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INTEGRABILITY AND L^1- CONVERGENCE OF REES-STANOJEVIĆ SUMS WITH GENERALIZED SEMI-CONVEX COEFFICIENTS OF NON-INTEGRAL ORDERS

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Abstract. Integrability and L^1-convergence of modified cosine sums introduced by Rees and Stanojević under a class of generalized semi-convex null coefficients are studied by using Cesàro means of non-integral orders.

1. Introduction

Let

$$g(x) = \sum_{k=1}^{\infty} a_k \cos kx \tag{1.1}$$

be the Fourier cosine series.

The problem of L^1-convergence of the Fourier cosine series (1.1) has been settled for various special classes of coefficients. Young [9] found that $a_n \log n = o(1)$, $n \to \infty$ is a necessary and sufficient condition for the L^1-convergence of the cosine series with convex ($\triangle^2 a_n \geq 0$) coefficients, and Kolmogorov [8] extended this result to the cosine series with quasi-convex $\big(\sum\limits_{n=1}^{\infty} n \left|\triangle^2 a_n\right| < \infty\big)$ coefficients and proved the following well known theorem:

Theorem 1.1. *If $\{a_n\}$ is a quasi-convex null sequence, then for the L^1-convergence of the cosine series* (1.1)*, it is necessary and sufficient that* $\lim\limits_{n\to\infty} a_n \log n = 0$.

Definition ([6])**.** A sequence $\{a_n\}$ is said to be semi-convex, if

$$a_n \to 0 \quad \text{as} \quad n \to \infty\,,$$

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and

$$\sum_{n=1}^{\infty} n\left|\triangle^2 a_{n-1}+\triangle^2 a_n\right|<\infty\,,\qquad (a_0=0) \tag{1.2}$$

where

$$\triangle^2 a_n = \triangle a_n - \triangle a_{n+1}\,.$$

It may be remarked here that every quasi-convex null sequence is semi- convex.

In 1968, Kano [6] generalized Theorem 1.1 in the following form:

Theorem 1.2. *If $\{a_k\}$ is semi-convex null sequence, then* (1.1) *is a Fourier series, or equivalently it represents an integrable function.*

Later on, Garrett and Stanojević [5] introduced modified cosine sums

$$g_n(x)=\frac{1}{2}\sum_{k=0}^{n}\triangle a_k+\sum_{k=1}^{n}\sum_{j=k}^{n}(\triangle a_j)\cos kx \tag{1.3}$$

and proved the following theorem.

Theorem 1.3. *Let $\{a_n\}$ be a null sequence of bounded variation. Then the sequence of modified cosine sums*

$$g_n(x)=S_n(x)-a_{n+1}D_n(x) \tag{1.4}$$

where $S_n(x)$ are the partial sums of the cosine series (1.1) *and $D_n(x)$ is the Dirichlet kernel, converges in L^1-norm to $g(x)$, the pointwise sum of the cosine series, if and only if for every $\epsilon>0$, there exists $\delta(\epsilon)>0$, independent of n, such that*

$$\int_0^{\delta}\sum_{k=n+1}^{\infty}|\triangle a_k D_k(x)|\,dx<\varepsilon\,,\quad \textit{for every}\quad n\,. \tag{1.5}$$

This result contains as a special case a number of classical and neo-classical results. In particular, in [5] the following corollary to Theorem 1.3 is proved.

Theorem 1.4. *Let $\{a_n\}$ be a null sequence of bounded variation satisfying condition* (1.5)*. Then the cosine series is the Fourier series of its sum $g(x)$ and $\|S_n-g\|=o(1)$, $n\to\infty$ is equivalent to $a_n\log n=o(1)$, $n\to\infty$.*

In [4] Garrett and Stanojević proved the following theorem:

Theorem 1.5. *If $\{a_n\}$ is a null quasi-convex sequence, then $g_n(x)$ converges to $g(x)$ in the L^1-norm.*

We generalize semi-convexity of null sequence in the following way:

Definition ([7])**.** A sequence $\{a_n\}$ is said to be generalized semi-convex, if

$$a_n\to 0\quad\text{as}\quad n\to\infty$$

and

$$\sum_{n=1}^{\infty} n^{\alpha}\left|(\triangle^{\alpha+1}a_{n-1}+\triangle^{\alpha+1}a_n)\right|<\infty\,,\quad \text{for}\quad \alpha>0\quad (a_0=0)\,. \tag{1.6}$$

For $\alpha = 1$, this class reduces to the class defined in [6].

In [7] Kaur, K. and Bhatia, S. S. proved the following theorem:

Theorem 1.6. *If*

$$a_n \to 0 \quad as \quad n \to \infty$$

and

$$\sum_{n=1}^{\infty} n^{\alpha} \left|(\triangle^{\alpha+1} a_{n-1} + \triangle^{\alpha+1} a_n)\right| < \infty \,,$$

where $\alpha > 0$ is an integral ($a_0 = 0$), then $g_n(x)$ converges to $g(x)$ in L^1-metric if and only if $\triangle a_n \log n = o(1)$, as $n \to \infty$.

The object of this paper is to show that the above mentioned Theorem of Kaur, K. and Bhatia, S. S. holds good for non-integral values of $\alpha > 0$.

