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ON SOME NONLINEAR ALTERNATIVES OF LERAY-SCHAUDER TYPE AND FUNCTIONAL INTEGRAL EQUATIONS

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Abstract. In this paper, some new fixed point theorems concerning the nonlinear alternative of Leray-Schauder type are proved in a Banach algebra. Applications are given to nonlinear functional integral equations in Banach algebras for proving the existence results. Our results of this paper complement the results that appear in Granas et. al. [12] and Dhage and Regan [10].

1. Introduction

Nonlinear functional integral equations have been discussed in the literature extensively, for a long time. See for example, Subramanyam and Sundersanam [15], Ntouyas and Tsamatos [14], Dhage and Regan [10] and the references therein, Recently, the present author, in a series of papers [4, 6] initiated the study of nonlinear integral equations in a Banach algebra via fixed point techniques. In this papers we study a new class of nonlinear functional integral equations for the existence theory via a new nonlinear alternative of Leray-Schauder-type to be developed in this paper. In particular, given a closed and bounded interval $J = [0,1] \subset \mathbb{R}$, $\mathbb{R}$ denotes the set of all real numbers, we study the existence of the nonlinear functional integral equation (in short FIE)

$$x(t) = k\big(t, x(\mu(t))\big) + \big[f\big(t, x(\nu(t))\big)\big]\Big(q(t) + \int_0^{\sigma(t)} g\big(s, x(\eta(s))\big)\, ds\Big) \tag{1.1}$$

for $t \in J$, where $q : J \to \mathbb{R}$, $f, g, h : J \times \mathbb{R} \to \mathbb{R}$ and $\mu, \nu, \sigma, \eta : J \to J$.

The FIE (1.1) is general in the sense that it includes the integral equations that have been studied in Dhage and Jahagirdar [9] and Dhage and Regan [10]. The special cases of the FIE (1.1) occur in some physical and biological processes, queing theory etc. For the details of such problems one is referred to Deimling [3], Chandrasekhar [2]. The rest of the paper is organized as follows. In section 2 and

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3 we shall develop a new fixed point theory and in section 4 we shall prove the existence results for the FIE (1.1) under Carathéodory condition.

2. Fixed point theory

It is known since long time that the fixed point theory has some nice applications to nonlinear differential and integral equations for proving the existence and uniqueness theorems. An interesting topological fixed point result that has been widely used while dealing with the nonlinear equations is the following variant of nonlinear alternative due to Laray and Schauder [16].

Theorem 2.1. *Let U and $\overline{U}$ denote respectively the open and closed subset of a convex set K of a normed linear space X such that $0 \in U$ and let $N : \bar{U} \to K$ be a compact and continuous operator. Then either*

(i) *the equation $x = Nx$ has a solution in $\bar{U}$, or*

(ii) *there exists a point $u \in \partial U$ such that $u = \lambda Nu$ for some $\lambda \in (0, 1)$, where ∂U is a boundary of U.*

Another important fixed point theorem that has been commonly used in the theory of non-linear differential and integral equations is the following generalization of Banach contraction mapping principle proved in Browder [1].

Theorem 2.2. *Let S be a closed convex and bounded subset of a Banach Space X and let $T : S \to S$ be a nonlinear contraction. Then T has a unique fixed point x^* and the sequence $\{T^n x\}$, $x \in X$, of successive iterations converges to x^*.*

Most recently the present author in [4] proved a fixed point theorem involving three operators in a Banach algebra by blending the Banach fixed point theorem with that of Schauder's fixed point principle.

Theorem 2.3 (Dhage [4])**.** *Let S be a closed, convex and bounded subset of a Banach algebra X and let $A, B, C : S \to S$ be three operators such that*

(a) *A and C are Lipschitzians with the Lipschitz constants α and β respectively,*

(b) *$\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(S)$, I being the identity operator on X, and the operator $\frac{I-C}{A} : X \to X$ is defined by $\left(\frac{I-C}{A}\right) x = \frac{x-Cx}{Ax}$,*

(c) *B is completely continuous, and*

(d) *$AxBy + Cx \in S \quad \forall\, x, y \in S$.*

Then the operator equation

$$AxBx + Cx = x \tag{2.1}$$

has a solution, whenever $\alpha M + \beta < 1$, where $M = \|B(S)\| = \sup\{\|Bx\| : x \in S\}$.

