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THE D-STABILITY PROBLEM FOR 4×4 REAL MATRICES

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Abstract. We give detailed discussion of a procedure for determining the robust D-stability of a 4×4 real matrix. The procedure begins from the Hurwitz stability criterion. The procedure is applied to two numerical examples.

1. Introduction

An $n \times n$ real matrix A is said to be D-stable (diagonally stable) if the product matrix DA is Hurwitz stable for each diagonal matrix

$$D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} \tag{1}$$

with positive diagonal entries d_i, $i = 1, \dots, n$. This concept is of importance in various contexts; see, for example, [1, 11, 7] for recent discussions of D-stability. This notion arises naturally in problems exhibiting different time scales. In fact, consider a problem of the form

$$\varepsilon_1 x_1' = f_1(x_1, \dots, x_n) \tag{2a}$$
$$\varepsilon_2 x_2' = f_2(x_1, \dots, x_n) \tag{2b}$$
$$\vdots \tag{2c}$$
$$\varepsilon_n x_n' = f_n(x_1, \dots, x_n) \tag{2d}$$

where $f_i(0, \dots, 0) = 0$, $i = 1, \dots, n$. Let A be the $n \times n$ matrix obtained by linearizing (2) at the origin $\underline{0} \in \mathbb{R}^n$. Then $\underline{0}$ is a linearly stable equilibrium of (2) for all positive values of parameters $\varepsilon_1, \dots, \varepsilon_n$ if and only if A is D-stable.

The goal of this paper is to discuss a procedure for determining the D-stability of 4×4 matrix, together with some examples illustrating the numerical implementation of this procedure. The starting point of our discussion is the paper [14], where the first steps of our procedure were sketched. As pointed out there, the problem of characterizing the D-stable $n \times n$ matrices is relatively simple if $n \leq 3$

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(but not trivial if $n = 3$; see [4] for an elegant description of the 3×3, D-stable, real matrices). However, for $n \geq 4$, the characterization problem appears considerably more complex. We note that C. R. Johnson [13] has given necessary and sufficient conditions for the D-stability of a 4×4 matrix which are related to ours in the sense that their point of departure is the Routh-Hurwitz condition, but which seem to be of a significantly more complicated nature. Also, it is not clear how to implement his criterion numerically.

It is shown in [14] that the problem of characterizing the D-stability of a real matrix A admits a polynomial decision procedure for each $n \geq 1$. Precisely, an $n \times n$ matrix A is D-stable if and only if a certain polynomial P in n variables $t_1, \ldots, t_n$ has no real zeros $(E_1, \ldots, E_n)$. The coefficients of P are certain fixed polynomials in the coefficients of A. It is well-known since the work of Tarski and Seidenberg [16] that there is a finite decision procedure for determining whether or not P has a real zero.

Thus our problem is *elementary*. However, as often happens, it is not immediately clear how to realize the decision procedure in a concrete way which can be carried out even with modern computational facilities: one must devise ways to limit the number and the complexity of the calculations to be performed.

We will discuss the characterization of both the D-stable and the *robustly* D-stable matrices. Recall that an $n \times n$ matrix A is said to be robustly D-stable if it together with all sufficiently near matrices A' are all D-stable. This concept seems more natural and important in applied problems and, perhaps surprisingly, seems somewhat easier to characterize than that of D-stability itself.

The following points will emerge from the discussion of our procedure for determining the D-stability and the robust D-stability of a 4×4 matrix. First, one can form a clear idea of the number of computations necessary to carry our procedure to completion for a general 4×4 matrix. Though this number is large, it can be concretely estimated in terms of the number of operations necessary to calculate several hundred determinants whose orders are ≤ 48 and whose entries are polynomials in one variable with degrees (almost) always ≤ 6, then to compare the number of sign variations in about 200 pairs of chains of length ≤ 24 of such determinants. This fact is perhaps worth emphasizing because general arguments such as those of Seidenberg's basic paper [16] give rise to decision procedures which require that a not-clearly-determined but certainly astronomical number of operations be performed. Second, the determination of the D-stability of a 4×4 matrix seems considerably more complicated than that of a 3×3 matrix. Moreover, it seems that the number of computational steps necessary to determine the D-stability of a generic $n \times n$ matrix must increase rapidly (at least exponentially) with the dimension n. The observation would seem to be consistent with the conjecture of [7] that the problem of characterizing D-stability is NP-hard.

