamma)^k$ is the k-th power of the map $\gamma : N \to \mathbf{R}$. Further, if we interpret X as the differentiation, then $(X\gamma - X\gamma(x))(\gamma)^{s-1}$ is a function on N which maps the point $x \in N$ into zero. So we can define linear natural operators $F_2, \dots, F_r : T \rightsquigarrow TT^{r*}$ by

$$F_s(X)(j_x^r\gamma) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \left[j_x^r\gamma + tj_x^r\left((X\gamma - X\gamma(x))(\gamma)^{s-1}\right)\right], \quad s = 2, \dots, r.$$

By [11], the flow operator $\mathcal{T}^{r*}$ and the operators $E_1, \dots, E_r, F_2, \dots, F_r$ form the basis over $\mathbf{R}$ of the vector space of all linear natural operators $T \rightsquigarrow TT^{r*}$. By Proposition 4 we have

Proposition 9. *All natural operators transforming pairs of connections on $Y \to M$ into a connection on $V^{T^{r*}}Y \to M$ are of the form*

$$(\Gamma_1,\Gamma_2) \mapsto \mathcal{V}^{T^{r*}}\Gamma_1 + c_0(\Gamma_1,\Gamma_2)^{T^{r*},\mathcal{T}^{r*}} + c_1(\Gamma_1,\Gamma_2)^{T^{r*},E_1} + \dots + c_r(\Gamma_1,\Gamma_2)^{T^{r*},E_r} + d_2(\Gamma_1,\Gamma_2)^{T^{r*},F_2} + \dots + d_r(\Gamma_1,\Gamma_2)^{T^{r*},F_r}\,, \quad c_i, d_i \in \mathbf{R}\,.$$

We remark that there are many papers which classify all natural operators $T \rightsquigarrow TF$ for particular natural bundles F, see e.g. [4], [6], [10]-[12], [14] and [15]. For example, P. Kobak [4] has determined all natural operators $T \rightsquigarrow TTT^*$ and J. Tomáš [14] has classified all natural operators $T \rightsquigarrow TT^*T^r_k$, where $T^r_kN = J^r_0(\mathbf{R}^k, N)$ is the bundle of k-dimensional velocities of order r. If we restrict ourselves only to linear natural operators, we can easily determine all natural operators transforming pairs of connections on $Y \to M$ into a connection on $V^FY \to M$.

4. Proof of Theorem 1

From now on $\mathbf{R}^{m,n}$ is the trivial bundle $\mathbf{R}^m \times \mathbf{R}^n$ over $\mathbf{R}^m$. The usual coordinates on $\mathbf{R}^{m,n}$ will be denoted by $x^1,\dots,x^m,y^1,\dots,y^n$. If $\tilde{D}$ is a natural operator of our type, then for given connections Γ_1 and Γ_2 on an $\mathcal{FM}_{m,n}$-object $Y \to M$ the difference

$$\Delta(\Gamma_1,\Gamma_2) = \tilde{D}(\Gamma_1,\Gamma_2) - \mathcal{V}^F\Gamma_1 : V^FY \times_M TM \to V(V^FY)$$

is a fiber linear map covering the identity on V^FY. So it remains to describe all natural operators of the type as Δ. Consider a natural operator D of the type as Δ. We prove some auxiliary lemmas.

Lemma 2. *Suppose that*

$$D\Big(\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^m\sum_{j=1}^n \sum_{|\alpha|+|\beta|\le K} \Gamma^j_{1i\alpha\beta}x^\alpha y^\beta dx^i \otimes \frac{\partial}{\partial y^j}\,, \sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^m\sum_{j=1}^n \sum_{|\alpha|+|\beta|\le K} \Gamma^j_{2i\alpha\beta}x^\alpha y^\beta dx^i \otimes \frac{\partial}{\partial y^j}\Big)(u,v) = 0$$

for any $K \in \mathbf{N}$, any $(u,v) \in (V^F\mathbf{R}^{m,n})_0 \times T_0\mathbf{R}^m$, any $\Gamma^j_{1i\alpha\beta}$ and any $\Gamma^j_{2i\alpha\beta}$ for i,j,α,β as indicated. Then $D = 0$.

