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THE NATURAL AFFINORS ON SOME FIBER PRODUCT PRESERVING GAUGE BUNDLE FUNCTORS OF VECTOR BUNDLES

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Dedicated to Professor Ivan Kolář on the occasion of his 70th bithday with respect and gratitude

Abstract. We classify all natural affinors on vertical fiber product preserving gauge bundle functors F on vector bundles. We explain this result for some more known such F. We present some applications. We remark a similar classification of all natural affinors on the gauge bundle functor F^* dual to F as above. We study also a similar problem for some (not all) not vertical fiber product preserving gauge bundle functors on vector bundles.

Introduction

Let m, n be fixed positive integers.

The category of vector bundles with m-dimensional bases and vector bundle maps with embeddings as base maps will be denoted by $\mathcal{VB}_m$.

The category of vector bundles with m-dimensional bases and n-dimensional fibers and vector bundle embeddings will be denoted by $\mathcal{VB}_{m,n}$.

Let $F : \mathcal{VB}_m \to \mathcal{FM}$ be a covariant functor. Let $B_{\mathcal{FM}} : \mathcal{FM} \to \mathcal{M}f$ and $B_{\mathcal{VB}_m} : \mathcal{VB}_m \to \mathcal{M}f$ be the base functors.

A gauge bundle functor on $\mathcal{VB}_m$ is a functor F as above satisfying:

(i) (*Base preservation*) $B_{\mathcal{FM}} \circ F = B_{\mathcal{VB}_m}$. Hence the induced projections form a functor transformation $\pi : F \to B_{\mathcal{VB}_m}$.

(ii) (*Localization*) For every inclusion of an open vector subbundle $i_{E|U} : E|U \to E$, $F(E|U)$ is the restriction $\pi^{-1}(U)$ of $\pi : FE \to B_{\mathcal{VB}_m}(E)$ to U and $Fi_{E|U}$ is the inclusion $\pi^{-1}(U) \to FE$.

(iii) (*Regularity*) F transforms smoothly parametrized systems of $\mathcal{VB}_m$-morphisms into smoothly parametrized families of $\mathcal{FM}$-morphisms.

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A gauge bundle functor $F : \mathcal{VB}_m \to \mathcal{FM}$ is of finite order r if from $j^r_x f = j^r_x g$ it follows $F_x f = F_x g$ for any $\mathcal{VB}_m$-objects $E_1 \to M$, $E_2 \to M$, any $\mathcal{VB}_m$-maps $f, g : E_1 \to E_2$ and any $x \in M_1$.

A gauge bundle functor F on $\mathcal{VB}_m$ is fiber product preserving if for any fiber product projections

$$E_1 \xleftarrow{\mathrm{pr}_1} E_1 \times_M E_2 \xrightarrow{\mathrm{pr}_2} E_2$$

in the category $\mathcal{VB}_m$,

$$FE_1 \xleftarrow{F\,\mathrm{pr}_1} F(E_1 \times_M E_2) \xrightarrow{F\,\mathrm{pr}_2} FE_2$$

are fiber product projections in the category $\mathcal{FM}$. In other words we have $F(E_1 \times_M E_2) = F(E_1) \times_M F(E_2)$.

A gauge bundle functor F on $\mathcal{VB}_m$ is called vertical if for any $\mathcal{VB}_m$-objects $E \to M$ and $E_1 \to M$ with the same basis, any $x \in M$ and any $\mathcal{VB}_m$-map $f : E \to E_1$ covering the identity of M the fiber restriction $F_x f : F_x E \to F_x E_1$ depends only on $f_x : E_x \to (E_1)_x$.

From now on we are interested in vertical fiber product preserving gauge bundle functors on $\mathcal{VB}_m$.

The most known example of vertical fiber product preserving gauge bundle functor F on $\mathcal{VB}_m$ is the so-called vertical r-jet prolongation functor $J^r_v : \mathcal{VB}_m \to \mathcal{FM}$, where for a $\mathcal{VB}_m$-object $p : E \to M$ we have a vector bundle $J^r_v E = \{j^r_x \gamma \mid \gamma$ is a local map $M \to E_x$, $x \in M\}$ and for a $\mathcal{VB}_m$-map $f : E_1 \to E_2$ covering $\underline{f} : M_1 \to M_2$ we have a vector bundle map $J^r_v f : J^r_v E_1 \to J^r_v E_2$, where $J^r_v f(j^r_x \gamma) = j^r_{\underline{f}(x)}(f \circ \gamma \circ \underline{f}^{-1})$ for $j^r_x \gamma \in J^r_v E_1$.

