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ON GENERALIZED "HAM SANDWICH" THEOREMS

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ABSTRACT. In this short note we utilize the Borsuk-Ulam Anitpodal Theorem to present a simple proof of the following generalization of the "Ham Sandwich Theorem":

Let $A_1, \ldots, A_m \subseteq \mathbb{R}^n$ be subsets with finite Lebesgue measure. Then, for any sequence $f_0, \ldots, f_m$ of $\mathbb{R}$-linearly independent polynomials in the polynomial ring $\mathbb{R}[X_1, \ldots, X_n]$ there are real numbers $\lambda_0, \ldots, \lambda_m$, not all zero, such that the real affine variety $\{x \in \mathbb{R}^n;\ \lambda_0 f_0(x) + \cdots + \lambda_m f_m(x) = 0\}$ simultaneously bisects each of subsets A_k, $k = 1, \ldots, m$. Then some its applications are studied.

The Borsuk-Ulam Antipodal Theorem (see e.g. [2, 12]) is the first really striking fact discovered in topology after the initial contributions of Poincaré and its fundamental role shows an enormous influence on mathematical research. A deep theory evolved from this result, including a large number of applications and a broad variety of diverse generalizations. In particular, as it was shown in [9], an interrelation between topology and geometry can be established by means of an appropriate version of the famous "Ham Sandwich" Theorem deduced from the Borsuk-Ulam Antipodal Theorem. It was pointed out in [6] that an existence of common hyperplane medians for random vectors can be proved from the "Ham Sandwich" Theorem as well.

The presented main result is probably known to some experts but its proof is much simpler than others in the literature and some consequences are easily deduced. Our paper grew up to answer the question posed in [6]; that is of which curves or manifolds other than straight lines or hyperplanes can serve as common medians for random vectors. To settle that question we make use of the result which is presented in later given Theorem 4.

Let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}^n$ the n-Euclidean space and $\mathbb{S}^n$ the n-sphere. The following theorem is well known (see e.g. [3, p.79] or [4, p.287]).

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Theorem 1 ("Ham Sandwich" Theorem)**.** *Given any subsets $A_1, \dots, A_n \subseteq \mathbb{R}^n$ with finite Lebesgue measure, there exists an $(n-1)$-hyperplane which simultaneously bisects each of subsets $A_1, \dots, A_n$.*

Its proof is based on the famous and with a broad spectrum of applications Borsuk-Ulam Antipodal Theorem ([2, 12]).

Theorem 2. *If $\Phi : \mathbb{S}^n \to \mathbb{R}^n$ is a continuous antipodal map then there is a point of $\mathbb{S}^n$ which maps into the origin of $\mathbb{R}^n$.*

A.H. Stone and J.W. Tukey show in [13] that a fuller use of the Borsuk-Ulam Anitpodal Theorem gives a more general fact and Arens' remarkable note [1] is to read as a gloss on [13] since a counterexample for the idea behind of the usual proof of the "Ham Sandwich" Theorem is provided.

We summarize [13] to present its extended version. Let $(X, \mu_1, \dots, \mu_m)$ be a space with signed measures and $f : X \times \mathbb{S}^m \to \mathbb{R}$ a real valued map such that:

(1) for each $\lambda \in \mathbb{S}^m$ the map $f(-,\lambda) : X \to \mathbb{R}$ is a μ_k-measurable map and vanishes only over a μ_k-measure zero set, $k = 1, \dots, m$;

(2) for each $x \in X$ the map $f(x,-) : \mathbb{S}^m \to \mathbb{R}$ is continuous;

(3) for each pair of diametrically opposite points $\lambda, -\lambda \in \mathbb{S}^n$, $f(x,\lambda)f(x,-\lambda) \leq 0$ almost everywhere in X with respect to all signed measures μ_k, $k = 1, \dots, m$.