2. Notation and formulae

In what follows, the following notations are used [10]:

Given a sequence S_0, $S_1, S_2, \ldots$, we define for every $\alpha = 0, 1, 2, \ldots$, the sequence S_0^{α}, $S_1^{\alpha}, S_2^{\alpha}, \ldots$, by the conditions:

$$S_n^0 = S_n$$
$$S_n^{\alpha} = S_0^{\alpha-1} + S_1^{\alpha-1} + S_2^{\alpha-1} + \cdots + S_n^{\alpha-1} \qquad (\alpha = 1, 2, \ldots, n = 0, 1, 2, \ldots)\,.$$

Similarly for $\alpha = 0, 1, 2, \ldots$, we define the sequence of numbers $A_0^{\alpha}, A_1^{\alpha}, A_2^{\alpha}, \ldots$, by the conditions

$$A_n^0 = 1\,,$$
$$A_n^{\alpha} = A_0^{\alpha-1} + A_1^{\alpha-1} + A_2^{\alpha-1} + \cdots + A_n^{\alpha-1} \qquad (\alpha = 1, 2, \ldots, \quad n = 0, 1, 2, \ldots)\,.$$

Consider $\sum a_n$ be a given infinite series. For any real number α the Cesàro sums of order α of $\sum a_n$ are defined by

$$S_n^{\alpha}(a_p) = S_n^{\alpha} = \sum_{p=0}^{n} A_{n-p}^{\alpha} a_p = \sum_{p=0}^{n} A_{n-p}^{\alpha-1} S_p\,,$$

where A_p^{α} denotes the binomial coefficients and are given by the following relations

$$\sum_{p=0}^{\infty} A_p^{\alpha} x^p = (1-x)^{-\alpha-1}$$

and S_n's are given by

$$\sum_{p=0}^{\infty} S_p^{\alpha} x^p = (1-x)^{-\alpha} \sum_{p=0}^{\infty} S_p x^p\,. \tag{2.1}$$

Also

$$A_n^\alpha = \sum_{p=0}^{n} A_p^{\alpha-1}, \qquad A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1}$$

$$A_n^\alpha = \binom{n+\alpha}{n} \simeq \frac{n^\alpha}{\Gamma(\alpha+1)} \qquad (\alpha \neq -1, -2, -3, \dots).$$

The Cesàro means $T_k^\alpha$of order α are then defined by

$$T_k^\alpha = \frac{S_k^\alpha}{A_k^\alpha}.$$

Also for $0 < x \leq \pi$, let

$$\begin{aligned}
\bar{D}_0(x) &= -\frac{1}{2}\cot\frac{x}{2} \\
\bar{S}_n(x) &= \bar{D}_0(x) + \sin x + \sin 2x + \cdots + \sin nx \\
\bar{S}_n^1(x) &= \bar{S}_0(x) + \bar{S}_1(x) + \bar{S}_2(x) + \cdots + \bar{S}_n(x) \\
\bar{S}_n^2(x) &= \bar{S}_0^1(x) + \bar{S}_1^1(x) + \bar{S}_2^1(x) + \cdots + \bar{S}_n^1(x) \\
&\vdots \\
\bar{S}_n^k(x) &= \bar{S}_0^{k-1}(x) + \bar{S}_1^{k-1}(x) + \bar{S}_2^{k-1}(x) + \cdots + \bar{S}_n^{k-1}(x).
\end{aligned}$$

The conjugate Cesàro means $\bar{T}_k^\alpha$ of order α are defined by

$$\bar{T}_k^\alpha = \frac{\bar{S}_k^\alpha}{A_k^\alpha}. \tag{2.2}$$

The following formulae will also be needed:

$$\bar{S}_n^\alpha(\bar{S}_p^r) = \bar{S}_n^{\alpha+r+1}, \tag{2.3}$$

$$\bar{S}_n^{\alpha+1} - \bar{S}_{n-1}^{\alpha+1} = \bar{S}_n^\alpha, \qquad \sum_{p=0}^{n} A_{n-p}^\alpha A_p^\beta = A_n^{\alpha+\beta+1}. \tag{2.4}$$

For any positive integer α the differences of order α of the sequence $\{a_n\}$ are defined by the equations

$$\triangle^1 a_n = a_n - a_{n+1},$$

$$\triangle^\alpha a_n = \triangle(\triangle^{\alpha-1} a_n), \quad n = 0, 1, 2, \dots.$$

For these differences we have

$$\triangle^\alpha a_n = \sum_{m=0}^{\alpha} A_m^{-\alpha-1} a_{n+m} = \sum_{m=0}^{\infty} A_m^{-\alpha-1} a_{n+m}, \tag{2.5}$$

since $A_m^{-\alpha-1} = 0$ for $m \geqslant \alpha + 1$.

If the series (2.5) are convergent for some α which is not a positive integer, then we denote the differences

$$\triangle^\alpha a_n = \sum_{m=0}^{\infty} A_m^{-\alpha-1} a_{n+m}, \qquad n = 0, 1, 2, \dots. \tag{2.6}$$

The broken differences $\triangle_n^\alpha a_p$ are defined by

$$\triangle_n^\alpha a_p = \sum_{m=0}^{n-p} A_m^{-\alpha-1} a_{p+m} \,. \tag{2.7}$$

By repeated partial summation of order α,

$$\sum_{p=0}^{n} a_p b_p = \sum_{p=0}^{n} \bar{S}_p^{\alpha-1}(a_p) \triangle_n^\alpha b_p \,. \tag{2.8}$$

If α is positive integer then we have

$$\sum_{p=0}^{n} a_p b_p = \sum_{p=0}^{n-\alpha} \bar{S}_p^{\alpha-1}(a_p) \triangle^\alpha b_p + \sum_{p=n-\alpha+1}^{n} \bar{S}_p^{\alpha-1}(a_p) \triangle_n^\alpha b_p \,. \tag{2.9}$$