Proof. It follows from a corollary of non-linear Peetre theorem (Corollary 19.8 in [7]). □

Lemma 3. *Suppose that*

$$D\Big(\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + y^\beta dx^{i_0} \otimes \frac{\partial}{\partial y^{j_0}}, \sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i}\Big)(u,v) = 0$$

and

$$D\Big(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i}, \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + y^\beta dx^{i_0} \otimes \frac{\partial}{\partial y^{j_0}}\Big)(u,v) = 0$$

for any $(u,v) \in (V^F\mathbf{R}^{m,n})_0 \times T_0\mathbf{R}^m$*, any* n*-tuple* β *and any* $i_0 = 1,\dots,m$ *and* $j_0 = 1,\dots,n$*. Then* $D = 0$*.*

Proof. Using the invariance of D with respect to the base homotheties $t\,\mathrm{id}_{\mathbf{R}^m} \times \mathrm{id}_{\mathbf{R}^n}$ for $t > 0$ we get the homogeneity condition

$$\begin{aligned}
&D\Big(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{|\alpha|+|\beta|\le K} t^{|\alpha|+1}\Gamma^j_{1i\alpha\beta}x^\alpha y^\beta dx^i \otimes \frac{\partial}{\partial y^j}\,,\\
&\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{|\alpha|+|\beta|\le K} t^{|\alpha|+1}\Gamma^j_{2i\alpha\beta}x^\alpha y^\beta dx^i \otimes \frac{\partial}{\partial y^j}\Big)(u,v)\\
&= tD\Big(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{|\alpha|+|\beta|\le K} \Gamma^j_{1i\alpha\beta}x^\alpha y^\beta dx^i \otimes \frac{\partial}{\partial y^j}\,,\\
&\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{|\alpha|+|\beta|\le K} \Gamma^j_{2i\alpha\beta}x^\alpha y^\beta dx^i \otimes \frac{\partial}{\partial y^j}\Big)(u,v)\,.
\end{aligned}$$

By the homogeneous function theorem, this type of homogeneity gives that

$$\begin{aligned}
&D\Big(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{|\alpha|+|\beta|\le K} \Gamma^j_{1i\alpha\beta}x^\alpha y^\beta dx^i \otimes \frac{\partial}{\partial y^j}\,,\\
&\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{|\alpha|+|\beta|\le K} \Gamma^j_{2i\alpha\beta}x^\alpha y^\beta dx^i \otimes \frac{\partial}{\partial y^j}\Big)(u,v)
\end{aligned}$$

depends linearly on $\Gamma^j_{1i(0)\beta}$ and $\Gamma^j_{2i(0)\beta}$ and is independent of $\Gamma^j_{1i\alpha\beta}$ and $\Gamma^j_{2i\alpha\beta}$ for $|\alpha| > 0$. So, the assumptions of the lemma imply the assumption of Lemma 2, which completes the proof. □

Lemma 4. *Suppose that*

$$D\Big(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + dx^{i_0} \otimes Y, \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i}\Big)(u,v) = 0$$

and

$$D\Big(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i}, \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + dx^{i_0} \otimes Y\Big)(u,v) = 0$$

for any $(u,v) \in (V^F\mathbf{R}^{m,n})_0 \times T_0\mathbf{R}^m$*, any* $i_0 = 1,\dots,m$ *and any vector field* Y *on* $\mathbf{R}^n$*. Then* $D = 0$*.*

Proof. Obviously, the assumptions of the lemma imply the assumptions of Lemma 3, which completes the proof. □

Lemma 5. *Suppose that*

$$D\Big(\sum_{i=1}^m dx^i\otimes\frac{\partial}{\partial x^i}+dx^1\otimes\frac{\partial}{\partial y^1},\sum_{i=1}^m dx^i\otimes\frac{\partial}{\partial x^i}\Big)(u,v)=0$$

and

$$D\Big(\sum_{i=1}^m dx^i\otimes\frac{\partial}{\partial x^i},\sum_{i=1}^m dx^i\otimes\frac{\partial}{\partial x^i}+dx^1\otimes\frac{\partial}{\partial y^1}\Big)(u,v)=0$$

for any $(u,v)\in(V^F\mathbf{R}^{m,n})_0\times T_0\mathbf{R}^m$. *Then* $D=0$.

Proof. Any non-vanishing vector field Y on $\mathbf{R}^n$ is locally $\frac{\partial}{\partial y^1}$ modulo a local diffeomorphism $\varphi:\mathbf{R}^n\to\mathbf{R}^n$. There exists a diffeomorphism $\psi:\mathbf{R}^m\to\mathbf{R}^m$ sending x^{i_0} into x^1. Using the invariance of D with respect to $\mathcal{FM}_{m,n}$-map $\psi\times\varphi$ we can see that the assumptions of the lemma imply the assumptions of Lemma 4 with non-vanishing Y. Then the regularity of D implies the assumptions of Lemma 4, which completes the proof. □

Lemma 6. *Suppose that*

$$D\Big(\sum_{i=1}^m dx^i\otimes\frac{\partial}{\partial x^i}+dx^1\otimes Y,\sum_{i=1}^m dx^i\otimes\frac{\partial}{\partial x^i}\Big)(u,v)=0$$

for any $(u,v)\in(V^F\mathbf{R}^{m,n})_0\times T_0\mathbf{R}^m$, *and any vector field* Y *on* $\mathbf{R}^n$. *Then* $D=0$.