Write $f^+(\lambda)$, $f^0(\lambda)$ and $f^-(\lambda)$ for the subsets of X on which $f(x,\lambda) \geq 0$, $= 0$ and ≤ 0, respectively. We say that $f^0(\lambda)$ *bisects* a μ_k-measurable subset $A \subseteq X$ with $|\mu_k(A)| < \infty$ if $\mu_k(f^+(\lambda) \cap A) = \mu_k(f^-(\lambda) \cap A) = \frac{1}{2}\mu_k(A)$, $k = 1, \dots, m$. But every signed measure can be represented as the difference of its upper and lower variations called the Jordan decomposition ([5, p.123]). Thus, by [13] the maps $\phi_k : \mathbb{S}^m \to \mathbb{R}$ given by $\phi_k(\lambda) = \mu_k(A \cap f^+(\lambda)) - \mu_k(A \cap f^-(\lambda))$ for $\lambda \in \mathbb{S}^m$, $k = 1, \dots, m$ are continuous odd functions. Therefore, the result in [13] yields

Theorem 3. *Given subsets $A_1, \dots, A_m \subseteq X$ in a space X with signed measures $\mu_1, \dots, \mu_m$, $|\mu_k(A_k)| < \infty$ and a map $f : X \times \mathbb{S}^m \to \mathbb{R}$ satisfying the properties above, there exists $\lambda \in \mathbb{S}^m$ such that $f^0(\lambda)$ simultaneously bisects each of subsets A_k with respect to signed measures μ_k, $k = 1, \dots, m$.*

Thus, the following corollary may be deduced from [13].

Corollary 1. *Let $f_0, \dots, f_m$ be real valued maps on X which are μ_k-measurable and linearly independent modulo subsets in X of μ_k-measure zero, $k = 1, \dots, m$ and $A_1, \dots, A_m \subseteq X$ be subsets with $|\mu_k(A_k)| < \infty$, $k = 1, \dots, m$. Then there exist real numbers $\lambda_0, \dots, \lambda_m$, not all zero, such that the set $\{x \in X;\ \lambda_0 f_0(x) + \cdots + \lambda_m f_m(x) = 0\}$ simultaneously bisects each of subsets A_k, $k = 1, \dots, m$.*

In particular, let (X, μ) be a measure space and $g_1, \dots, g_m$ be μ-integrable real valued maps on X. For a μ-measurable subset $A \subseteq X$ put $\mu_k(A) = \int_A g_k d\mu$, $k = 1, \dots, m$. Then $\mu_1, \dots, \mu_m$ are signed measures and a generalization of the result presented in [9] can be derived.

Corollary 2. *Let (X, μ) be a measure space, $f : X \times \mathbb{S}^n \to \mathbb{R}$ is a map satisfying the properties above for $\mu_1 = \cdots = \mu_m = \mu$ and $A_1, \dots, A_m \subseteq X$ be μ-measurable* *subsets with* $|\mu(A_k)| < \infty$, $k = 1, \ldots, m$. *Then for* μ*-integrable real valued maps* $g_1, \ldots, g_m$ *on* X *there exist real numbers* $\lambda_0, \ldots, \lambda_m$, *not all zero, such that for* $\lambda = (\lambda_0, \ldots, \lambda_m)$

$$\int_{\{x \in A_k;\ f(x,\lambda) \leq 0\}} g_k d\mu = \int_{\{x \in A_k;\ f(x,\lambda) \geq 0\}} g_k d\mu = \frac{1}{2} \int_{A_k} g_k d\mu,$$

$k = 1, \ldots, m$.

Let now $\mathbb{R}[X_1, \ldots, X_n]$ be the polynomial ring over $\mathbb{R}$ of n-variables. Then, we may formulate the following theorem as a consequence of the results above.

Theorem 4. *Let* $\mu_1, \ldots, \mu_m$ *be signed measures on* $\mathbb{R}^n$ *and* $A_1, \ldots, A_m \subseteq \mathbb{R}^n$ *subsets with* $|\mu_k(A_k)| < \infty$, *all polynomial functions are* μ_k*-measurable, and real affine varieties in* $\mathbb{R}^n$ *determined by nonzero polynomials in the ring* $\mathbb{R}[X_1, \ldots, X_n]$ *are* μ_k*-zero subsets,* $k = 1, \ldots, m$. *Then for any sequence* $f_0, \ldots, f_m$ *of* $\mathbb{R}$*-linearly independent polynomials in the ring* $\mathbb{R}[X_1, \ldots, X_n]$ *there exist real numbers* $\lambda_0, \ldots, \lambda_m$, *not all zero, such that the real affine variety determined by the polynomial* $f = \lambda_0 f_0 + \cdots + \lambda_m f_m$ *simultaneously bisects each of subsets* A_k *with respect to the signed measure* μ_k, $k = 1, \ldots, m$.