Another example is the vertical Weil functor V^A on $\mathcal{VB}_m$ corresponding to a Weil algebra A, where for a $\mathcal{VB}_m$-object $p : E \to M$ we have $V^A E = \cup_{x \in M} T^A(E_x)$ and for a $V\mathcal{B}_m$-map $f : E_1 \to E_2$ we have $V^A f = \cup_{x \in M_1} T^A(f_x) : V^A E_1 \to V^A E_2$. The functor V^A is equivalent to $E \otimes A$.

The fiber product $F_1 \times_{B_{\mathcal{VB}_m}} F_2 : \mathcal{VB}_m \to \mathcal{FM}$ of vertical fiber product preserving gauge bundle functors $F_1, F_2 : \mathcal{VB}_m \to \mathcal{FM}$ is again a vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$.

In [8], we proved that every fiber product preserving gauge bundle functor F on $\mathcal{VB}_m$ has values in $\mathcal{VB}_m$. (More precisely, the fiber sum map $+ : E \times_M E \to E$, the fiber scalar multiplication $\lambda_t : E \to E$ for $t \in \mathbf{R}$ and the zero map $0 : E \to E$ are $\mathcal{VB}_m$-map and we can apply F. We obtain $F(+) : FE \times_M FE \to FE$, $F(\lambda_t) : FE \to FE$ and $F(0) : FE \to FE$. Then $\big(F(+), F(\lambda_t), F0\big)$ is a vector bundle structure on FE.) Then we can compose such functors. The composition of vertical fiber product preserving gauge bundle functors on $\mathcal{VB}_m$ is again a vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$.

If F is a vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$, then $(F^*)^* : \mathcal{VB}_m \to \mathcal{FM}$, $(F^*)^*(E) = (FE^*)^*$, $(F^*)^*(f) = (Ff^*)^*$ is a vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$ (E^* denote the dual vector bundle of E).

In [8], we classified all fiber product preserving gauge bundle functors F on $\mathcal{VB}_m$ of finite order r in terms of triples (V, H, t), where V is a finite-dimensional vector space over $\mathbf{R}$, $H : G^r_m \to \mathrm{GL}(V)$ is a smooth group homomorphism from $G^r_m = inv J^r_0(\mathbf{R}^m, \mathbf{R}^m)_0$ into $\mathrm{GL}(V)$ and $t : \mathcal{D}^r_m \to \mathrm{gl}(V)$ is a G^r_m-equivariant unity preserving associative algebra homomorphism from $\mathcal{D}^r_m = J^r_0(\mathbf{R}^m, \mathbf{R})$ into $\mathrm{gl}(V)$. Moreover, we proved that all fiber product preserving gauge bundle functors F on $\mathcal{VB}_m$ are of finite order. Analyzing the construction on (V, H, t) one can easily seen that the triple (V, H, t) corresponding to a vertical F in question has trivial $t : \mathcal{D}^r_m \to \mathrm{gl}(V)$, $t(j^r_x \gamma) = \gamma(0)\,\mathrm{id}$, $j^r_0\gamma \in \mathcal{D}^r_m$. Then by Fact 5 and Theorem 2 in [8] it follows that all vertical fiber product preserving gauge bundle functors on $\mathcal{VB}_m$ can be constructed (up to $\mathcal{VB}_m$-equivalence) as follows.

Let $V : \mathcal{M}f_m \to \mathcal{VB}$ be a vector natural bundle. For any $\mathcal{VB}_m$-object $p : E \to M$ we put $F^V E = E \otimes_M VM$ and for any $\mathcal{VB}_m$-map $f : E_1 \to E_2$ covering $\underline{f} : M_1 \to E_2$ we put $F^V f = f \otimes_{\underline{f}} V\underline{f} : F^V E_1 \to F^V E_2$. The correspondence $F^V : \mathcal{VB}_m \to \mathcal{FM}$ is a vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$. (For example, if $V : \mathcal{M}f_m \to \mathcal{VB}$ is the natural vector bundle corresponding to the standard G^r_m-space $\mathcal{D}^r_m$, then F^V is equivalent with J^r_v. If $V : \mathcal{M}f_m \to \mathcal{VB}$ is the trivial vector natural bundle with the standard fiber A, then F^V is equivalent to V^A.)