3. Lemmas

We need the following lemmas for the proof of our result:

Lemma 3.1 ([3]). *If $\alpha \geqslant 0$, $p \geqslant 0$,*

(i) $\epsilon_n = o(n^{-p})$,

(ii) $\sum\limits_{n=0}^{\infty} A_n^{\alpha+p} \left|\triangle^{\alpha+1}\epsilon_n\right| < \infty$,

then

(iii) $\sum\limits_{n=0}^{\infty} A_n^{\lambda+p} \left|\triangle^{\lambda+1}\epsilon_n\right| < \infty$, *for* $-1 \leq \lambda \leq \alpha$ *and*

(iv) $A_n^{\lambda+p}\triangle^\lambda \epsilon_n$ *is of bounded variation for* $0 \leq \lambda \leq \alpha$ *and tends to zero as* $n \to \infty$.

Lemma 3.2 ([1]). *Let r be the real number $\geqslant 0$. If the sequence $\{\epsilon_n\}$ satisfies the conditions:*

(i) $\epsilon_n = O(1)$ *and*

(ii) $\sum\limits_{n=1}^{\infty} n^r \left|\triangle^{r+1}\epsilon_n\right| < \infty$,

then

$$\triangle^\beta \epsilon_n = \sum_{m=0}^{\infty} A_m^{r-\beta} \triangle^{r+1} \epsilon_{n+m} \,, \quad \textit{for} \quad \beta > 0 \,.$$

Lemma 3.3 ([2]). *If $0 \leqslant \delta \leq 1$ and $0 \leq m < n$, then*

$$\Big| \sum_{i=0}^{m} A_{n-i}^{\delta-1} S_i \Big| \leq \max_{0\leq p\leq m} \left|S_p^\delta\right| .$$

Lemma 3.4 ([10]). *Let $\bar{S}_n(x)$ and $\bar{T}_n^\alpha$ be the n-th partial sum and conjugate Cesàro mean of order $\alpha > 0$, respectively, of the series*

$$\bar{D}_0(x) + \sin x + \sin 2x + \sin 3x + \cdots + \sin nx + \ldots$$

Then

(i) $\int\limits_0^\pi \left|\bar{S}_n(x)\right| dx \sim \log n$,

(ii) $\int\limits_0^\pi \left|\bar{T}_n^\alpha\right| dx$ *remains bounded for all* n.

4. Main result

The main result of this paper is the following theorem:

Theorem 4.1. *If*

$$a_n \to 0 \qquad as \quad n \to \infty$$

and

$$\sum_{n=1}^{\infty} n^\alpha \left|(\triangle^{\alpha+1} a_{n-1} + \triangle^{\alpha+1} a_n)\right| < \infty ,$$

where $\alpha > 0$ *is non-integral* $(a_0 = 0)$, *then* $g_n(x)$ *converges to* $g(x)$ *in* L^1*-metric if and only if* $\triangle a_n \log n = o(1)$, *as* $n \to \infty$.

Proof. We have

$$\begin{aligned}
g_n(x) &= \frac{1}{2}\sum_{k=0}^{n} \triangle a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} (\triangle a_j)\cos kx \\
&= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - a_{n+1} D_n(x) \\
&= \sum_{k=1}^{n} a_k \cos kx - a_{n+1} D_n(x) \qquad (a_0 = 0) \\
&= \sum_{k=1}^{n-1} (a_{k-1} - a_{k+1}) \frac{\sin kx}{2\sin x} + a_{n-1}\frac{\sin nx}{2\sin x} \\
&\quad + a_n \frac{\sin(n+1)x}{2\sin x} - a_{n+1} D_n(x) ,
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
D_n(x) &= \frac{\sin nx + \sin(n+1)x}{2\sin x} , \\
g_n(x) &= \sum_{k=1}^{n-1} (a_{k-1} - a_{k+1}) \frac{\sin kx}{2\sin x} + a_{n-1}\frac{\sin nx}{2\sin x} + a_n \frac{\sin(n+1)x}{2\sin x} \\
&\quad - a_{n+1}\frac{\sin nx}{2\sin x} - a_{n+1}\frac{\sin(n+1)x}{2\sin x} \\
&= \frac{1}{2\sin x}\sum_{k=1}^{n} (\triangle a_{k-1} + \triangle a_k)\sin kx + \triangle a_n \frac{\sin(n+1)x}{2\sin x} .
\end{aligned}$$