Put μ for a given measure on $\mathbb{R}^n$ vanishing on all real affine varieties determined by nonzero polynomials in the ring $\mathbb{R}[X_1, \ldots, X_n]$ and let $g_1, \ldots, g_m$ be μ-integrable real valued maps on $\mathbb{R}^n$. Then by Corollary 2, for any sequence $f_0, \ldots, f_m$ of $\mathbb{R}$-linearly independent polynomials in the ring $\mathbb{R}[X_1, \ldots, X_n]$ there exist real numbers $\lambda_0, \ldots, \lambda_m$, not all zero, such that

$$\int_{\{x \in \mathbb{R}^n;\ \lambda_0 f_0(x) + \cdots + \lambda_m f_m(x) \leq 0\}} g_k d\mu = \int_{\{x \in \mathbb{R}^n;\ \lambda_0 f_0(x) + \cdots + \lambda_m f_m(x) \geq 0\}} g_k d\mu$$
$$= \frac{1}{2} \int_{\mathbb{R}^n} g_k d\mu\,,$$

$k = 1, \ldots, m$.

In particular, for $n = 1$ we get a solution of a generalized moment problem a special case of which has been examined in [7] and smartly reproved in [8].

Corollary 3. *Let* $\mu_1, \ldots, \mu_m$ *be measures on the unit interval* $[0, 1]$ *vanishing on all single point subsets and* $g_1, \ldots, g_m$ *be functions on* $[0, 1]$ *such that* g_k *is* μ_k*-*-*integrable,* $k = 1, \ldots, m$. *Then there are real numbers* $0 = x_0 < x_1 \cdots < x_{l+1} = 1$, $l \leq m$ *and such that*

$$\sum_{i=0}^{l} (-1)^i \int_{x_i}^{x_{i+1}} g_k d\mu_k = 0\,,$$

$k = 1, \ldots, m$.

Proof. For $\mathbb{R}$-linearly independent polynomials $f_0 = 1, f_1 = X, \ldots, f_m = X^m$ in the polynomial ring $\mathbb{R}[X]$, by the arguments above there exist real numbers

$\lambda_0, \lambda_1, \ldots, \lambda_m$, not all zero, and such that

$$\int_{\{x\in[0,1];\, \lambda_0+\lambda_1 x+\cdots+\lambda_m x^m \le 0\}} g_k d\mu = \int_{\{x\in[0,1];\, \lambda_0+\lambda_1 x+\cdots+\lambda_m x^m \ge 0\}} g_k d\mu$$

$$= \frac{1}{2}\int_0^1 g_k d\mu\,,$$

$k = 1, \ldots, m$. Take $x_1, \ldots, x_l$, $l \le m$, to be the all real roots in $[0,1]$ of the polynomial $f = \lambda_0 + \lambda_1 X + \cdots + \lambda_m X^m$ and the result follows. □

Let $I \subseteq \mathbb{R}[X_1, \ldots, X_n]$ be an ideal and $V(I)$ the associated real affine variety. To deduce the next result we need

Lemma 1. *If $I \subseteq \mathbb{R}[X_1, \ldots, X_n]$ is a nonzero ideal then the real variety $V(I)$ is a subset of zero Lebesgue measure in $\mathbb{R}^n$.*

Proof. First observe that $V(I) \subseteq V(f)$ for any polynomial f in I, where $V(f)$ is the real affine variety associated with the principal ideal (f). Therefore, we may assume that $I = (f)$ for a nonzero polynomial f in $\mathbb{R}[X_1, \ldots, X_n]$. Take now a positive integer l greater than any of the exponents of the powers occurring in f. After the substitution $X_1 = X_1'$ and $X_k' = X_k + X_1^{l^{k-1}}$, $k = 2, 3, \ldots, n$ the monomial $rX_1^{i_1} \cdots X_n^{i_n}$ takes the form

$$r{X_1}'^{(i_1+i_2 l+\cdots+i_n l^{n-1})} + \alpha(X_1', X_2', \ldots, X_n')\,,$$

the degree of polynomial α with respect to X_1' being less than $i_1 + i_2 l + \cdots + i_n l^{n-1}$.