Let F be a gauge bundle functor on $\mathcal{VB}_m$. A $\mathcal{VB}_{m,n}$-natural affinor B on F is a system of $\mathcal{VB}_{m,n}$-invariant affinors $B : TFE \to TFE$ on FE for any $\mathcal{VB}_{m,n}$-object E. The invariance means that $B \circ TFf = TFf \circ B$ for any $\mathcal{VB}_{m,n}$-map f.

In the present paper we describe all $\mathcal{VB}_{m,n}$-natural affinors B on vertical fiber product preserving gauge bundle functors F on $\mathcal{VB}_m$. We prove that $B : TFE \to TFE$ is of the form

$$B = \lambda\,\mathrm{Id} + \mathrm{Mod}(A)$$

for a real number λ and a fiber bilinear $\mathcal{VB}_{m,n}$-natural transformation $A : TM \times_M FE \to FE$, where $\mathrm{Mod}(A)$ is the $\mathcal{VB}_{m,n}$-natural affinor corresponding to A (see Example 2) and Id is the identity affinor.

In Section 3, we explain this main result for some more known vertical fiber product preserving gauge bundle functors F on $\mathcal{VB}_m$. Thus for J^r_v we reobtain the result from [15] saying that the vector space of all $\mathcal{VB}_{m,n}$-natural affinors on J^r_v is 2-dimensional.

In Section 4, we remark a similar classification of $\mathcal{VB}_{m,n}$-natural affinors on a gauge bundle functor F^* dual to a vertical fiber product preserving gauge bundle functor F on $\mathcal{VB}_m$.

In Section 5, we remark a similar classification of $\mathcal{VB}_{m,n}$-natural affinors for some (not all) not vertical fiber product preserving gauge bundle functors F on $\mathcal{VB}_m$ (as the r-jet prolongation gauge bundle functor J^r on $\mathcal{VB}_m$ and the vector r-tangent gauge bundle functor $T^{(r)\,\mathrm{fl}}$ on $\mathcal{VB}_m$). Thus a similar result as the main result for not necessarily vertical F is very very probably.

Natural affinors can be used to study torsions of connections, see [5]. That is why they have been classified in many papers, [1] – [4], [6], [8] – [16], e.t.c.

The trivial vector bundle $\mathbf{R}^m \times \mathbf{R}^n$ over $\mathbf{R}^m$ with standard fiber $\mathbf{R}^n$ will be denoted by $\mathbf{R}^{m,n}$. The coordinates on $\mathbf{R}^m$ will be denoted by $x^1, \dots, x^m$. The fiber coordinates on $\mathbf{R}^{m,n}$ will be denoted by $y^1, \dots, y^n$.

All manifolds are assumed to be finite dimensional and smooth. Maps are assumed to be smooth, i.e. of class $\mathcal{C}^\infty$.

1. The main result

Let F be a fiber product preserving gauge bundle functor on $\mathcal{VB}_m$. We are going to present examples of $\mathcal{VB}_{m,n}$-natural affinors on F.

Example 1 (*The identity affinor***).** For any $\mathcal{VB}_{m,n}$-object E we have the identity map $\mathrm{Id} : TFE \to TFE$. The family Id is a $\mathcal{VB}_{m,n}$-natural affinor on FE.

Example 2. Suppose we have a family A of fiber bilinear maps $A : TM \times FE \to FE$ covering the identity of M for any $\mathcal{VB}_{m,n}$-object $E \to M$ such that $Ff \circ A = A \circ (T\underline{f} \times_{\underline{f}} Ff)$ for any $\mathcal{VB}_{m,n}$-map $f : E_1 \to E_2$ covering $\underline{f} : M_1 \to M_2$, i.e. we have a fiber bilinear $\mathcal{VB}_{m,n}$-natural transformation $A : TM \times_M FE \to FE$, where TM is the tangent bundle of M and FE is the vector bundle as is explained in Introduction. For any $\mathcal{VB}_{m,n}$-object $p : E \to M$ we define $\mathrm{Mod}(A) : TFE \to TFE$ by

$$\mathrm{Mod}(A)(v) = \frac{d}{dt}_0 (y + tA(T\pi(v), y)) \in T_y FY\,, \quad v \in T_y FE\,,\ y \in FE\,,$$

where $\pi : FE \to M$ is the bundle projection. Then $\mathrm{Mod}(A)$ is a $\mathcal{VB}_{m,n}$-natural affinor on F. We call $\mathrm{Mod}(A)$ the $\mathcal{VB}_{m,n}$-natural affinor on F corresponding to A (the modification of A).