It is shown in Dhage [4] that the above Theorem 2.3 is useful in the existence theory of certain nonlinear integral equations in Banach algebras. Now we shall obtain the nonlinear-alternative versions of Theorem 2.3 and a fixed point theorem established is Nashed and Wong [13] under suitable conditions, Before going to the main results, we give some preliminary definitions needed in sequel.

Definition 2.1. A mapping $T : X \to X$ is called $\mathcal{D}$**-Lipschitzian** if there exists a continuous and nondecreasing function $\phi_T : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\|Tx - Ty\| \le \phi_T(\|x - y\|) \tag{2.2}$$

for all $x, y \in X$ with $\phi_T(0) = 0$. Sometimes we call the function ϕ_T a $\mathcal{D}$**-function** of T on X. If $\phi_T(r) = \alpha r$ for some constant $\alpha > 0$, then T is called a **Lipschitzian** with a Lipschitz constant α and further if $\alpha < 1$, then T is called a **contraction** with the contraction constant α. Again if ϕ_T satisfies $\phi_T(r) < r$, $r > 0$, then T is called a **nonlinear** $\mathcal{D}$**-contraction** on X.

Remark 2.1. It is clear that every $\mathcal{D}$-contraction implies nonlinear contraction and nonlinear contraction implies $\mathcal{D}$-Lipschitzian but the reverse implications may not hold.

Definition 2.2. An operator $T : X \to X$ is called **compact** if $\overline{T(X)}$ is a compact subset of X, and it is called **completely continuous** if it is continuous and for any bounded subset S of X, $\overline{T(S)}$ is a compact subset of X.

It is clear that every compact operator is totally bounded but the converse may not be true. However the two notions are equivalent on a bounded subset of X.

Theorem 2.4. *Let U and $\bar{U}$ denote respectively the open bounded and closed bounded subset of a Banach algebra X, and let $A, C : X \to X$ and $B : \overline{U} \to X$ be three operators satisfying*

(a) *A and C are with the $\mathcal{D}$-functions ϕ_A and ϕ_C respectively,*
(b) *$\left(\frac{I}{A}\right)^{-1}$ exists, I being the identity operator on X, and the operator $\frac{I}{A} : X \to X$ is defined by $\left(\frac{I}{A}\right)x = \frac{x}{Ax}$,*
(c) *B is completely continuous, and*
(d) *$M\phi_A(r) + \phi_C(r) < r$ for $r > 0$, where $M = \|B(\overline{U})\|$.*

Then either

(i) *the operator equation $AxBx + Cx = x$ has a solution in $\overline{U}$, or*
(ii) *there exists an $u \in \partial U$ such that*

$$\lambda A\Big(\frac{u}{\lambda}\Big)Bu + \lambda C\Big(\frac{u}{\lambda}\Big) = u$$

for some $\lambda \in (0, 1)$, where ∂U in the boundary of U.

Proof. Now the desired conclusion of the theorem follows by an application of Theorem 2.1, where the operator N is defined by

$$N = \left(\frac{I - C}{A}\right)^{-1} B\,. \tag{2.3}$$

From hypothesis (d), it follows that $\phi_C(r) < r$ for $r > 0$, and so by Theorem 2.2 $(I - C)^{-1}$ exists on X. Again the operator $\left(\frac{I}{A}\right)^{-1}$ exists in view of hypothesis (b).

Therefore the operator $\left(\frac{I-C}{A}\right)^{-1} = \left(\frac{I}{A}\right)^{-1}(I-C)^{-1}$ exists on $\overline{U}$. We show that the operator N given by (2.3) is well defined. We claim that

$$\left(\frac{I-C}{A}\right)^{-1} B : \overline{U} \to X\,.$$

It is enough to prove that

$$B(\overline{U}) \subset \left(\frac{I-C}{A}\right)(X)\,.$$

Let $y \in \overline{U}$ be fixed point. Define a mapping $A_y : X \to X$ by

$$A_y(x) = AxBy + Cx\,. \tag{2.4}$$

Let $x_1, x_2 \in X$. Then by (a),

$$\begin{aligned}
\|A_y(x_1) - A_y(x_2)\| &\le \|Ax_1By - Ax_2By\| + \|Cx_1 - Cx_2\| \\
&\le \|Ax_1 - Ax_2\|\|By\| + \|Cx_1 - Cx_2\| \\
&\le \|B(\overline{U})\|\phi_A(\|x_1 - x_2\|) + \phi_C(\|x_1 - x_2\|) \\
&\le M\phi_A(\|x_1 - x_2\|) + \phi_C(\|x_1 - x_2\|) \\
&= \psi(\|x_1 - x_2\|)\,,
\end{aligned}$$

where $\psi(r) = M\phi_A(r) + \phi_C(r) < r$ if $r > 0$.