We apply our method to two examples in order to illustrate various aspects of the D-stability concept. First, we consider Bessel matrices. Then, we examine a matrix studied in [14], with the purpose of illustrating the calculation of the determinants alluded to above. Our work with this and other matrices indicates that the calculation of those determinants of order less than 40 can be carried out and that their sign variations can be determined. For matrices with certain *symmetry* properties (related to a certain discriminant; see the example in Section 4), one need only consider determinants of order less than 40, and for these, the D-stability property can be studied with our methods. We remark further that our procedure can sometimes be rendered very short by introducing supplementary algebraic manipulations, which are sufficient for D-stability, at an appropriate point of the calculations. In particular, our *supplemented* procedure can be used to determine the D-stability of certain matrices when such well-known sufficient conditions for D-stability as positive definiteness or diagonal Lyapunov stability do not hold.

We refer to the paper [15] for another approach to developing a procedure to determine the D-stability of a 4×4 matrix. Though the starting point in [15] is the same as ours, namely to show that a certain cubic polynomial of three variables is positive on the positive orthant, there a polynomial programming method is introduced, whereas we analyze chains of Hankel determinants. We plan to compare the two approaches in further work.

2. Characterization of D-stable and robustly D-stable matrices

We will discuss in some detail a method suggested by [14] which yields necessary and sufficient conditions for the D-stability and the robust D-stability of an $n \times n$ matrix, $n \geq 4$. Thus, let $A = \|a_{ij}\|_{i,j=1}^{n}$ be a real matrix, and let D be a diagonal matrix as in (1). The matrix A is called robustly D-stable if, for all $n \times n$ matrices A' whose Euclidean distance from A is sufficiently small, the product matrix DA' is stable in the Hurwitz sense, i.e. all its eigenvalues lie in the (open) left half-plane. We will emphasize the question of robust D-stability, and will usually leave it to the reader to work out how the conditions we introduce should be modified if the D-stability property is of interest.

The method of [14] gives rise to a polynomial decision procedure for determining whether or not a given matrix A is robustly D-stable. The procedure consists of finitely many steps, in each of which one checks whether certain polynomial equalities and inequalities are valid. If at some step at least one equality-inequality group fails to hold, then the matrix A is not robustly D-stable. It will be seen that the procedure is quite simple for $n \leq 3$. However, the case $n = 4$ is markedly more complex from the computational point of view. The method can be extended to dimensions $n \geq 5$ to yield a polynomial decision procedure, but it will be clear that, for this method, with each increase in dimension the complexity of the calculations increases quickly. Our discussion can be regarded as further evidence that the problem of deciding whether a matrix is robustly D-stable is an NP-hard problem [7].

We restrict attention to the case $n = 4$; the reader will be able to apply our method to the case $n = 3$; see [4] for a characterization of D-stability when $n = 3$. We assume throughout that A itself is Hurwitz stable. The first step is, as in [14], to combine the classical Hurwitz stability criterion with Orlando's Theorem [9] to deduce that *A is D-stable if and only if the third Hurwitz determinant $H_3(DA)$ is* *positive for all matrices DA with D being as in* (1):

$$H_3(DA) > 0 \tag{3}$$

It is more convenient to write out $H_3(DA)$ explicitly. Thus, put

$$\begin{aligned} m_i &= a_{ii} \\ m_{ij} &= i, j^{th} \text{ principal minor of } A \\ m_{ijk} &= i, j, k^{th} \text{ principal minor of } A \\ M &= \det A \end{aligned}$$

where $1 \le i < j < k \le 4$. Setting

$$\begin{aligned} p_0 &= d_1 d_2 d_3 d_4 M \\ p_1 &= \sum_{1 \le i < j < k \le 4} d_i d_j d_k m_{ijk} \\ p_2 &= \sum_{1 \le i < j \le 4} d_i d_j m_{ij} \\ p_3 &= \sum_{1 \le i \le 4} d_i m_i \end{aligned}$$

one has

$$H_3(DA) = -p_0 p_3^2 - p_1^2 + p_1 p_2 p_3 \,. \tag{4}$$

Following [14], observe that H_3 is homogeneous of degree 6 in $d_1, \ldots, d_4$. In fact, in dimension n, $H_{n-1}(DA)$ is homogeneous of dimension $n(n-1)/2$ in $d_1, \ldots, d_n$. One is led to *homogenize* (3) by dividing by d_4^6 and setting $x = d_1/d_4$, $y = d_2/d_4$ and $z = d_3/d_4$. Define $b(x, y, z) = H_3(DA)/d_4^6$. Then, for all $x, y, z > 0$, (3) is equivalent to

$$b(x, y, z) > 0 \tag{5}$$

where $b(x, y, z) = B_3(y, z)x^3 + B_2(y, z)x^2 + B_1(y, z)x + B_0(y, z)$. Note that B_3 is quadratic in y and z, and that B_2, B_1 and B_0 are cubic in y and z.