For example, in the case of $F = J^r_v$ we have a fiber bilinear $\mathcal{VB}_{m,n}$-natural transformation $A^r_v : TM \times J^r_v E \to J^r_v E$, $A^r_v(w, j^r_x\sigma) = j^r_x(w\sigma)$, $w \in T_xM$, $x \in M$, $\sigma : M \to E_x$, $w\sigma \in E_x$ is the differential of σ with respect to w and $w\sigma : M \to E_x$ is the constant map.

The main result of the present paper is the following classification theorem.

Theorem 1. *Let F be a vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$. Any $\mathcal{VB}_{m,n}$-natural affinor B on F is the form*

$$B = \lambda\,\mathrm{Id} + \mathrm{Mod}(A)$$

for some real number λ and some fiber bilinear $\mathcal{VB}_{m,n}$-natural transformation $A : TM \times_M FE \to FE$.

Thus for $F = J^r_v$ we reobtain the result from [15] saying that any $\mathcal{VB}_{m,n}$-natural affinor on J^r_v is a linear combination with real coefficients of the identity affinor and $\mathrm{Mod}(A^r_v)$ (see Corollary 5 bellow).

We end this section by the following observation.

Let F be of the form F^V for some natural vector bundle $V : \mathcal{M}f_m \to \mathcal{VB}$ (see Introduction). Let $C : TM \times_M VM \to VM$ be an $\mathcal{M}f_m$-natural fiber bilinear transformation. Then we have a $\mathcal{VB}_{m,n}$-natural fiber bilinear transformation $A^C : TM \times_M F^V E \to F^V E$,

$$A^C(v, e \otimes y) = e \otimes C(v, y) \ ,$$

$y \in V_x M$, $e \in E_x$, $v \in T_x M$, $x \in M$.

Proposition 1. *Let $V : \mathcal{M}f_m \to \mathcal{VB}$ be a natural vector bundle. Any $\mathcal{VB}_{m,n}$-natural fiber bilinear transformation $A : TM \times_M F^V E \to FV^E$ is of the form A^C for some $\mathcal{M}f_m$-natural fiber bilinear transformation $C : TM \times_M VM \to VM$.*

Proof of Proposition 1. By the $\mathcal{VB}_{m,n}$-invariance, A is determined by the $\mathcal{M}f_m$-natural fiber bilinear transformation

$$TM \times_M VM \ni (v, y) \to \big\langle A\big(v, e_1(\pi^T(v)\big) \otimes y), e_1^*\big(\pi^T(v)\big)\big\rangle \in VM \,,$$

where $e_1, \dots, e_n$ is the usual basis of sections of the trivial vector bundle $M \times \mathbf{R}^n$ and $e_1^*, \dots, e_n^*$ is the dual basis, and $\pi^T : TM \to M$ is the tangent bundle projection.

2. Proof of Theorem 1

We fix a basis in the vector space $F_0\mathbf{R}^{m,n}$.

Step 1. Consider

$$T\pi \circ B : (TF\mathbf{R}^{m,n})_0 \tilde{=} \mathbf{R}^m \times F_0\mathbf{R}^{m,n} \times F_0\mathbf{R}^{m,n} \to T_0\mathbf{R}^m \,,$$

where $\pi : FE \to M$ is the bundle projection. Using the invariance of B with respect to the fiber homotheties we deduce that $T\pi \circ B(a, u, v) = T\pi \circ B(a, tu, tv)$ for any $u, v \in F_0\mathbf{R}^{m,n}$, $a \in \mathbf{R}^m$, $t \neq 0$. Then $T\pi \circ B(a, u, v) = T\pi \circ B(a, 0, 0)$ for u, v, a as above. Then using the invariance of B with respect to $C \times \mathrm{id}_{\mathbf{R}^n}$ for linear isomorphisms C of $\mathbf{R}^n$ we deduce that $T\pi \circ B(a, 0, 0) = \lambda a$ for some real number λ. Then replacing B by $B - \lambda\,\mathrm{Id}$ we have $T\pi \circ B(a, u, v) = 0$ for any a, u, v as above. Then B is of vertical type.