Hence by an application of Theorem B yields that there is a unique point $x^* \in X$ such tat

$$Ax^*By + Cx^* = x^*$$

or

$$Ax^*By = (I-C)x^*\,,$$

i.e.,

$$By = \left(\frac{I-C}{A}\right)x^*\,.$$

Hence $\left(\frac{I-C}{A}\right)^{-1}B$ defines a mapping

$$\left(\frac{I-C}{A}\right)^{-1} B : \overline{U} \to X\,.$$

Next we show that $\left(\frac{I-C}{A}\right)^{-1}$ is a continuous mapping on $B(\overline{U})$. Let $\{x_n\}$ be any sequence in $B(\overline{U})$ such that $x_n \to x$.

Let

$$\left(\frac{I-C}{A}\right)^{-1}(x_n) = y_n \Rightarrow x_nAy_n + Cy_n = y_n$$

and

$$\left(\frac{I-C}{A}\right)^{-1}(x) = y \Rightarrow xAy + cy = y\,.$$

Now

$$\begin{aligned}
\|y_n - y\| &= \|x_n Ay_n + Cy_n - xAy - Cy\| \\
&\le \|x_n Ay_n - xAy\| + \|Cy_n - Cy\| \\
&\le \|x_n Ay_n - x_n Ay\| + \|x_n Ay - xAy\| + \|Cy_n - Cy\| \\
&\le \|x_n\|\|Ay_n - Ay\| + \|Ay\|\|x_n - x\| + \|Cy_n - Cy\| \\
&\le M\phi_A(\|y_n - y\|) + \|Ay\|\|x_n - x\| + \phi_C(\|y_n - y\|)\,.
\end{aligned}$$

Therefore

$$\limsup_n \|y_n - y\| \le M\phi_A\big(\limsup_n \|y_n - y\|\big) + \phi_C\big(\limsup_n \|y_n - y\|\big)\,.$$

If $\limsup_n \|y_n - y\| \neq 0$, then we get a contradiction to the hypothesis (d). Hence

$$\lim_n \Big\|\Big(\frac{I-C}{A}\Big)^{-1}(x_n) - \Big(\frac{I-C}{A}\Big)^{-1}(x)\Big\| = \lim_n \|y_n - y\| = 0\,.$$

This shows that the operator $(\frac{I-C}{A})^{-1}$ is continuous on $B(\overline{U})$. Since N is a composite of a continuous and a completely continuous mappings, it is completely continuous operator on $\overline{U}$. Now an application of Theorem 2.1 yields that either the operator equation $\big(\frac{I-C}{A}\big)^{-1}Bx = x$ has a solution in $\overline{U}$ or there exists a point $u \in \partial U$ such that $\lambda\big(\frac{I-C}{A}\big)^{-1}Bu = u$ for some $\lambda \in (0,1)$. This further implies that either the operator equation $AxBx + Cx = x$ has a solution in $\overline{U}$ or there exists an $u \in \partial U$ such that $\lambda A\big(\frac{u}{\lambda}\big)Bu + \lambda C\big(\frac{u}{\lambda}\big) = u$ for some $\lambda \in (0,1)$, that is, either conclusion (i) or (ii) hold. This completes the proof. □

Remark 2.2. We note that in a Banach algebra X, the operator $\big(\frac{I}{A}\big)^{-1}$ exists if $\frac{I}{A}$ is well defined and one-to-one.

An interesting special case of Theorem 2.4 in the applicable form is

Corollary 2.1. *Let $\mathcal{B}(0,r)$ and $\mathcal{B}[0,r]$ respectively denote the open and closed balls centered at origin of radius $r > 0$ in a Banach algebra X and let $A, B, C : X \to X$ be three operators such that*

(a) *$\frac{I}{A}$ is well defined and one-to-one,*

(b) *A and C are Lipschitzians with the Lipschitz constants α and β respectively,*

(c) *B is completely continuous, and*

(d) *$\alpha M + \beta < 1$, where $M = \|B(B[0,r])\|$.*

Then either

(i) *the operator equation $AxBx + Cx = x$ has a solution in $\mathcal{B}[0,r]$, or*