We now begin listing conditions which are necessary for robust D-stability of a given Hurwitz stable matrix A. It will be convenient to separate them into classes $\mathcal{C}_I$, $\mathcal{C}_{II}$, $\mathcal{C}_{III}$ and $\mathcal{C}_{IV}$ corresponding to certain *primitive* Conditions I, II, III and IV, respectively. Each such class will consist of polynomial equalities and inequalities in the entries of A, certain subclasses of which are joined by the logical connective *or* while others are joined by the logical connective *and.* In general, given any such class $\mathcal{C}$ of polynomial equalities and inequalities in 16 variables $a_{11}, \ldots, a_{44}$, there is natural sense in which a given 4×4 matrix either *satisfies* $\mathcal{C}$ or *does not satisfy* $\mathcal{C}$.

The first *primitive* condition is as below:

Condition I. $b(x, 1, 1) > 0$, $\forall x > 0$.

It is clear that, if A is D-stable, then Condition I holds. One can write a class $\mathcal{C}_I$ of polynomial relations as described above, involving the coefficients of A

(or rather, the quantities $m_1, \ldots, m_{12}, \ldots, m_{123}, \ldots, M$) which are necessary and sufficient for the validity of Condition I, for all matrices A' sufficiently near A. In other words, Condition I holds for all matrices A' sufficiently near A if and only if A satisfies $\mathcal{C}_I$.

In what follows we will write $Q_+ = \{(y,z) \in \mathbb{R}^2 | y, z > 0\}$.

Condition II. $B_3(y,z) > 0$ and $B_0(y,z) > 0$, $\forall (y,z) \in Q_+$.

This condition is necessary for the robust D-stability of A, because if, for example, $B_3(y_*, z_*) = 0$ for some $(y_*, z_*) \in Q_+$, then every neighborhood of A contains a point A' for which there exists $(y', z') \in Q_+$ such that $B_3^{A'}(y', z') < 0$ (here and below, we use a superscript when it is necessary to indicate the dependence of an object on A). Then we would have $b^{A'}(x, y', z') < 0$ for large x, and (5) would be violated for A'. One can write down a class $\mathcal{C}_{II}$ of polynomial relations as described above, involving the quantities $m_1, \ldots, M$ so that Condition II holds for all matrices sufficiently near A if and only if A satisfies $\mathcal{C}_{II}$.

Assume now that A satisfies Conditions I and II. One verifies as in [14] that A is D-stable if and only if, for each $(y,z) \in Q_+$, the polynomial $b(x,y,z)$ admits no positive double root x_*. Now, the presence of *complex* double roots (necessarily real in our case) is equivalent to the vanishing of the classical discriminant function

$$\Delta(y,z) \equiv \Delta = B_1^2 B_2^2 - 4B_0 B_2^3 - 4B_1^3 B_3 - 27 B_0^2 B_3^2 + 18 B_0 B_1 B_2 B_3. \tag{6}$$

As in [14], one can show that $b(x,y,z)$ admits no positive double root x_* if and only if *neither* of the following problems admits a solution $(y,z) \in Q_+$:

$$\Delta(y,z) = 0, \quad B_2(y,z) \le 0,$$
$$\Delta(y,z) = 0, \quad B_1(y,z) \le 0.$$

We are led to introduce the following conditions:

Condition III. If $\Delta(y,z) = 0$ for $(y,z) \in Q_+$, then $B_1(y,z) > 0$.

Condition IV. If $\Delta(y,z) = 0$ for $(y,z) \in Q_+$, then $B_2(y,z) > 0$.

Assuming that A satisfies $\mathcal{C}_I$ and $\mathcal{C}_{II}$, our problem is now the following. We must determine a class $\mathcal{C}_{III}$ of polynomial relations as described earlier, involving the coefficients of A, so that Condition III holds for all matrices A' sufficiently near A if and only if A satisfies conditions $\mathcal{C}_{III}$. We must also determine an analogous class of polynomial relations $\mathcal{C}_{IV}$.