Applying Abel's transformation, we have

$$\begin{aligned}
g_n(x) &= \frac{1}{2\sin x}\Big[\sum_{k=1}^{n-1}(\triangle^2 a_{k-1}+\triangle^2 a_k)\sum_{v=1}^{k}\sin vx\Big]\\
&\quad + \frac{1}{2\sin x}\Big[(\triangle a_{n-1}+\triangle a_n)\sum_{v=1}^{n}\sin vx\Big] + \triangle a_n\frac{\sin(n+1)x}{2\sin x}\\
&= \frac{1}{2\sin x}\Big[\sum_{k=1}^{n-1}(\triangle^2 a_{k-1}+\triangle^2 a_k)(\bar{S}_k^0(x)-\bar{S}_0(x))\Big]\\
&\quad + \frac{1}{2\sin x}\big[(\triangle a_{n-1}+\triangle a_n)\big(\bar{S}_n^0(x)-\bar{S}_0(x)\big)\big] + \triangle a_n\frac{\sin(n+1)x}{2\sin x}\\
&= \frac{1}{2\sin x}\Big[\sum_{k=1}^{n-1}(\triangle^2 a_{k-1}+\triangle^2 a_k)\bar{S}_k^0(x) - \sum_{k=1}^{n-1}(\triangle^2 a_{k-1}+\triangle^2 a_k)\bar{S}_0(x)\Big]\\
&\quad + \frac{1}{2\sin x}\left[(\triangle a_{n-1}+\triangle a_n)\bar{S}_n^0(x) - (\triangle a_{n-1}+\triangle a_n)\bar{S}_0(x)\right]\\
&\quad + \triangle a_n\frac{\sin(n+1)x}{2\sin x}\\
&= \frac{1}{2\sin x}\Big[\sum_{k=1}^{n-1}(\triangle^2 a_{k-1}+\triangle^2 a_k)\bar{S}_k^0(x) + (\triangle a_{n-1}+\triangle a_n)\bar{S}_n^0(x) + a_2\bar{S}_0(x)\Big]\\
&\quad + \triangle a_n\frac{\sin(n+1)x}{2\sin x}\,. \qquad (4.2)
\end{aligned}$$

As $\alpha > 0$ is non-integral. Let $\alpha = r+\delta$, r is the integral part of α, and δ is the fractional part, $0<\delta<1$.

Case (i). Let $r=0$.

Applying Abel's transformation of order $-\delta+1$, we have by (2.8)

$$\begin{aligned}
&\sum_{k=1}^{n-1}\bar{S}_k^{\delta-1}(x)\left(\triangle^{\delta+1}a_{k-1}+\triangle^{\delta+1}a_k\right)\\
&\qquad = \sum_{k=1}^{n-1}\bar{S}_k(x)\sum_{m=1}^{n-(k+1)}A_m^{\delta-2}(\triangle^{\delta+1}a_{m+k-1}+\triangle^{\delta+1}a_{m+k})
\end{aligned}$$

Also by Lemma 3.2, we have

$$\begin{aligned}
\sum_{k=1}^{n-1}\bar{S}_k^{\delta-1}(x)\left(\triangle^{\delta+1}a_{k-1}+\triangle^{\delta+1}a_k\right) &= \sum_{k=1}^{n-1}\bar{S}_k(x)\Big\{\left(\triangle^2 a_{k-1}+\triangle^2 a_k\right)\\
&\qquad - \sum_{m=n-k}^{\infty}A_m^{\delta-2}(\triangle^{\delta+1}a_{m+k-1}+\triangle^{\delta+1}a_{m+k})\Big\}\\
&= \sum_{k=1}^{n-1}\bar{S}_k(x)\left(\triangle^2 a_{k-1}+\triangle^2 a_k\right) - R_n(x)
\end{aligned}$$

where

$$R_n(x) = \sum_{k=1}^{n-1} \bar{S}_k(x)\{A_{n-k}^{\delta-2}(\triangle^{\delta+1}a_{n-1} + \triangle^{\delta+1}a_n) + A_{n-k+1}^{\delta-2}(\triangle^{\delta+1}a_n + \triangle^{\delta+1}a_{n+1}) + \dots\}\,.$$

Therefore

$$\frac{1}{2\sin x}\sum_{k=1}^{n-1} \bar{S}_k(x)\left(\triangle^2 a_{k-1} + \triangle^2 a_k\right) = \frac{1}{2\sin x}\Big\{\sum_{k=1}^{n-1} \bar{S}_k^{\delta-1}(x)\left(\triangle^{\delta+1}a_{k-1} + \triangle^{\delta+1}a_k\right) + R_n(x)\Big\}$$

and consequently by (4.2)

$$g_n(x) = \frac{1}{2\sin x}\Big\{\sum_{k=1}^{n-1} \bar{S}_k^{\delta-1}(x)\left(\triangle^{\delta+1}a_{k-1} + \triangle^{\delta+1}a_k\right) + R_n(x) + (\triangle a_{n-1} + \triangle a_n)\bar{S}_n^0(x) + a_2\bar{S}_0(x)\Big\} + \triangle a_n \frac{\sin(n+1)x}{2\sin x}\,.$$

When $r = 0$, then $\alpha = \delta$ and

$$g(x) = \lim_{n\to\infty} g_n(x) = \frac{1}{2\sin x}\Big\{\sum_{k=1}^{\infty} \bar{S}_k^{\delta-1}(x)\left(\triangle^{\delta+1}a_{k-1} + \triangle^{\delta+1}a_k\right) + a_2\bar{S}_0(x)\Big\}$$

Therefore,

$$\begin{aligned}\int_0^\pi \big|g(x) - g_n(x)\big|\,dx &\le \int_0^\pi \Big|\frac{1}{2\sin x}\Big\{\sum_{k=n}^{\infty} \bar{S}_k^{\delta-1}(x)(\triangle^{\delta+1}a_{k-1} + \triangle^{\delta+1}a_k) \\ &\quad - R_n(x) - (\triangle a_{n-1} + \triangle a_n)\bar{S}_n^0(x)\Big\}\Big|\,dx + \int_0^\pi \Big|\triangle a_n \frac{\sin(n+1)x}{2\sin x}\Big|\,dx \\ &\leqslant C\Big\{\sum_{k=n}^{\infty} \big|(\triangle^{\delta+1}a_{k-1} + \triangle^{\delta+1}a_k)\big| \int_0^\pi \big|\bar{S}_k^{\delta-1}(x)\big|\,dx \\ &\quad + |(\triangle a_{n-1} + \triangle a_n)| \int_0^\pi \big|\bar{S}_n^0(x)dx\big|\Big\} \\ &\quad + C\int_0^\pi |R_n(x)|\,dx + \int_0^\pi \Big|\triangle a_n \frac{\sin(n+1)x}{2\sin x}\Big|\,dx\end{aligned}$$