(ii) *there exists an $u \in X$ with $\|u\| = r$ such that $\lambda A\big(\frac{u}{\lambda}\big)Bu + \lambda C\big(\frac{u}{\lambda}\big) = u$ for some $\lambda \in (0,1)$.*

When $C \equiv 0$, we obtain

Theorem 2.5. *Let U and $\overline{U}$ denote respectively the open bounded and closed bounded subset of a Banach algebra X, and let $A : X \to X$ and $B : \overline{U} \to X$ be three operators satisfying*

(a) *A is $\mathcal{D}$-Lipschitzian with the $\mathcal{D}$-function ϕ_A,*

(b) *$\left(\frac{I}{A}\right)^{-1}$ exists, I being the identity operator on X, and the operator $\frac{I}{A}$: $X \to X$ is defined by $\left(\frac{I}{A}\right)x = \frac{x}{Ax}$,*

(c) *B is completely continuous, and*

(d) *$M\phi_A(r) < r$ for $r > 0$, where $M = \|B(\overline{U})\|$.*

Then either

(i) *the operator equation $AxBx = x$ has a solution in $\overline{U}$, or*

(ii) *there exists an $u \in \partial U$ such that*

$$\lambda A\Big(\frac{u}{\lambda}\Big)Bu = u$$

for some $\lambda \in (0,1)$, where ∂U in the boundary of U.

Corollary 2.2 (Dhage and Regan [10])**.** *Let $\mathcal{B}(0,r)$ and $\mathcal{B}[0,r]$ denote respectively the open and closed balls centered at origin of radius $r > 0$ in a Banach algebra X and let $A, B : X \to X$ be three operators such that*

(a) *$\frac{I}{A}$ is well defined and one-to-one,*

(b) *A is Lipschitzian with the Lipschitz constants α,*

(c) *B is completely continuous, and*

(d) *$\alpha M < 1$, where $M = \|B(\mathcal{B}[0,r])\|$.*

Then either

(i) *the operator equation $AxBx = x$ has a solution in $\mathcal{B}[0,r]$, or*

(ii) *there exists a $u \in X$ with $\|u\| = r$ such that $\lambda A\big(\frac{u}{\lambda}\big)Bu = u$ for some $\lambda \in (0,1)$.*

3. Another direction

In this section, we obtain some nonlinear alternatives of Leray-Schauder type involving the two operators in a Banach space X. A slight generalization of a nonlinear alternative established in Dhage and Regan [10] is

Theorem 3.1. *Let U and $\overline{U}$ denote respectively the open and closed subsets of a Banach space X such that $0 \in U$, and let $A, B : X \to X$ and $B : \overline{U} \to X$ satisfy*

(a) *A is nonlinear $\mathcal{D}$-contraction, and*

(a) *B is compact and continuous.*

Then either

(i) *the operator equation $Ax + Bx = x$ has a solution in $\overline{U}$, or*

(ii) *there exists an $u \in \partial U$ such that $\lambda A(\frac{u}{\lambda}) + \lambda Bu = u$ for $\lambda \in (0,1)$, where ∂U is the boundary of $\overline{U}$.*

Proof. The desired conclusion of the theorem follows by a direct application of Theorem 2.1 to the mapping $N = (I - A)^{-1}B$. Since A is nonlinear contraction, by Theorem 2.2, the sequence $\{A^n x\}$, $x \in X$, of successive iteration of A at x converges to the unique fixed point of A, Therefore $(I - A)^{-1}$ exists and is continuous on X. Now the map $(I - A)^{-1}B$ which is a composite of a continuous and completely continuous operator, is completely continuous on X, and in particular on $\overline{U}$. Hence by an application of Theorem 2.1 yields that either the operator equation $(I - A)^{-1}Bx = x$ has a solution in $\overline{U}$, or the operator equation $\lambda(I - A)^{-1}Bx = x$ has a solution on the boundary $\partial\bar{U}$ of $\overline{U}$ for some $\lambda \in (0, 1)$. This further implies that either (i) the operator equation $Ax + Bx = x$ has a solution in $\overline{U}$, or (ii) there exists an $u \in \partial\overline{U}$ such that $\lambda A(\frac{u}{\lambda}) + \lambda Bu = u$ for $\lambda \in (0, 1)$. This completes the proof. □

An interesting corollary to Theorem 3.1 is

Corollary 3.1. (Dhage and Regan [10]) *Let $\mathcal{B}(0, r)$ and $\mathcal{B}[0, r]$ respectively denote the open and closed ball in a Banach space X and let $A, B : X \to X$ satisfy*