Proof. The assumption of the lemma implies the first assumption of Lemma 5. Further, using the invariance of D with respect to $\mathcal{FM}_{m,n}$-map $(x^1,\dots,x^m,-y^1+x^1,y^2,\dots,y^n)$ we obtain the second assumption of Lemma 5. Finally, Lemma 5 completes the proof. □

Lemma 7. *Suppose that*

$$D\Big(\sum_{i=1}^m dx^i\otimes\frac{\partial}{\partial x^i}+dx^1\otimes Y,\sum_{i=1}^m dx^i\otimes\frac{\partial}{\partial x^i}\Big)\Big(u,\frac{\partial}{\partial x^1}(0)\Big)=0$$

for any $u\in(V^F\mathbf{R}^{m,n})_0$, *and any vector field* Y *on* $\mathbf{R}^n$. *Then* $D=0$.

Proof. Any vector $v\in T_0\mathbf{R}^m$ with $d_0x^1(v)\neq0$ is proportional to $\frac{\partial}{\partial x^1}(0)$ modulo a diffeomorphism $\psi:\mathbf{R}^m\to\mathbf{R}^m$ preserving x^1. Using the invariance of D with respect to $\mathcal{FM}_{m,n}$-map $\psi\times\mathrm{id}_{\mathbf{R}^n}$ we see that the assumption of the lemma implies the assumption of Lemma 6 with $d_0x^1(v)\neq0$. Then using the regularity of D we obtain the assumption of Lemma 6, which completes the proof. □

Let Y be a vector field on an n-manifold N. Define a vector field $L^D(Y)$ on $F(N)$ by

(14)

$$L^D(Y)(u)=D\Big(\sum_{i=1}^m dx^i\otimes\frac{\partial}{\partial x^i}+dx^1\otimes Y,\sum_{i=1}^m dx^i\otimes\frac{\partial}{\partial x^i}\Big)\Big(u,\frac{\partial}{\partial x^1}(0)\Big)\in T_uF(N)$$

for any $u\in(V^F(\mathbf{R}^m\times N))_0=F(N)$, where we use the obvious identification $V_u(V^F(\mathbf{R}^m\times N))=T_uF(N)$.

Lemma 8. *The $\mathcal{M}f_n$-natural operator $L^D : T \rightsquigarrow TF$ is linear.*

Proof. The $\mathcal{M}f_n$-naturality is a simple consequence of the invariance of D with respect to $\mathcal{FM}_{m,n}$-maps of the form $\mathrm{id}_{\mathbf{R}^m} \times \varphi$. Further, by the invariance of D with respect to the base homotheties $t\,\mathrm{id}_{\mathbf{R}^m} \times \mathrm{id}_{\mathbf{R}^n}$ for $t > 0$ we get the homogeneity condition $D(tY)(u) = tD(Y)(u)$. So, the linearity is an immediate consequence of the homogeneous function theorem. □

Lemma 9. *We have*

$$D\Big(\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + dx^1 \otimes Y, \sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i}\Big)\Big(u, \frac{\partial}{\partial x^1}(0)\Big)$$
$$= \Big(\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + dx^1 \otimes Y, \sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i}\Big)^{F,L^D}\Big(u, \frac{\partial}{\partial x^1}(0)\Big)$$

for any $u \in (V^F\mathbf{R}^{m,n})_0$ and $Y \in \mathcal{X}(\mathbf{R}^n)$, where $(\Gamma_1, \Gamma_2)^{F,L}$ was defined in Section 2.

Proof. Observe that $v^\Gamma = v + Y$ if $\Gamma = \sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + dx^1 \otimes Y$ and $v = \frac{\partial}{\partial x^1}(0)$. □

Now, using Lemma 7 we see that $D(\Gamma_1, \Gamma_2) = (\Gamma_1, \Gamma_2)^{F,L^D}$. Therefore $\tilde{D} = \mathcal{V}^{F,L^\Delta}$ and the proof of Theorem 1 is complete. □

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