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BEST APPROXIMATION FOR NONCONVEX SET IN q-NORMED SPACE

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Abstract. Some existence results on best approximation are proved without starshaped subset and affine mapping in the set up of q-normed space. First, we consider the closed subset and then weakly compact subsets for said purpose. Our results improve the result of Mukherjee and Som [11] and Jungck and Sessa [7] and some known results [4], [9], [12] are obtained as consequence. To achieve our goal, we have introduced a property known as "Property(A)".

1. Introduction

Fixed point theorems have been used at many places in approximation theory. One of them is while existence of best approximation is proved. Later on, number of results were developed using fixed point theorem to prove the existence of best approximation. However, the result given by Meinardus [10] was the fundamental result in this direction. An excellent reference can be seen in [18]. An other celebrated result was due to Brosowski [1] also in fact extended the result of Meinardus [10]. Hicks and Humpheries [5], Jungck and Sessa [7], Latif [9], Mukherjee and Som [11], Sahab, Khan and Sessa [14], Singh [15, 16, 17], Subramanyam [20] were some other authors who worked in this direction under different conditions following the line made by Meinardus [10].

In a paper [15], Singh relaxed the condition of linearity of mapping and convexity of set but later, he observed [16] that only the nonexpansiveness is necessary to prove best approximation while appliying fixed point theorem. Similary, Hicks and Humpheries said in their paper [5] that the element for the set of best approximation be not necessarily in the interior of set.

Next, Sahab, Khan and Sessa [14] improved the hypothesis of Hicks and Humpheries [5] using two mappings, one linear and other nonexpansive. They took this idea from Park [13]. In an other paper, Jungck and Sessa [7] futher weakened the hypothesis of Sahab, Khan and Sessa [14] by replacing the condition of linearity

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by affineness to prove the existance of best approximation in normed linear space. However, they used weak continuity of the mapping for such purpose in the second result. Recently, Latif [9] has removed the of weak continuity from the hypothesis of Jungck and Sessa [7] and obtained the result in q-normed space.

Here, it is important to remark that Dotson [3] proved the existence of fixed point for nonexpansive mapping and thus extended his result under non-convex condition [4]. This idea was used by Mukherjee and Som [11] to prove existence of best approximation. In this way, they extended the result of Singh [15] without starshapedness condition.

The object of this paper is to prove the existence of best approximation applying common fixed point theorem without starshapedness condition of subset and affineness condition of mapping in the setup of q-normed space. In our opinion these two conditions are not required of the theorem of Mukherjee and Som [11] even if, we consider the concept of relatively nonexpansive mapping, i.e., $\|Tx - Ty\| \leq \|Ix - Iy\|$ defined under the subset of q-normed space. For this purpose, we have used the property of nonconvexity given by Dotson [4]. We infact, improve the results of Mukherjee and Som [11] and Jungck and Sessa [7] for closed subset and weakly compact subset in q-normed space. While doing so, however, we need to prove such result first for closed subset by using result of Smoluk [19] and then we proved it for weakly compact subset by using Jungck result [6]. To achive our goal, we have introduced a property known as "Property(A)".

2. Preliminaries

To prove our results, we need the following:

Definition 2.1 ([8])**.** Let X be a linear space. A q-norm on X is a real-valued function $\|\,.\,\|_q$ on X with $0 < q \leq 1$, satisfying the following conditions:

(a) $\|x\|_q \geq 0$ and $\|x\|_q = 0$ iff $x = 0$,
(b) $\|\lambda x\|_q = |\lambda|^q \|x\|_q$,
(c) $\|x + y\|_q \leq \|x\|_q + \|y\|_q$,

for all $x, y \in X$ and all scalars λ. The pair $(X, \|.\|_q)$ is called a q-normed space. It is a metric space with $d_q(x, y) = \|x - y\|_q$ for all $x, y \in X$, defining a translation invariant metric d_q on X. If $q = 1$, we obtain the concept of a normed linear space. It is well-known that the topology of every Hausdorff locally bounded topological linear space is given by some q-norm, $0 < q \leq 1$. The spaces l_q and $L_q[0, 1]$, $0 < q \leq 1$ are q-normed space. A q-normed space is not necessarily a locally convex space.