$$= C\Big\{ \sum_{k=n}^{\infty} A_k^{\delta-1} \left|(\triangle^{\delta+1} a_{k-1} + \triangle^{\delta+1} a_k)\right| \int_0^{\pi} \left|\bar{T}_k^{\delta-1}(x)\right| \, dx$$

$$+ \left|(\triangle a_{n-1} + \triangle a_n)\right| \int_0^{\pi} \left|\bar{S}_n^0(x) dx\right| \Big\}$$

$$+ C \int_0^{\pi} |R_n(x)| \, dx + \int_0^{\pi} \left|\triangle a_n \frac{\sin(n+1)x}{2\sin x}\right| dx$$

$$\leqslant C_1 \sum_{k=n}^{\infty} A_k^{\delta-1} \left|(\triangle^{\delta+1} a_{k-1} + \triangle^{\delta+1} a_k)\right| + \left|(\triangle a_{n-1} + \triangle a_n)\right|$$

$$+ C \int_0^{\pi} |R_n(x)| \, dx + \int_0^{\pi} \left|\triangle a_n \frac{\sin(n+1)x}{2\sin x}\right| dx$$

$$= o(1) + C_1 \left|(a_{n-1} - a_{n+1})\right| + C \int_0^{\pi} |R_n(x)| \, dx + \int_0^{\pi} \left|\triangle a_n \frac{\sin(n+1)x}{2\sin x}\right| dx \,, \tag{4.3}$$

by Lemma 3.1 and 3.4.

Now for the estimate of $\int_0^{\pi} |R_n(x)| \, dx$, we have

$$\int_0^{\pi} |R_n(x)| \, dx = \int_0^{\pi} \Big| \Big(\sum_{k=1}^{n-1} \bar{S}_k(x) A_{n-k}^{\delta-2}\Big)(\triangle^{\delta+1} a_{n-1} + \triangle^{\delta+1} a_n)$$

$$+ \Big(\sum_{k=1}^{n-1} \bar{S}_k(x) A_{n-k+1}^{\delta-2}\Big)(\triangle^{\delta+1} a_n + \triangle^{\delta+1} a_{n+1}) + \cdots \Big| \, dx$$

$$\leqslant \int_0^{\pi} \left|(\triangle^{\delta+1} a_{n-1} + \triangle^{\delta+1} a_n)\right| \Big| \sum_{k=1}^{n-1} \bar{S}_k(x) A_{n-k}^{\delta-2} \Big| \, dx$$

$$+ \int_0^{\pi} \left|(\triangle^{\delta+1} a_n + \triangle^{\delta+1} a_{n+1})\right| \Big| \sum_{k=1}^{n-1} \bar{S}_k(x) A_{n-k+1}^{\delta-2} \Big| \, dx + \ldots$$

$$\leqslant \int_0^{\pi} \left|(\triangle^{\delta+1} a_{n-1} + \triangle^{\delta+1} a_n)\right| \max_{1 \leqslant p \leqslant n-1} \left|\bar{S}_p^{\delta-1}(x)\right| \, dx$$

$$+ \int_0^{\pi} \left|(\triangle^{\delta+1} a_n + \triangle^{\delta+1} a_{n+1})\right| \max_{1 \leqslant p \leqslant n} \left|\bar{S}_p^{\delta-1}(x)\right| \, dx + \ldots \,,$$

by Lemma 3.3,

$$
\begin{aligned}
&= \left|\triangle^{\delta+1}a_{n-1}+\triangle^{\delta+1}a_n)\right| A_n^{\delta-1}\int\limits_0^{\pi}\max_{1\leqslant p\leqslant n-1}\left|\bar{T}_p^{\delta-1}(x)\right|\,dx\\
&\quad+\left|(\triangle^{\delta+1}a_n+\triangle^{\delta+1}a_{n+1})\right| A_{n+1}^{\delta-1}\int\limits_0^{\pi}\max_{1\leqslant p\leqslant n}\left|\bar{T}_p^{\delta-1}(x)\right|\,dx+\dots\\
&\le CA_n^{\delta-1}\left|(\triangle^{\delta+1}a_{n-1}+\triangle^{\delta+1}a_n)\right|\\
&\quad+CA_{n+1}^{\delta-1}\left|(\triangle^{\delta+1}a_n+\triangle^{\delta+1}a_{n+1})\right|+\dots\\
&\le CA_n^{\delta}\left|(\triangle^{\delta+1}a_{n-1}+\triangle^{\delta+1}a_n)\right|\\
&\quad+CA_{n+1}^{\delta}\left|(\triangle^{\delta+1}a_n+\triangle^{\delta+1}a_{n+1})\right|+\dots\\
&=o(1)+o(1)+\dots\\
&=o(1)\,,\qquad\text{by Lemma 3.1 and 3.4.}
\end{aligned}
$$

Moreover, since

$$\int_0^{\pi}\Big|\frac{\sin(n+1)x}{2\sin x}dx\Big|\le C\log n\,,\qquad n\ge 2\,.$$

Therefore,

$$\int_0^{\pi}\Big|\triangle a_n\frac{\sin(n+1)x}{2\sin x}\,dx\Big|\sim\triangle a_n\log n\,.$$

Thus, by (4.3),

$$\lim_{n\to\infty}\int\limits_0^{\pi}|g(x)-g_n(x)|\,dx=o(1)\,,\quad\text{if and only if}\qquad(4.4)$$

$$\triangle a_n\log n=o(1)\,,\quad\text{as}\quad n\to\infty.$$

Case (ii). Let $r\geqslant 1$.