We will determine the appropriate class of polynomial relations $\mathcal{C}_{III}$. It will be clear that the determination of the class $\mathcal{C}_{IV}$ can be carried out in much the same way. It will also be clear that the determination of the classes $\mathcal{C}_I$ and $\mathcal{C}_{II}$ is much easier.

To better understand the quantities we must deal with, we write out certain terms of the polynomials B_3, B_2, B_1 and B_0. We put

$$\begin{aligned} B_3(y,z) &= B_3^{(2)}y^2 + B_3^{(1)}y + B_3^{(0)} \\ B_2(y,z) &= B_2^{(3)}y^3 + B_2^{(2)}y^2 + B_2^{(1)}y + B_2^{(0)} \\ B_1(y,z) &= B_1^{(3)}y^3 + B_1^{(2)}y^2 + B_1^{(1)}y + B_1^{(0)} \\ B_0(y,z) &= B_0^{(3)}y^3 + B_0^{(2)}y^2 + B_0^{(1)}y \end{aligned}$$

where

$$\begin{aligned} B_3^{(2)}(z) &= m_1m_{12}m_{123}z + m_1m_{12}m_{124} \\ B_2^{(3)}(z) &= m_2m_{12}m_{123}z + m_2m_{12}m_{124} = \varepsilon_1 z + \varepsilon_2 \\ B_1^{(3)}(z) &= m_2m_{23}m_{123}z^2 + (m_2m_{12}m_{234} + m_2m_{23}m_{124} \\ &\qquad + m_2m_{24}m_{123} - m_2^2M)z + m_2m_{24}m_{124} = \varepsilon_3z^2 + \varepsilon_4z + \varepsilon_5 \\ B_0^{(3)}(z) &= m_2m_{23}m_{234}z^2 + m_2m_{24}m_{234}z = \varepsilon_6z^2 + \varepsilon_7z \\ B_3^{(0)}(z) &= m_1m_{13}m_{134}z^2 + m_1m_{14}m_{134}z = \varepsilon_{13}z^2 + \varepsilon_{14}z \\ B_2^{(0)}(z) &= m_3m_{13}m_{134}z^3 + (m_4m_{13}m_{134} + m_3m_{14}m_{134} \\ &\qquad + m_1m_{34}m_{134} - m_{134}^2)z^2 + m_4m_{14}m_{134}z = \varepsilon_{10}z^3 + \varepsilon_{11}z^2 + \varepsilon_{12}z \\ B_1^{(0)}(z) &= m_3m_{34}m_{134}z^3 + m_4m_{34}m_{134}z^2 = \varepsilon_8z^3 + \varepsilon_9z^2 \\ B_0^{(1)}(z) &= m_3m_{34}m_{234}z^3 + m_4m_{34}m_{234}z^2 \end{aligned}$$

Observe that $B_3^{(2)}(z) > 0$, $\varepsilon_6z^2 + \varepsilon_7z > 0$, $\varepsilon_{13}z^2 + \varepsilon_{14}z > 0$, and $B_0^{(1)}(z) > 0$ for all $z > 0$; see the discussion of Condition II.

Next we consider the discriminant $\Delta(y,z)$. It is of degree 12 in y; we can write

$$\Delta(y,z) = \sum_{i=0}^{12} \delta_i(z)y^i$$

where

$$\begin{aligned} \delta_{12}(z) &= (\varepsilon_1z+\varepsilon_2)^2\left\{(\varepsilon_3z^2+\varepsilon_4z+\varepsilon_5)^2 - 4(\varepsilon_1z+\varepsilon_2)(\varepsilon_6z^2+\varepsilon_7z)\right\} \\ \delta_0(z) &= (\varepsilon_8z+\varepsilon_9)^2\left\{(\varepsilon_{10}z^3+\varepsilon_{11}z^2+\varepsilon_{12}z)^2 - 4(\varepsilon_{13}z^2+\varepsilon_{14}z)(\varepsilon_8z^3+\varepsilon_9z^2)\right\} \end{aligned}$$

with $\varepsilon_1, \ldots, \varepsilon_{14}$ being as given above. Let us assume for the time being that $\Delta(y, z_*)$ does not vanish identically in y for any $z_* > 0$. Then, as observed in [14], Condition III is equivalent to

$$I_0^\infty\left(\frac{\Delta'(y,z)}{\Delta(y,z)}\right) = I_0^\infty\left(\frac{B_1(y,z)\Delta'(y,z)}{\Delta(y,z)}\right). \tag{7}$$

Here, the prime $'$ indicates $\frac{d}{dy}$, and I_0^∞ is the Cauchy index computed for $0 < y < \infty$ for each fixed positive z. We will obtain the appropriate conditions $\mathcal{C}_{III}$ beginning from (7).