Definition 2.2 ([9])**.** Let X be a q-normed space and let C be a nonempty subset of X. Let $x \in X$. An element $y \in C$ is called a best C-approximation to $x \in X$ if

$$\|x - y\|_q = d_q(x, C) = \inf\{\|x - z\|_q : z \in C\}.$$

The set of best C-approximations to x is denoted by D and is defined as

$$D = \{z \in C : \|x - z\|_q = d_q(x, C)\}.$$

Definition 2.3 ([9]). A subset C in q-normed space X is said to be starshaped, if there exists at least one point $p \in C$ such that $\lambda x + (1-\lambda)p \in C$, for all $x \in C$ and $0 \leq \lambda \leq 1$. In this case p is called the starcenter of C.
Each convex set is starshaped with respect to each of its points, but not conversely.

Definition 2.4 ([9]). If $T : C \mapsto C$, where C is a subset of q-normed space X and $\|Tx - Ty\|_q \leq \|x - y\|_q$ for $x, y \in C$, then T is called a nonexpansive map .
A map $T : C \mapsto C$ is said to be I-contraction, if there exists a self-map I on C and a real number $k \in (0, 1)$ such that

$$\|Tx - Ty\|_q \leq [k]^q \|Ix - Iy\|_q \,,$$

for all $x, y \in C$.

If in the above inequality $k = 1$, then T is called I-nonexpansive.

Recall that, if X is a topological linear space, then its continuous dual space X' is said to separate the points of X, if for each $x \neq 0$ in X, there exists an $I \in X'$ such that $Ix \neq 0$. In this case the weak topology on X is well-defined [9]. We mention that, if X is not locally convex, then X' need not separates the points of X. For example, if $X = L_q[0, 1]$, $0 < q < 1$, then $X' = \{0\}$. However,there are some non-locally convex spaces (such as the q-normed space l_q, $0 < q < 1$) whose dual separates the points [8].

Definition 2.5 ([9]). Let X be a complete q-normed space whose dual X' separates the points of X. A map $T : C \mapsto X$ $(C \subseteq X)$ is said to be demiclosed iff whenever $\{x_n\}$ is a sequence in C converging weakly to $x \in C$ and $\{Tx_n\}$ converges strongly to $y \in X$, then $Tx = y$.

Definition 2.6 ([21]). The space X is said to be an opial space, if for every sequence $\{x_n\}$ in X weakly convergent to $x \in X$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\|_q < \liminf_{n\to\infty} \|x_n - y\|_q$$

holds for all $y \neq x$.

We give the definition providing the notion of (S)-convex structure introduced by Dotson [4].

Definition 2.7. A family of maps $\{f_\alpha\}_{\alpha\in X}$ is said to be a (S)-convex structure on q-normed space X, if it satisfies the following conditions:

(i) $f_\alpha : [0, 1] \mapsto X$, i.e. f_α is a map from $[0, 1]$ into X for each $\alpha \in X$,
(ii) $f_\alpha(1) = \alpha$ for each $\alpha \in X$,
(iii) $f_\alpha(t)$ is a jointly continuous in (α, t), i.e., $f_\alpha(t) \mapsto f_{\alpha_0}(t_0)$ for $\alpha \mapsto \alpha_0$ in X and $t \mapsto t_0$ in $[0, 1]$,
(iv) if f is a map from X into itself, then for any $x \in X$, $f_{Tx}(t) \subseteq Tx$ for all $t \in [0, 1]$,
(v) $\|f_\alpha(t) - f_\beta(t)\|_q \leq [\phi(t)]^q \|\alpha - \beta\|_q$, where ϕ is a function from $[0, 1]$ into itself.

Now, we give the definition "Property (A)" for (S)-convex structure.

Definition 2.8. A self mapping T of X is said to satisfy the Property (A), if for any $t \in [0,1]$, for all $x \in X$ and for all f_x, we have $T(f_x(t)) = f_{Tx}(t)$, where $\{f_x(t)\}$ is defined as above.

Throughout, this paper $F(T)$ denotes the fixed point set of mapping T.
We also use the following result:

Theorem 2.9 ([19]). *Let C be a closed subset of a metric space X and let I and T be self maps of C with $T(C) \subset I(C)$. If $\mathrm{cl}(T(C))$ (closure of T) is complete, I is continuous, I and T are commuting and T is I contraction. Then I and T have a unique common fixed point.*

3. Main result

First, we prove our main result for closed subset of this paper.