We have

$$g_n(x)=\frac{1}{2\sin x}\sum_{k=1}^{n}(\triangle a_{k-1}+\triangle a_k)\sin kx+\triangle a_n\frac{\sin(n+1)x}{2\sin x}\,.$$

Applying Abel's transformation of order r,

$$
\begin{aligned}
g_n(x)&=\frac{1}{2\sin x}\Big\{\sum_{k=1}^{n-r}\left(\triangle^{r+1}a_{k-1}+\triangle^{r+1}a_k\right)\bar{S}_k^{r-1}(x)\\
&\quad+\sum_{k=1}^{r}\left(\triangle^{k}a_{n-k}+\triangle^{k}a_{n-k+1}\right)\bar{S}_{n-k+1}^{k-1}(x)+a_2\bar{S}_0(x)\Big\}\\
&\quad+\triangle a_n\frac{\sin(n+1)x}{2\sin x}\,.
\end{aligned}\qquad(4.5)
$$

Again applying Abel's transformation of order -δ, we obtain

$$\sum_{k=1}^{n-1} \bar{S}_k^{\alpha-1}(x)(\triangle^{\alpha+1}a_{k-1}+\triangle^{\alpha+1}a_k)$$
$$=\sum_{k=1}^{n-1}\bar{S}_k^{r-1}(x)\sum_{m=0}^{n-(k+1)}A_m^{\delta-1}(\triangle^{\alpha+1}a_{m+k-1}+\triangle^{\alpha+1}a_{m+k}).$$

By Lemma 3.2,

$$\frac{1}{2\sin x}\sum_{k=1}^{n-1}\bar{S}_k^{\alpha-1}(x)(\triangle^{\alpha+1}a_{k-1}+\triangle^{\alpha+1}a_k)$$
$$=\frac{1}{2\sin x}\Big\{\sum_{k=1}^{n-1}\bar{S}_k^{r-1}(x)(\triangle^{r+1}a_{k-1}+\triangle^{r+1}a_k)-R_n(x)\Big\}, \tag{4.6}$$

where

$$\begin{aligned}R_n(x)&=\sum_{k=1}^{n-1}\bar{S}_k^{r-1}(x)\{A_{n-k}^{\delta-1}(\triangle^{\alpha+1}a_{n-1}+\triangle^{\alpha+1}a_n)\\&\quad+A_{n-k+1}^{\delta-1}(\triangle^{\alpha+1}a_n+\triangle^{\alpha+1}a_{n+1})+\dots\}\\&=\Big(\sum_{k=1}^{n-1}\bar{S}_k^{r-1}(x)A_{n-k}^{\delta-1}\Big)(\triangle^{\alpha+1}a_{n-1}+\triangle^{\alpha+1}a_n)\\&\quad+\Big(\sum_{k=1}^{n-1}\bar{S}_k^{r-1}(x)A_{n-k+1}^{\delta-1}\Big)(\triangle^{\alpha+1}a_n+\triangle^{\alpha+1}a_{n+1})+\dots\end{aligned}$$

Replacing n by $n-r+1$ in (4.6), we have

$$\frac{1}{2\sin x}\sum_{k=1}^{n-r}\bar{S}_k^{\alpha-1}(x)(\triangle^{\alpha+1}a_{k-1}+\triangle^{\alpha+1}a_k)$$
$$=\frac{1}{2\sin x}\Big\{\sum_{k=1}^{n-r}\bar{S}_k^{r-1}(x)(\triangle^{r+1}a_{k-1}+\triangle^{r+1}a_k)-R_{n-r+1}(x)\Big\}. \tag{4.7}$$

Now by (4.5) and (4.7), we have

$$\begin{aligned}g_n(x)&=\frac{1}{2\sin x}\Big\{\sum_{k=1}^{n-r}\bar{S}_k^{\alpha-1}(x)(\triangle^{\alpha+1}a_{k-1}+\triangle^{\alpha+1}a_k)+R_{n-r+1}(x)\\&\quad+\sum_{k=1}^{r}(\triangle^k a_{n-k}+\triangle^k a_{n-k+1})\bar{S}_{n-k+1}^{k-1}(x)+a_2\bar{S}_0(x)\Big\}\\&\quad+\triangle a_n\frac{\sin(n+1)x}{2\sin x}.\end{aligned} \tag{4.8}$$