(a) *A is contraction, and*

(b) *B is completely continuous.*

Then either

(i) *the operator equation $Ax + Bx = x$ has a solution in $\mathcal{B}[0, r]$, or*

(ii) *there exists an element $u \in X$ with $\|u\| = r$ such that $\lambda A(\frac{u}{\lambda}) + \lambda Bu = u$ for some $\lambda \in (0, 1)$.*

Theorem 3.2. *Let U and $\overline{U}$ respectively denote the open and closed ball in a Banach space X such that $0 \in U$. Let $A, B : X \to X$ and $B : \overline{U} \to X$ satisfy*

(a) *A is linear and bounded,*

(b) *there exists a positive integer p such that A^p is a nonlinear $\mathcal{D}$-contraction, and*

(c) *B is compact and continuous.*

Then either the conclusion (i) *or* (ii) *of Theorem 3.1 holds.*

Proof. The desired conclusion of the theorem follows by a direct application of Theorem 2.1 to the mapping $N : \overline{U} \to X$ defined by

$$N = (I - A)^{-1}B\,.$$

Now

$$(I - A)^{-1} = (I - A^p)^{-1}\Big(\sum_{j=0}^{p-1} A^j\Big)\,.$$

By hypothesis (b), A^p is a nonlinear contraction, so by Theorem 2.2, the mapping $(I - A^p)^{-1}$ exists on X and consequently the mapping $(I - A)^{-1}$ exists on X. Moreover from hypothesis (a) it follows that $(I - A)^{-1}$ is continuous on X. The rest of the proof is similar to Theorem 3.1. We omit the details. □

A special case to Theorem 3.2 in its applicable form is

Corollary 3.2. *Let $\mathcal{B}(0,r)$ and $\mathcal{B}[0,r]$ denote respectively the open and closed ball, in a Banach space X centered at origin and of radius $r>0$. Let $A,B: X\to X$ satisfy*

(a) *A is linear and bounded,*
(b) *there exists a positive integer p such that A^p is a contraction, and*
(c) *B is completely continuous.*

Then either conclusion (i) *or* (ii) *of Theorem 3.2 holds.*

Finally we give an important remark which is useful in the applications to the theory of nonlinear differential equations.

Remark 3.1. Note that in Theorems 2.4 and 2.5, the sets U and $\overline{U}$ are assumed to be the open and closed subsets in a Banach space X respectively, however, it is enough to assume that U and $\overline{U}$ are subsets of some closed convex subset K of the Banach algebra X. The same argument is also valid for Theorems 3.1 and 3.2.

4. Existence theory

We shall seek the solution of the FIE (1.1) in the space $BM(J,\mathbb{R})$ of all bounded and measurable real-valued functions on J. Define a norm $\|\cdot\|$ in $BM(J,\mathbb{R})$ by

$$\|x\| = \max_{t\in J} |x(t)|\,. \tag{4.1}$$

Clearly $BM(J,\mathbb{R})$ becomes a Banach algebra with this supremum norm. We need the following definition in the sequel.

Definition 4.1. A mapping $\beta : J\times\mathbb{R}\to\mathbb{R}$ is said to satisfy L^1-*Carathéodory* condition or simply is called L^1-*Carathéodory* if

(a) $t\to\beta(t,x)$ is measurable for each $x\in\mathbb{R}$,
(b) $x\to\beta(t,x)$ is almost everywhere continuous for $t\in J$, and
(c) for each real number $r>0$, there exists a function $h_r\in L^l(J,\mathbb{R})$ such that
$$|\beta(t,x)|\le h_r(t) \quad \text{a.e.} \quad t\in J$$
for all $x\in\mathbb{R}$ with $|x|\le r$.

We consider the following assumptions:

(H_0) The functions $\mu,\nu,\sigma,\eta : J\to J$ are continuous.
(H_1) The function $k : J\times\mathbb{R}\to\mathbb{R}$ is continuous and there exists a bounded function $\gamma : J\to\mathbb{R}$ with bound $\|\gamma\|$ such that
$$|k(t,x)-k(t,y)|\le\gamma(t)|x-y| \quad \text{a.e.} \quad t\in J$$
for all $x,y\in\mathbb{R}$.
(H_2) The function $x\to\dfrac{x}{f(t,x)}$ is well defined for all $t\in J$ and $\dfrac{x}{f(t,x)} = \dfrac{y}{f(t,y)} \Rightarrow x=y$ for all $t\in J$.