Theorem 3.1. *Let X be a q-normed space with a (S)-convex structure. Let $T, I : X \mapsto X$ and $C \subseteq X$ such that $T(\partial C) \subseteq C$. Let $x_0 \in F(T) \cap F(I)$. Suppose T is I-nonexpansive on $D' = D \cup \{x_0\}$, I satisfies Property (A), I is continuous, $TI = IT$ on D, $\mathrm{cl}(T(D))$ (closure of T) is compact on D. Also assume, range of f_α is contained in $I(D)$. If D is nonempty, closed and if $I(D) \subseteq D$, then*

$$D \cap F(T) \cap F(I) \neq \phi\,.$$

Proof. First, we show that T is a self map on D, i.e., $T : D \mapsto D$. Let $y \in D$, then $Iy \in D$, since $I(D) \subseteq D$. Also, by Lemma 2.3 [9] $y \in \partial C$. Also $Ty \in C$, since $T(\partial C) \subseteq C$. Now since $Tx_0 = x_0$ and T is I-nonexpansive map, we have

$$\|Ty - x_0\|_q = \|Ty - Tx_0\|_q \leq \|Iy - Ix_0\|_q\,.$$

As $Ix_0 = x_0$, we therefore have

$$\|Ty - Tx_0\|_q \leq \|Iy - x_0\|_q = d_q(x_0, C)\,,$$

since $Iy \in D$. This implies that Ty is also closest to x_0, so $Ty \in D$. Choose $k_n \in (0,1)$ such that $\{k_n\} \to 1$. Then define T_n as

$$T_n(x) = f_{Tx}(k_n) \quad \text{for all} \quad x \in D\,.$$

T_n is a well-defined map from D into D for each n. Also, since range of f_α is contained in $I(D)$, it is easy to see that $T_n(D) \subseteq I(D)$. Since T commutes with I and I satisfies Property (A), for each $x \in D$, we have

$$T_n(Ix) = f_{T(Ix)}(k_n) = f_{I(Tx)}(k_n) = I(f_{Tx}(k_n)) = IT_n(x)\,.$$

Thus, $T_nI = IT_n$ for all $n \in N$ and for all $x \in D$. Also, for each n and for all $x, y \in D$, we have

$$\begin{aligned}\|T_n(x) - T_n(y)\|_q &= \|f_{Tx}(k_n) - f_{Ty}(k_n)\|_q \\ &\leq [\phi(t)]^q \|Tx - Ty\|_q \\ &\leq [\phi(t)]^q \|Ix - Iy\|_q\,,\end{aligned}$$

i.e.,

$$\|T_n(x) - T_n(y)\|_q \leq [\phi(t)]^q \|Ix - Iy\|_q$$

for all $x, y \in D$. Therefore each T_n is I-contraction. Since $\mathrm{cl}(T(D))$ is compact, each $\mathrm{cl}(T_n(D))$ is compact. It follows from continuity of I and by the Theorem 2.9,

$$x_n = T_n x_n = I x_n \quad \text{for all} \quad n \in N.$$

As $\mathrm{cl}(T(D))$ is compact and $\{Tx_n\}$ is sequence in it, so $\{Tx_n\}$ has a subsequence $\{Tx_m\}$ converging, e.g., to $y \in cl(T(D))$.

$$x_m = T_m x_m = f_{Tx_m}(k_m)$$

converges to y. By the continuity of T, $\{Tx_m\}$ converges to Ty. But Tx_m tends to y by the assumption,

$$T_m x_m = f_{Tx_m}(k_m) \to f_{Ty}(1) = Ty\,, \quad \text{as} \quad m \mapsto \infty\,.$$

Thus,

$$Ty = y\,.$$

Also from the continuity of I, we have

$$Iy = I(\lim x_m) = \lim Ix_m = \lim x_m = y\,, \quad \text{as} \quad m \mapsto \infty\,,$$

i.e., $Iy = y$. Hence

$$D \cap F(T) \cap F(I) \neq \phi\,.$$

This completes the proof. □

To proof Theorem 3.3 in which we consider weakly compact subset, we use following result:

Theorem 3.2 ([6]). *Let (X, d) be a compact metric space and $T, I : X \to X$ be two commuting mappings such that $T(X) \subseteq I(X)$, I is continuous, and $d(Tx, Ty) < (Ix, Iy)$, whenever $Ix \neq Iy$. Then $F(T) \cap F(I)$ is singleton.*

Next result we prove for weakly compact subset as below:

Theorem 3.3. *Let X be a complete q-normed space whose dual separates the points of X with a (S)-convex structure. Let $T, I : X \mapsto X$ and $C \subseteq X$ such that $T(\partial C) \subseteq C$. Let $x_0 \in F(T) \cap F(I)$. Suppose T is I-nonexpansive on $D' = D \cup \{x_0\}$, I satisfies Property (A), I is weakly continuous, $TI = IT$ on D. Also assume, range of f_α is contained in $I(D)$. If D is nonempty, weakly compact and if $I(D) \subseteq D$, then $D \cap F(T) \cap F(I) \neq \phi$, provided $I - T$ is demiclosed.*