Among the sequences of exponents of the monomials occurring in f there exists the greatest one (under the lexicographical order), from which, after expressing in terms of $X_1', X_2', \ldots, X_n'$, we can isolate the monomial $s{X'}_1^N$ so that the equation $f(X_1, \ldots, X_n) = 0$ takes the form

$$f'(X_1', X_2', \ldots, X_n') = sX_1'^N + \beta(X_1', X_2', \ldots, X_n') = 0\,,$$

where the coefficient s is a nonzero real number and the degree of the polynomial β with respect to X_1' is less than N.

Put ℓ_n for Lebesgue measure in $\mathbb{R}^n$. Then $\ell_n(V(f)) = \ell_n(V(f'))$, since the Jacobian of the induced polynomial transformation $x_1 = x_1'$ and $x_k = x_k' + x_1^{l^{k-1}}$, $k = 2, 3, \ldots, n$ of the space $\mathbb{R}^n$ is equal to 1. On the other hand, for a fixed point $(x_2', \ldots, x_n')$ in $\mathbb{R}^{n-1}$ the characteristic function $\chi_{V(f')}(-, x_2', \ldots, x_n')$ takes a finite number of nonzero values. Therefore, from the Fubini Theorem ([5, p.148]),

$$\ell_n(V(f')) = \int_{\mathbb{R}^n} \chi_{V(f')} = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \chi_{V(f')} = 0\,.$$

Finally we derive that $\ell_n\big(V(f)\big) = 0$. □

In particular, $\mathbb{R}$-linearly independent polynomials $f_0, \ldots, f_m \in \mathbb{R}[X_1, \ldots, X_n]$ are also linearly independent modulo any subset in $\mathbb{R}^n$ of Lebesgue measure zero.

Theorem 5. *Let $A_1, \ldots, A_m \subseteq \mathbb{R}^n$ be subsets with finite Lebesgue measure. Then, for any sequence $f_0, \ldots, f_m$ of $\mathbb{R}$-linearly independent polynomials in $\mathbb{R}[X_1, \ldots, X_n]$ there are real numbers $\lambda_0, \ldots, \lambda_m$, not all zero, such that the real affine variety $\{x \in \mathbb{R}^n;\ \lambda_0 f_0(x) + \cdots + \lambda_m f_m(x) = 0\}$ simultaneously bisects each of subsets A_k, $k = 1, \ldots, m$.*

Taking $f_0 = 1,\ f_1 = X_1, \ldots, f_n = X_n$ we get the "Ham Sandwich" Theorem (see e.g. [3, p.79] or [4, p.287]). Moreover, for $f_0 = 1,\ f_1 = X_1, \ldots, f_n = X_n$ and $f_{n+1} = X_1^2 + \cdots + X_n^2$ we obtain

Corollary 4 (cf. [13])**.** *Any $(n+1)$ subsets in $\mathbb{R}^n$ with finite Lebesgue measure can be bisected by an $(n-1)$-sphere in $\mathbb{R}^n$.*

The fact above, for $n = 2$, has been proved in [10] and mentioned in [11, p.145] as well.

Observe that for a positive integer m, as $\mathbb{R}$-linearly independent polynomials $f_0, \ldots, f_m$ we can take some monomials $f_k = X_1^{i_0(k)} \cdots X_n^{i_n(k)}$, $k = 0, \ldots, m$ of appropriately small degree. Namely, the set of solution in positive integers of the equation $i_1 + \cdots + i_n = k$ is equal to $\binom{k+n-1}{n-1}$. Therefore, if $d(m)$ is a positive integer such that

$$m < \sum_{k=0}^{d(m)} \binom{k+n-1}{n-1}$$

then we can take for f_k, $k = 0, \ldots, m$ monomials of degree $\leq d(m)$. In particular, we obtain that any $2n + \binom{n}{2}$ subsets in $\mathbb{R}^n$ with finite Lebesgue measure can be bisected by a hyperquadric in $\mathbb{R}^n$.

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