There are (at least) two ways to study the Cauchy indices in (7). One method consists of writing out generalized Sturm chains; this seems complicated even in principle because of the many singular cases which can occur as z varies. A second method consists of counting sign variations in a chain of Hankel determinants [9]. This method has the advantage that there are fixed rules for calculating the sign variations even when some of the Hankel determinants are zero. We will apply this method. To do so, we set $y = t^2$ and consider the following Cauchy indices:

$$V_z = I_{-\infty}^{\infty}\left(\frac{\Delta'(t^2,z)}{\Delta(t^2,z)}\right), \qquad \widehat{V}_z = I_{-\infty}^{\infty}\left(\frac{B_1(t^2,z)\Delta'(t^2,z)}{\Delta(t^2,z)}\right). \tag{8}$$

Note that we have put $\Delta'(t^2,z)$ and not $t\Delta'(t^2,z)$ in the numerators of the quantities defining V_z and $\widehat{V}_z$; this has the advantage that $t = 0$ contributes neither to V_z nor to $\widehat{V}_z$ (because if $y = 0$ is a zero of $\Delta(\cdot, z)$, then t factors from $\Delta'(t^2,z)/\Delta(t^2,z)$ with a negative *even* exponent). It follows that (7) is equivalent to the equality of V_z and $\widehat{V}_z$.

We are led to formulate the following condition:

Condition $\mathbf{III_1}$ Whenever $z > 0$ has the property that $\Delta(y,z)$ does not vanish identically in y, then $V_z = \widehat{V}_z$.

It is clear from the preceding discussion that Condition III_1 is necessary in order that A together with all matrices A' sufficiently near A satisfy (7) for all $z > 0$ for which $\Delta(y,z)$ does not vanish identically in y. On the other hand, suppose that one can formulate a class $\mathcal{C}_{(1)}$ of polynomial relations in the coefficients of A, certain subclasses of which are joined by the connective *and* while others are joined by the connective *or*, such that Condition $\mathrm{III}_{(1)}$ holds for all A' near A if and only if A satisfies $\mathcal{C}_{(1)}$. Then, if A satisfies $\mathcal{C}_{(1)}$, all matrices A' sufficiently near A satisfy (7) whenever $z > 0$ has the property that $\Delta(y,z)$ does not vanish identically in y.

We consider briefly how to express the appropriate class of conditions $\mathcal{C}_{(1)}$. Formally we can write

$$\frac{\Delta'(y,z)}{\Delta(y,z)} = \frac{\sum_{k=1}^{\infty} k\delta_k(z)y^{k-1}}{\sum_{k=0}^{\infty}\delta_k(z)y^k} \tag{9}$$

$$\delta_{12}^3(z)\frac{B_1(y,z)\Delta'(y,z)}{\Delta(y,z)} = \frac{\sum_{k=0}^{12}\gamma_k(z)y^k}{\sum_{k=0}^{12}\delta_k(z)y^k} + w(y,z) \tag{10}$$

where $w(y,z)$ is a polynomial quadratic in y. From [9] it is known that

$$V_z = m - 2V(1, D_1(z), \ldots, D_m(z))$$
$$\widehat{V}_z = \widehat{m} - 2V(1, \widehat{D}_1(z), \ldots, \widehat{D}_{\widehat{m}}(z))$$

where m and $\widehat{m}$ are the ranks of the Hankel forms (in the variable t) corresponding to the *proper* rational functions obtained by reducing the rational functions on the right-hand sides of (9) and (10), respectively. Here, D_s ($\widehat{D}_s$) is a $2s \times 2s$ determinant whose calculation reduces to that of several $s \times s$ determinants due to the presence of zeros; see page 214 of [9], and note that we write D_s instead of

∇_{2s}. Let us observe that $\widehat{m} \leq m$ ($\widehat{m}$ can be zero). Let us further observe that, if $\widehat{m} < m$, then $\Delta(t^2, z)$ and $B_1(t^2, z)$ must have a common zero t_*. If $t_* \neq 0$, then $\Delta(y, z)$ and $B_1(y, z)$ have a common positive zero $y_* = t_*$, and A is not D-stable. The possibility $t_* = 0$ cannot be excluded, but it is relatively easy to check if A has the property that $y = 0$ is a root of $B(\cdot, z)$ and of $\Delta(\cdot, z)$ for some $z > 0$.