Proof. First, we show that T is a self map on D, i.e., $T : D \mapsto D$. Let $y \in D$, then $Iy \in D$, since $I(D) \subseteq D$. Also, by Lemma 2.3 [9] $y \in \partial C$. Also $Ty \in C$, since $T(\partial C) \subseteq C$. Now since $Tx_0 = x_0$ and T is I-nonexpansive map, we have

$$\|Ty - x_0\|_q = \|Ty - Tx_0\|_q \leq \|Iy - Ix_0\|_q\,.$$

As $Ix_0 = x_0$, we therefore have

$$\|Ty - Tx_0\|_q \leq \|Iy - x_0\|_q = d_q(x_0, C)\,,$$

since $Iy \in D$. This implies that Ty is also closest to x_0, so $Ty \in D$. Choose $k_n \in (0,1)$ such that $\{k_n\} \to 1$. Then define T_n as

$$T_n(x) = f_{Tx}(k_n) \quad \text{for all} \quad x \in D\,.$$

T_n is a well-defined map from D into D for each n. Also, since the range of f_α is contained in $I(D)$, it is easy to see that $T_n(D) \subseteq I(D)$. Since T commutes with I and I satisfies Property (A), for each $x \in D$, we have

$$T_n(Ix) = f_{T(Ix)}(k_n) = f_{I(Tx)}(k_n) = I(f_{Tx}(k_n)) = IT_n(x)\,.$$

Thus, $T_nI = IT_n$, for all $n \in N$ and for all $x \in D$. Also, for each n and for all $x, y \in D$, we have

$$\begin{aligned}\|T_n(x) - T_n(y)\|_q &= \|f_{Tx}(k_n) - f_{Ty}(k_n)\|_q \\ &\leq [\phi(t)]^q \|Tx - Ty\|_q \\ &\leq [\phi(t)]^q \|Ix - Iy\|_q\,,\end{aligned}$$

i.e.,

$$\|T_n(x) - T_n(y)\|_q \leq [\phi(t)]^q \|Ix - Iy\|_q\,,$$

for all $x, y \in D$. Therefore, it follows from continuity of I and by the Theorem 3.2,

$$x_n = T_n x_n = I x_n \quad \text{for all} \quad n \in N\,.$$

Also, since D is weakly compact, there exists a subsequence of $\{x_n\}$ in D, denoted by $\{x_m\}$, converging weakly to a point, say, $y \in D$. From the weakly continuity of I, we have

$$Iy = I(\lim x_m) = \lim I x_m = \lim x_m = y\,, \quad \text{as} \quad m \mapsto \infty\,,$$

i.e., $Iy = y$. Let

$$y_m = x_m - Tx_m = T_m x_m - Tx_m = f_{Tx_m}(k_m) - Tx_m\,,$$

we have

$$y_m = x_m - Tx_m = f_{Ty}(1) - Ty = Ty - Ty = 0\,. \tag{3.1}$$

Now, $I - T$ is demiclosed at 0 and the sequence $\{x_m\}$ converges weakly to y. Also, from 3.1, $y_m \to 0$ where $y_m = x_m - Tx_m$. Thus, $0 = (I - T)y$ implies that $y = Ty$. Hence y is fixed point of T in D. Hence

$$D \cap F(T) \cap F(I) \neq \phi\,.$$

This completes the proof. □

Remark 3.4. If we consider, $I =$ identity mapping and $q = 1$, then Theorem 3.1 is a special case of Theorem 2 of Mukherjee and Som [11].

Remark 3.5. Theorem 3.3 is improvement and extension of Mukherjee and Som [11] to q-normed space for weak topology.

Remark 3.6. Theorem 3.1 and Theorem 3.3 are improvement and extension of Jungck and Sessa [7] to q-normed space without starshapedness condition of subset and affineness of mapping.

Remark 3.7. Theorem 3.1 and Theorem 3.3 are extension of Nashine [12] to q-normed space without starshapedness condition of subset.

Remark 3.8. Theorem 3.3 extends Theorem 2.4 of Latif [9].

Remark 3.9. Theorem 3.1 and Theorem 3.3 are extension and application of Dotson [4] to q-normed space.

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