Therefore

$$\begin{aligned}
\int_0^{\pi} |g(x)-g_n(x)|\,dx &\leq \int_0^{\pi} \Big|\frac{1}{2\sin x}\Big\{\sum_{k=n-r+1}^{\infty} \bar{S}_k^{\alpha-1}(x)(\triangle^{\alpha+1}a_{k-1}+\triangle^{\alpha+1}a_k) \\
&\quad - R_{n-r+1}(x) - \sum_{k=1}^{r}(\triangle^k a_{n-k}+\triangle^k a_{n-k+1})\bar{S}_{n-k+1}^{k-1}(x)\Big\}\Big|\,dx \\
&\quad + \int_0^{\pi} \Big|\triangle a_n \frac{\sin(n+1)x}{2\sin x}\Big|\,dx \\
&\leq C\sum_{k=n-r+1}^{\infty} \big|(\triangle^{\alpha+1}a_{k-1}+\triangle^{\alpha+1}a_k)\big| \int_0^{\pi} \big|\bar{S}_k^{\alpha-1}(x)\big|\,dx \\
&\quad + \int_0^{\pi} |R_{n-r+1}(x)|\,dx \\
&\quad + \sum_{k=1}^{r}\big|(\triangle^k a_{n-k}+\triangle^k a_{n-k+1})\big| \int_0^{\pi} \big|\bar{S}_{n-k+1}^{k-1}(x)\big|\,dx \\
&\quad + \int_0^{\pi} \Big|\triangle a_n \frac{\sin(n+1)x}{2\sin x}\Big|\,dx \\
&= C\sum_{k=n-r+1}^{\infty} A_k^{\alpha-1}\big|(\triangle^{\alpha+1}a_{k-1}+\triangle^{\alpha+1}a_k)\big| \int_0^{\pi} \big|\bar{T}_k^{\alpha-1}(x)\big|\,dx \\
&\quad + \int_0^{\pi} |R_{n-r+1}(x)|\,dx \\
&\quad + \sum_{k=1}^{r} A_{n-k+1}^{k-1}\big|(\triangle^k a_{n-k}+\triangle^k a_{n-k+1})\big| \int_0^{\pi} \big|\bar{T}_{n-k+1}^{k-1}(x)\big|\,dx \\
&\quad + \int_0^{\pi} \Big|\triangle a_n \frac{\sin(n+1)x}{2\sin x}\Big|\,dx \\
&\leq C_1\sum_{k=n-r+1}^{\infty} A_k^{\alpha}\big|(\triangle^{\alpha+1}a_{k-1}+\triangle^{\alpha+1}a_k)\big| + C\int_0^{\pi} |R_{n-r+1}(x)|\,dx \\
&\quad + C_1\sum_{k=1}^{r} A_{n-k+1}^{k-1}\big|(\triangle^k a_{n-k}+\triangle^k a_{n-k+1})\big| + \int_0^{\pi} \Big|\triangle a_n \frac{\sin(n+1)x}{2\sin x}\Big|\,dx
\end{aligned}$$

$$= o(1) + o(1) + C\int_0^\pi |R_{n-r+1}(x)|\,dx + \int_0^\pi \left|\triangle a_n \frac{\sin(n+1)x}{2\sin x}\right| dx$$

$$= o(1) + C\int_0^\pi |R_{n-r+1}(x)|\,dx + \int_0^\pi \left|\triangle a_n \frac{\sin(n+1)x}{2\sin x}\right| dx \tag{4.9}$$

by the hypothesis of the theorem and Lemma 3.1.

But

$$\int_0^\pi |R_{n-r+1}(x)|\,dx \le \int_0^\pi \Big|\Big(\sum_{k=1}^{n-r} \bar{S}_k^{r-1}(x) A_{n-r-k+1}^{\delta-1}\Big)(\triangle^{\alpha+1} a_{n-r} + \triangle^{\alpha+1} a_{n-r+1})\Big|\,dx$$

$$+ \int_0^\pi \Big|\Big(\sum_{k=1}^{n-r} \bar{S}_k^{r-1}(x) A_{n-r-k+2}^{\delta-1}\Big)(\triangle^{\alpha+1} a_{n-r+1} + \triangle^{\alpha+1} a_{n-r+2})\Big|\,dx$$

$$+ \int_0^\pi \Big|\Big(\sum_{k=1}^{n-r} \bar{S}_k^{r-1}(x) A_{n-r-k+3}^{\delta-1}\Big)(\triangle^{\alpha+1} a_{n-r+2} + \triangle^{\alpha+1} a_{n-r+3})\Big|\,dx$$

$$\vdots$$

$$\le \sum_{k=1}^{n-r} A_{n-r-k+1}^{\delta-1} A_k^{r-1} \int_0^\pi \left|\bar{T}_k^{r-1}(x)\right| dx \left|(\triangle^{\alpha+1} a_{n-r} + \triangle^{\alpha+1} a_{n-r+1})\right|$$

$$+ \sum_{k=1}^{n-r} A_{n-r-k+2}^{\delta-1} A_k^{r-1} \int_0^\pi \left|\bar{T}_k^{r-1}(x)\right| dx \left|(\triangle^{\alpha+1} a_{n-r+1} + \triangle^{\alpha+1} a_{n-r+2})\right|$$

$$+ \sum_{k=1}^{n-r} A_{n-r-k+3}^{\delta-1} A_k^{r-1} \int_0^\pi \left|\bar{T}_k^{r-1}(x)\right| dx \left|(\triangle^{\alpha+1} a_{n-r+2} + \triangle^{\alpha+1} a_{n-r+3})\right|$$

$$\vdots$$

$$\le C_1 \sum_{k=1}^{n-r} A_{n-r-k+1}^{\delta-1} A_k^{r-1} \left|(\triangle^{\alpha+1} a_{n-r} + \triangle^{\alpha+1} a_{n-r+1})\right|$$

$$+ C_1 \sum_{k=1}^{n-r} A_{n-r-k+2}^{\delta-1} A_k^{r-1} \left|(\triangle^{\alpha+1} a_{n-r+1} + \triangle^{\alpha+1} a_{n-r+2})\right|$$