We now indicate a way to formulate conditions expressing the equality of V_z and $\widehat{V}_z$ for all $z > 0$ for which $\Delta(y, z)$ does not vanish identically in y. Let $1 \leq m \leq 24$ be an integer. One can write down a class $\mathcal{C}^m_{(1)}$ of polynomial relations in the coefficients of A, certain subclasses of which are joined by the connective *and* and others are joined by the connective *or*, such that A satisfies $\mathcal{C}^m_{(1)}$ if and only if whenever $z > 0$ satisfies $\delta_{12}(z) \neq 0$, $D_m(z) \neq 0$, $D_{m+1}(z) = \cdots = D_{24}(z) = 0$, then for all A' sufficiently near A the relation $V_z^{A'} = \widehat{V}_z^{A'}$ is valid. In writing down the classes $\mathcal{C}^{24}_{(1)}, \ldots, \mathcal{C}^1_{(1)}$ one will take account of the considerations of the preceding paragraph, and will make a multitude of other considerations as well.

These classes of polynomial relations have a complex structure. However, they are rendered simpler by certain circumstances when one sets about verifying them for a given matrix A. One such circumstance is the following. Suppose that $\delta_{12}(z)$ does not vanish identically in z, and let $m_* = \max\{i \geq 1 | D_i(z)$ does not vanish identically$\}$. It is clear that m_* might be less than 24 because Δ' and Δ may have a non-constant common polynomial factor; we will see an example of this phenomenon in Section 4. Then if $m < m_*$, the class $\mathcal{C}^m_{(1)}$ is satisfied by A if and only if it is satisfied by all matrices A' near A. This is because $D_{m_*}(z)$ has only finitely many zeros.

We conjecture that a second simplifying circumstance is present. Namely, we believe that, if A is D-stable, if $\delta_{12}(z)$ and $\delta_0(z)$ do not vanish identically in z, and if m_* is as above, then $V_z = \widehat{V}_z$ for all positive z satisfying $\delta_{12}(z) \neq 0$, $\delta_0(z) \neq 0$, and $D_{m_*}(z) \neq 0$ if and only if for each $i = 1, 2, \ldots, m_* - 2$ there holds $D_i(z) = 0 \Rightarrow D_{i+1}(z) \neq 0$ and $\widehat{D}_i(z) = 0 \Rightarrow D_{i+1}(z) \neq 0$.

Now let $\mathcal{C}_{(1,12)}$ be the class of conditions $\mathcal{C}^{24}_{(1)} \wedge \cdots \wedge \mathcal{C}^1_{(1)}$ where $\wedge$ denotes logical *and*. We construct further classes of conditions $\mathcal{C}_{(1,11)}, \cdots, \mathcal{C}_{(1,1)}$ corresponding to the possibilities that $\delta_{12}(z) = 0, \ldots, \delta_{i+1}(z) = 0$, $\delta_i(z) \neq 0$ where $1 \leq i \leq 11$. Put finally $\mathcal{C}_{(1)} = \mathcal{C}_{(1,12)} \wedge \cdots \wedge \mathcal{C}_{(1,1)}$.

Let $\mathcal{C}_{(2)}$ be a class of polynomial equalities and inequalities in the coefficients of A, certain subclasses of which are joined by the connective *and* while others are joined by the connective *or*, such that A satisfies $\mathcal{C}_{(2)}$ if and only if all matrices A' sufficiently near A have the following property: if for some $z > 0$ the discriminant $\Delta(y, z)$ vanishes identically in y, then $B_1(y, z) > 0$ for all $y > 0$. Put $\mathcal{C}_{(III)} = \mathcal{C}_{(1)} \wedge \mathcal{C}_{(2)}$. We see that A satisfies $\mathcal{C}_{(III)}$ if and only if Condition III holds for all A' sufficiently near A.