(H_3) The function $f : J \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$ is continuous and there exists a bounded function $\alpha : J \to \mathbb{R}$ with bound $\|\alpha\|$ such that

$$|f(t,x) - f(t,y)| \leq \alpha(t)|x - y| \quad \text{a.e.} \quad t \in J\,.$$

(H_4) $q : J \to \mathbb{R}$ is continuous.

(H_5) g is L^1-Carathéodory.

(H_6) There exists a continuous and nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ and a function $p \in L^1(J,\mathbb{R})$, $p(t) > 0$ a.e. $t \in J$ such that

$$|g(t,x)| \leq p(t)\psi(|x|) \quad \text{a.e.} \quad t \in J$$

for all $x \in \mathbb{R}$.

Theorem 4.1. *Suppose that the assumptions (H_0)–(H_6) hold. Further if there exists a real number $r > 0$ such that*

$$\begin{cases} \|\alpha\|\big(Q + \|p\|_{L^1}\psi(r)\big) + \|\gamma\| < 1, \\ r > \frac{K+F(Q+\|p\|_{L^1}\psi(r))}{1-[\|\alpha\|(Q+\|p\|_{L^1}\psi(r))+\|\gamma\|]}\,, \end{cases} \tag{4.2}$$

where $Q = \sup_{t\in J} |q(t)|$, $K = \sup_{t\in J} |k(t,0)|$ and $F = \sup_{t\in J} |f(t,0)|$, then the FIE (1.1) *has a solution on J.*

Proof. Define an open ball $\mathcal{B}(0,r)$ centered at origin of radius $r > 0$, where r satisfies the inequalities in (4.2). Now consider the mapping A, B, C on $BM(J,\mathbb{R})$ defined by

$$\begin{cases} Ax(t) = f\big(t, x(\nu(t))\big), \quad t \in J, \\ Bx(t) = q(t) + \displaystyle\int_0^{\sigma(t)} g\big(s, x(\eta(s))\big)\,ds, \quad t \in J, \quad \text{and} \\ Cx(t) = k\big(t, x(\mu(t))\big)\,, \quad t \in J\,. \end{cases} \tag{4.3}$$

Then the problem FIE (1.1) is equivalent to the operator equation,

$$Ax(t)Bx(t) + Cx(t) = x(t)\,, \quad t \in J\,.$$

We shall show that the mappings A, B and C satisfy all the conditions of Corollary 2.1 on $BM(J,\mathbb{R})$. Clearly A, B and C define the mappings $A, B, C : BM(J,\mathbb{R}) \to BM(J,\mathbb{R})$. As $f(t,x) \neq 0$ for all $(t,x) \in J \times \mathbb{R}$, the operator $\frac{I}{A}$ is well defined on $BM(J,\mathbb{R})$. Further in view of hypotheis (H_2) it follows that the operator $\frac{I}{A}$ is one-to-one on $BM(J,\mathbb{R})$. We shall show that A and C are Lipschitzians on $BM(J,\mathbb{R})$. Let $x, y \in BM(J,\mathbb{R})$. Then by (H_1) and (H_3),

$$\begin{aligned} |Ax(t) - Ay(t)| &= |f\big(t, x(\nu(t))\big) - f\big(t, y(\nu(t))\big)| \\ &\leq \alpha(t)|x\big(\nu(t)\big) - y\big(\nu(t)\big)| \\ &\leq \|\alpha\|\|x - y\|\,, \end{aligned}$$

i.e.,

$$\|Ax - Ay\| \le \|\alpha\| \|x - y\|\,,$$

and

$$\begin{aligned}|Cx(t) - Cy(t)| &= |k\big(t, x(\mu(t))\big) - k\big(t, y(\mu(t))\big)|\\ &\le \gamma(t)|x\big(\mu(t)\big) - y\big(\mu(t)\big)|\\ &\le \|\gamma\| \|x - y\|\,,\end{aligned}$$

i.e.,

$$\|Cx - Cy\| \le \|\gamma\|\ \|x - y\|\,.$$

Thus A and B are Lipschitzians with the Lipschitz constants $\|\alpha\|$ and $\|\gamma\|$ respectively. Now we shall show that B is completely continuous mapping on $\mathcal{B}[0, r]$. Since σ and η are continuous and g is Carathéodory, it follows from dominated convergence theorem that B is continuous on $\mathcal{B}[0, r]$. See Granas et. al. [12]. Let $\{x_n\}$ be a sequence in $\mathcal{B}[0, r]$, then $\|x_n\| \le r$ for all $n \in \mathbb{N}$; so by (H_5),