$$+ C_1 \sum_{k=1}^{n-r} A_{n-r-k+3}^{\delta-1} A_k^{r-1} \left|\triangle^{\alpha+1} a_{n-r+2} + \triangle^{\alpha+1} a_{n-r+3})\right|$$

$$\vdots$$

$$
\begin{aligned}
&\leq C_1 \sum_{k=1}^{n+1-r} A_{n+1-r-k}^{\delta-1} A_k^{r-1} \left|(\triangle^{\alpha+1} a_{n-r} + \triangle^{\alpha+1} a_{n-r+1})\right| \\
&\quad + C_1 \sum_{k=1}^{n+2-r} A_{n+2-r-k}^{\delta-1} A_k^{r-1} \left|(\triangle^{\alpha+1} a_{n-r+1} + \triangle^{\alpha+1} a_{n-r+2})\right| \\
&\quad + C_1 \sum_{k=1}^{n+3-r} A_{n+3-r-k}^{\delta-1} A_k^{r-1} \left|(\triangle^{\alpha+1} a_{n-r+2} + \triangle^{\alpha+1} a_{n-r+3})\right| \\
&\quad \vdots \\
&\leq C_2 A_{n+1-r}^{r+\delta-1} \left|(\triangle^{\alpha+1} a_{n-r} + \triangle^{\alpha+1} a_{n-r+1})\right| \\
&\quad + C_2 A_{n+2-r}^{r+\delta-1} \left|(\triangle^{\alpha+1} a_{n-r+1} + \triangle^{\alpha+1} a_{n-r+2})\right| \\
&\quad + C_2 A_{n+3-r}^{r+\delta-1} \left|(\triangle^{\alpha+1} a_{n-r+2} + \triangle^{\alpha+1} a_{n-r+3})\right| \\
&\quad \vdots \\
&\leq C_2 A_{n+1-r}^{r+\delta} \left|(\triangle^{\alpha+1} a_{n-r} + \triangle^{\alpha+1} a_{n-r+1})\right| \\
&\quad + C_2 A_{n+2-r}^{r+\delta} \left|(\triangle^{\alpha+1} a_{n-r+1} + \triangle^{\alpha+1} a_{n-r+2})\right| \\
&\quad + C_2 A_{n+3-r}^{r+\delta} \left|(\triangle^{\alpha+1} a_{n-r+2} + \triangle^{\alpha+1} a_{n-r+3})\right| \\
&\quad \vdots \\
&= C_2 A_{n+1-r}^{\alpha} \left|(\triangle^{\alpha+1} a_{n-r} + \triangle^{\alpha+1} a_{n-r+1})\right| \\
&\quad + C_2 A_{n+2-r}^{\alpha} \left|(\triangle^{\alpha+1} a_{n-r+1} + \triangle^{\alpha+1} a_{n-r+2})\right| \\
&\quad + C_2 A_{n+3-r}^{\alpha} \left|(\triangle^{\alpha+1} a_{n-r+2} + \triangle^{\alpha+1} a_{n-r+4})\right| \\
&\quad \vdots \\
&= o(1) + o(1) \\
&\quad \vdots \\
&= o(1)\,,
\end{aligned}
$$

by the hypothesis of the theorem.

Hence (4.9) implies

$$
\lim_{n\to\infty} \int_0^\pi |g(x) - g_n(x)|\, dx = o(1)\,, \tag{4.10}
$$

if and only if $\triangle a_n \log n = o(1)$, as n$\to \infty$.

Thus by (4.4) and (4.10)

$$
\lim_{n\to\infty} \int_0^\pi |g(x) - g_n(x)|\, dx = o(1)\,, \tag{4.11}
$$

if and only if $\triangle a_n \log n = o(1)$, as n$\to \infty$, where α is non-integral. $\square$

References

[1] Andersen, A. F., *On extensions within the theory of Cesàro summability of a classical convergence theorem of Dedekind*, Proc. London Math. Soc. **8** (1958), 1–52.

[2] Bosanquet, L. S., *Note on the Bohr-Hardy theorem*, J. London Math. Soc. **17** (1942), 166–173.

[3] Bosanquet, L. S., *Note on convergence and summability factors* (III), Proc. London Math. Soc. (1949), 482–496.

[4] Garrett, J. W. and Stanojević, Č. V., *On integrability and L^1-convergence of certain cosine sums*, Notices, Amer. Math. Soc. **22** (1975), A–166.

[5] Garrett, J. W. and Stanojević, Č. V., *On L^1-convergence of certain cosine sums*, Proc. Amer. Math. Soc. **54** (1976), 101–105.

[6] Kano, T., *Coefficients of some trigonometric series*, J. Fac. Sci. Shihshu University **3** (1968), 153–162.

[7] Kaur, K. and Bhatia, S. S., *Integrability and L-convergence of Rees-Stanojević sums with generalized semi-convex coefficients*, Int. J. Math. Math. Sci. **30**(11) (2002), 645–650.

[8] Kolmogorov, A. N., *Sur l'ordere de grandeur des coefficients de la series de Fourier–Lebesque*, Bull. Polon. Sci. Ser. Sci. Math. Astronom. Phys. (1923) 83–86.

[9] Young, W. H., *On the Fourier series of bounded functions*, Proc. London Math. Soc. **12**(2) (1913), 41–70.

[10] Zygmund, A., *Trigonometric series*, Volume 1, Vol. II, Cambridge University Press.

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