One can derive a class $\mathcal{C}_{(IV)}$ of conditions corresponding to Condition IV in an analogous way. Putting $\mathcal{C} = \mathcal{C}_I \wedge \cdots \wedge \mathcal{C}_{IV}$, we see that A satisfies $\mathcal{C}$ if and only if A is robustly D-stable. Clearly, the class $\mathcal{C}$ has a complicated structure. However, we note that if the discriminant Δ happens to have a degree lower than

12 in y, then our procedure will, in general, be less complex. We also wish to point out that our procedure is derived from the simple condition $b(x, y, z) > 0$ for positive values of x, y, and z, which in turn translates quickly into Condition III and Condition IV.

The considerations given above suggest that the D-stability problem is of a highly complex computational nature. They also indicate that the problem of characterizing D-stability is substantially more complex than that of determining robust D-stability (see [10], which was corrected in [5]), since the robustness property renders vacuous a large number of singular cases.

3. A hybrid procedure

The theoretical results of the previous section have been implemented as a hybrid (numerical-symbolic) computer program which uses some basic commands of MATLAB 5.3 and its MAPLE powered Symbolic Math Toolbox. Integrating two different computation environments allows us to use their respective superiorities in order to overcome some disadvantages of both. The main steps of the proposed procedure are as follows:

(1) Compute symbolically the determinant of the matrix $(\lambda I - DA)$ where λ is a symbolic variable, I is the 4×4 identity matrix, A is a given 4×4 matrix of reals, and D is the diagonal matrix with symbolic entries d_1, d_2, d_3 and d_4 which are specified to be positive.

(2) Define as p_3 the coefficient of λ^3, as p_2 the coefficient of λ^2, as p_1 the coefficient of λ, and as p_0 the sum of the remaining terms.

(3) Obtain $H_3(DA)$ using (4).

(4) Divide the resulting expression by d_4^6, and suppress d_4 and its powers; hereafter d_1, d_2 and d_3 mean d_1/d_4, d_2/d_4 and d_3/d_4, respectively.

(5) Substitute d_1, d_2 and d_3 by x, y and z, respectively.

(6) Define as B_3 the coefficient of x^3, as B_2 the coefficient of x^2, as B_1 the coefficient of x, and as B_0 the sum of the remaining terms.

(7) If B_3, B_2, B_1 and B_0 are always positive then stop. Else, if Condition I or Condition II is violated then stop. Else, proceed with Condition III and Condition IV.

The implementation of Steps 1-6 above is quite simple, and it runs reasonably fast for a given A matrix. These statements are true also for Step 7, but only if Conditions III and IV need not be checked. The main difficulty associated with these two conditions has a computational nature, and arises when the determinants of the Hankel matrices, whose entries are polynomials in z, are to be computed. Recall from Section 2 that, in general, the determinants of 24 Hankel matrices of order $n \times n$, where $n = 2, 4, 6, \ldots, 48$, need to be computed. However, it may not always be possible to succeed in these computations which means that there is an implicit restriction on the A matrices that can be treated. This restriction is, of course, imposed by the limitations of MATLAB and MAPLE and, as such, is inevitable. Another difficulty arises when $\delta_{12}(z) = 0$ for some $z > 0$. On the other hand, these positive z values for which $\delta_{12}(z) = 0$ can be isolated by simple MATLAB and MAPLE commands. Then, the presence of D-stability (robust D-stability) can be detected by studying certain conditions which we do not wish to indicate in detail here. These conditions take account of whether some, or all, of $\delta_{11}(z), \ldots, \delta_0(z)$ vanish as well. Finally, it may, of course, be possible to determine the positivity of $H_3(DA)$ for all positive x, y and z by direct observation, in which case Steps 6 and 7 are not necessary. Moreover, as already noted in the Introduction, other simplifications may also occur.

4. Numerical examples

As a first example, we consider the Bessel matrices. They are stable for any order $n \times n$. Moreover, for *large* values of n, their simple eigenvalues lie in the left half-plane along a well-determined curve. For $n = 4$, the Bessel matrix is

$$B = \begin{bmatrix} -1 & -0.5774 & 0 & 0 \\ 0.5774 & 0 & -0.2582 & 0 \\ 0 & 0.2582 & 0 & -0.1690 \\ 0 & 0 & 0.1690 & 0 \end{bmatrix}$$

By means of our procedure, we find that at Step 5

$$H_3(DB) = \frac{1}{45} x^3 y^2 z$$

which is clearly positive for all positive x, y and z. Therefore, B is immediately deemed to be D-stable, and the procedure is stopped without carrying out Steps 6 and 7 if D-stability is the property of interest. The matrix is, however, not robustly D-stable because every neighborhood of A contains a matrix A' such that the corresponding coefficient B_0 has the property that $B_0(y', z') < 0$ for some $y', z' > 0$.