$$\begin{aligned}\|Bx_n\| &\le \sup_{t\in J} |q(t)| + \sup_{t\in J} \int_0^{\sigma(t)} |g\big(s, x_n(\eta(s))\big)|\, ds\\ &\le Q + \int_0^1 h_r(s)\, ds\\ &= Q + \|h_r\|_{L^1}\,,\end{aligned}$$

which shows that $\{Bx_n\}$ is a uniformly bounded sequence in $BM(J, \mathbb{R})$. Next we show that $\{Bx_n\}$ is a equi-continuous set. Let $t, \tau \in J$. Then we have

$$\begin{aligned}|Bx_n(t) - Bx_n(\tau)| &\le |q(t) - q(\tau)| + \Big|\int_{\sigma(\tau)}^{\sigma(t)} g\big(s, x(\eta(s))\big)\, ds\Big|\\ &\le |q(t) - q(\tau)| + |m(t) - m(\tau)|\,,\end{aligned}$$

where $m(t) = \displaystyle\int_0^{\sigma(t)} h_r(s)\, ds$.

Since q and m are uniformly continuous functions on J, we conclude that $\{Bx_n\}$ is a equi-continuous set. Hence B is a completely continuous mapping from $B[0, r]$ into $BM(J, \mathbb{R})$ by Arzela-Ascoli theorem. Again by (H_5),

$$\begin{aligned}M &= \|B(\mathcal{B}[0, r])\|\\ &= \sup\{\|Bx\| : x \in \mathcal{B}[0, r]\}\\ &= \sup_{x\in\mathcal{B}[0,r]} \Big\{\sup_{t\in J} |Bx(t)|\Big\}\\ &\le \sup_{x\in\mathcal{B}[0,r]} \Big\{\sup_{t\in J} |q(t)| + \sup_{t\in J} \int_0^{\sigma(t)} |g\big(s, x(\eta(s))\big)|\, ds\Big\}\end{aligned}$$

$$\leq \sup_{x\in\mathcal{B}[0,r]} \Big\{ Q + \int_0^1 p(s)\psi(\|x\|)\,ds \Big\}$$
$$\leq Q + \|p\|_{L^1}\psi(r)$$

and so

$$\|\alpha\|M + \|\gamma\| = \|\alpha\|\big(Q + \|p\|_{L^1}\psi(r)\big) + \|\gamma\| < 1$$

in view of first inequality in condition (4.2).

Thus all the conditions of Corollary 2.1 are satisfied and hence an application of it yields that either the conclusion (i) or (ii) holds. Now we shall show that the conclusion (ii) is not possible. Indeed if u is a solution of the operator equation

$$\lambda A\Big(\frac{u}{\lambda}\Big)Bu + \lambda C\Big(\frac{u}{\lambda}\Big) = u$$

with $\|u\| = r$ for some $\lambda \in (0,1)$, then u is also a solution of the FIE

$$u(t) = \lambda\Big[k\big(t, \frac{1}{\lambda}u(\mu(t))\big)\Big] + \lambda\Big[f\big(t, \frac{1}{\lambda}u(\nu(t))\big)\Big]\Big(q(t) + \int_0^{\sigma(t)} g\big(s, u(\eta(s))\big)\,ds\Big) \tag{4.4}$$

for some $\lambda \in (0,1)$.

But for any solution u of (4.4) with $\|u\| = r$ and $\lambda \in (0,1)$ one has

$$\begin{aligned}
|u(t)| &\leq \Big|\lambda k\Big(t, \frac{1}{\lambda}(\mu(t))\Big) - \lambda k(t,0)\Big| + \lambda|k(t,0)| \\
&\quad + \Big(\Big|\lambda f\Big(t, \frac{1}{\lambda}u(\nu(t))\Big) - \lambda f(t,0)\Big| + \lambda|f(t,0)|\Big) \\
&\quad \times \Big(|q(t)| + \int_0^{\sigma(t)} |g(s, u(\eta(s)))|\,ds\Big) \\
&\leq \gamma(t)|u\big(\mu(t)\big)| + |k(t,0)| \\
&\quad + (|\alpha(t)||u(\nu(t)| + |f(t,0)|)\Big(|q(t)| + \int_0^1 p(s)\psi\big(|u(\eta(s))|\big)\,ds\Big) \\
&\leq \|\gamma\|\|u\| + K + (\|\alpha\|\|u\| + F)\big(Q + \|p\|_{L^1}\psi(\|u\|)\big)\,, \\
\|u\| &\leq \big[\|\alpha\|\big(Q + \|p\|_{L^1}\psi(\|u\|)\big) + \|\gamma\|\big]\|u\| + F\big(Q + \|p\|_{L^1}\psi(\|u\|)\big) + K\,.
\end{aligned}$$