Next, consider the matrix

$$A = \begin{bmatrix} -1 & 0 & q & 0 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

where q is a real parameter. It is easy to show that A is stable for $q > -8/3$. From [14], it is also known that A is D-stable if $q \geq -1$. Here, the aim is not only to verify this result by utilizing the procedure outlined in the previous section, but also to see if Conditions III and IV can actually be checked in this example. To this end, we set $q = -1$ from now on. It turns out that

$$B_3(y, z) = (z + 1)y^2 + y \tag{11a}$$

$$B_2(y, z) = (1 + z)y^3 + (z^2 + 2z + 2)y^2 + (2z + 1)y \tag{11b}$$

$$B_1(y, z) = (z^2 + 2z + 1)y^3 + (z^3 + 2z^2 + 3z + 1)y^2 + (2z^2 + z)y \tag{11c}$$

$$B_0(y, z) = (z^2 + z)y^3 + (z^3 + 2z^2 + z)y^2 + (z^3 + z^2)y \tag{11d}$$

which are all clearly positive for all $y, z > 0$, and the procedure is stopped at this point because of the first condition in Step 7. It is clear that the matrix A is D-stable. It is not quite so obvious that A is not robustly D-stable; however, this follows if we note first that $B_3^{(0)}(z)$ vanishes identically, then check that each neighborhood of A contains a matrix A' such that the corresponding coefficient $B_3^{(0)}(z)$ is negative for some positive value z' ($B_3^{(0)}(z') < 0$).

In any case, we choose to ignore these conditions; our aim is rather to determine whether, for this matrix A, the calculations involved in checking Conditions III and IV take a *reasonable* amount of time to execute. In fact, Conditions III and IV are quite sophisticated and, thus, their verification requires much computational effort. However, in the present case, certain simplifications occur. In fact, for the A matrix under consideration, the coefficient functions $\delta_3(z), \ldots, \delta_0(z)$ are not present, i.e. they are zero. Therefore, one would expect the denominator of the rational function

$$\frac{\Delta'(t^2, z)}{\Delta(t^2, z)} \tag{12}$$

which was previously given in (8) to be of degree 16, and the corresponding Hankel matrices to have sizes $n \times n$ with n being $2, 4, 6, \ldots, 32$. However, it turns out that the denominator in (12) is of order 10. The reason for this extra reduction is due to the common factors

$$(z+1)y + z^2 + z + 1$$
$$(z^2 - 1)y^2 + zy + 1$$

between $\Delta'(y, z)$ and $\Delta(y, z)$. Consequently, the largest Hankel matrix has dimensions 20×20. Similar reductions occur also in the rational function given on the right in (8), and its equivalent in Condition IV. It was observed that computing the determinants of the Hankel matrices, and determining the positive z values (if there exist any) at which these determinants vanish, demands most of the total computation time which, for the present example, was about 22 minutes on a Pentium III driven PC. At this point, it must be pointed out that the leading coefficient of $\Delta(y, z)$ is $\delta_{12}(z) = (z+1)^4(z-1)^2$ which vanishes at $z = 1$. Recall from the previous section that this type of singularity cannot be handled systematically by the current version of the software implementation of the proposed procedure. However, this is not a cause for concern since such singularities can be easily singled out by calling simple commands. The code fragment which achieves this is

```
rts = solve(d);
drts = double(rts);
inx = find((real(drts) <= 0) | (imag(drts) ~= 0));
rts(inx) = [];
```

where d is the determinant of a Hankel matrix. The first command solves for the roots of d. Needless to say that these roots are the z values at which d vanishes. The second command, namely `double`, converts the roots from symbolic to numeric format for subsequent use. Next, by invoking `find`, the roots with negative real parts and those with imaginary parts are determined. The singularity $z = 1$ gives rise to conditions on A which must be satisfied if A is D-stable (robustly D-stable). In particular, these conditions take account of the possibility that $\Delta(y, z) = 0$ admits positive roots y which are *near* ∞ if z is *near* 1; in fact, one must have $B_2^{(3)}(1) > 0$ and $B_1^{(3)}(1) > 0$. For the A matrix under study, these conditions are actually satisfied by A and by all matrices A' near A.

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