Letting $\|u\| = r$ in the above inequality yields

$$r \leq \frac{K + F(Q + \|p\|_{L^1}\psi(r))}{1 - [\|\alpha\|(Q + \|p\|_{L^1}\psi(r)) + \|\gamma\|]},$$

which is the contradiction to the second inequality in (4.2). Therefore the conclusion (i) holds, and consequently the FIE (1.1) has a solution u on J with $\|u\| \leq r$. This completes the proof. □

As an application, we consider the following nonlinear functional differential equation (in short FDE)

$$\left(\frac{x(t) - k(t, x(\mu(t)))}{f(t, x(\nu(t)))}\right)' = g\big(t, x(\eta(s))\big) \quad \text{a.e.} \quad t \in J \tag{4.5}$$

satisfying the initial condition

$$x(0) = \xi \in \mathbb{R}\,, \tag{4.6}$$

where $k, g : J \times \mathbb{R} \to \mathbb{R}$, $f : J \times \mathbb{R} \to \mathbb{R}\setminus\{0\}$ and $\mu, \nu, \eta : J \to J$ are continuous with $\nu(t) = 0 = \mu(0)$.

By a solution of the FDE (4.5)–(4.6) we mean an absolutely continuous function $x : J \to \mathbb{R}$ that satisfies the equations (4.5)–(4.6) on J.

Let $C(J, \mathbb{R})$ denote the space of all continuous functions on J. Clearly $C(J, \mathbb{R}) \subset BM(J, \mathbb{R})$. The existence result for the FDE (4.5)–(4.6) is

Theorem 4.2. *Suppose that the assumptions* (H_1)*–*(H_3) *and* (H_3)*–*(H_5) *hold. Further if there exists a real number* $r > 0$ *such that the inequalities in* (4.2) *hold with* $Q = \left|\frac{\xi - k(0,\xi)}{f(0,\xi)}\right|$*, then the* FDE (4.5)–(4.6) *has a solution on* J.

Proof. The FDE (4.5)–(4.6) is equivalent to the integral equation.

$$x(t) = k\big(t, x(\mu(t))\big) + \big[f\big(t, x(\nu(t))\big)\big]\Big(\frac{\xi - k(0,\xi)}{f(0,\xi)} + \int_0^t g\big(s, x(\eta(s))\big)\,ds\Big) \tag{4.7}$$

for $t \in J$.

Now the desired conclusion of the theorem follows by a direct application of Theorem 4.1 with $q(t) = \frac{\xi - k(0,\xi)}{f(0,\xi)}$ for all $t \in J$, $\sigma(t) = t$ and the space $BM(J, \mathbb{R})$ replaced with $C(J, \mathbb{R})$. The proof is complete. □

Further in an analogous way to Theorem 4.1, we can obtain the existence results for the functional integral equations,

$$x(t) = q(t) + \int_0^{\mu(t)} f\big(s, x(\nu(s))\big)\,ds + \int_0^{\sigma(t)} g\big(s, x(\eta(s)\big)\,ds$$

and

$$x(t) = q(t) + \int_0^{t} k\big(t, s)x(\nu(s)\big)\,ds + \int_0^{\sigma(t)} g\big(s, x(\eta(s)\big)\,ds$$

by the application of Corollaries 3.1 and 3.2, respectively. Some of the results in this direction may be found in Dhage [7] and Dhage and Ntouyas [8].

Remark 4.1. Finally while concluding this paper, we remark if $k(t, x) \equiv 0$ on $J \times \mathbb{R}$, then Theorem 4.1 reduces to the existence results proved in Dhage and Regan [10] for the functional integral equation

$$x(t) = \big[f\big(t, x(\nu(t))\big]\Big(q(t) + \int_0^{\sigma(t)} g\big(s, x(\eta(s))\big)ds\Big)\,.$$

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