Step 2. Consider

$$\mathrm{pr}_2 \circ B : (TF\mathbf{R}^{m,n})_0 \tilde{=} \mathbf{R}^m \times F_0\mathbf{R}^{m,n} \times F_0\mathbf{R}^{m,n} \to F_0\mathbf{R}^{m,n} \,,$$

where $(VF\mathbf{R}^{m,n})_0 \cong F_0\mathbf{R}^{m,n} \times F_0\mathbf{R}^{m,n} \to F_0\mathbf{R}^{m,n}$ is the projection onto the second (essential) factor. Using the invariance of B with respect to the fiber homotheties we deduce that $\mathrm{pr}_2 \circ B(a, tu, tv) = t\,\mathrm{pr}_2 \circ B(a, u, v)$ for a, u, v as in Step 1. Then $\mathrm{pr}_2 \circ B(a, u, v)$ is a system of linear combinations of the coefficients of u and v with coefficients being smooth maps in a because of the homogeneous function theorem. On the other hand, since B is an affinor, $\mathrm{pr}_2 \circ B(a, u, v)$ is a system of linear combinations of the coefficients of a and v with coefficients being smooth functions in u. Then

$$(*) \qquad \mathrm{pr}_2 \circ B(a,u,v) = G(a,u) + H(v)$$

for some bilinear map G and some linear map H.

Let $\Phi : \mathbf{R}^{m,n} \to \mathbf{R}^{m,n}$ be a $\mathcal{VB}_{m,n}$-map such that $\Phi(x,v) = (x, e^{x^1} v)$, $(x,y) \in \mathbf{R}^{m,n}$. Then Φ sends $\frac{\partial}{\partial x^1}$ into $\frac{\partial}{\partial x^1} + L$, where L is the Liouville vector field on $\mathbf{R}^{m,n}$. Then using the invariance of B with respect to Φ we obtain

$$F\Phi(G(e_1, F\Phi^{-1}(v))) = G(e_1, v) + H(v)\,,$$

where $e_1 = (1,0,\dots,0) \in \mathbf{R}^m$. Since F is vertical, $F_0\Phi = \mathrm{id}$. Hence $H(v) = 0$, and

$$pr_2 \circ B(a,u,v) = G(a,u)\,.$$

Then by the $\mathcal{VB}_{m,n}$-invariance of B we obtain the equivariant condition

$$F_0 f(G(a,u)) = G(T_0\underline{f}(a), F_0 f(u))$$

for any a,u as above and any $\mathcal{VB}_{m,n}$-map $f : \mathbf{R}^{m,n} \to \mathbf{R}^{m,n}$ preserving $0 \in \mathbf{R}^m$. Hence there is a $\mathcal{VB}_{m,n}$-natural fiber bilinear transformation $A : TM \times_M FE \to FE$ corresponding to G. It is easy to see that $B = \mathrm{Mod}(A)$. $\square$

3. Applications

Let $T^{(p,q)} = \otimes^q T^* \otimes \otimes^p T : \mathcal{M}f_m \to \mathcal{VM}$ be the natural vector bundle of tensor fields of type (p,q) over m-manifolds. Let $F^{(p,r)} = F^{T^{(p,r)}} : \mathcal{VB}_m \to \mathcal{FM}$, $F^{(p,r)}E = E \otimes_M T^{(p,r)}M$, $F^{(p,q)}f = f \otimes_{\underline{f}} T^{(p,q)}\underline{f}$ be the corresponding vertical fiber product preserving gauge bundle functor (see Introduction).

Suppose that $C : TM \times_M T^{(p,r)}M \to T^{(p,q)}M$ is a fiber bilinear $\mathcal{M}f_m$-natural transformation. Using the invariance of C with respect to base homotheties on $\mathbf{R}^{m,n}$ one can easily deduce that $C = 0$. Thus we have the following corollary

Corollary 1. *Any $\mathcal{VB}_{m,n}$-natural affinor on $F^{(p,q)}$ as above is a constant multiple of the identity affinor.*

Similarly, any $\mathcal{M}f_m$-natural fiber bilinear transformation $C : TM \times_M M \to M$, where M is treated as the zero vector bundle over M, is zero. Thus we have

Corollary 2. *Any $V\mathcal{B}_{m,n}$-natural affinor on the vertical Weil bundle V^A is a constant multiple of the identity affinor.*

Let $T^{(r)} = (J^r(\cdot, \mathbf{R})_0)^* : \mathcal{M}f_m \to \mathcal{VB}$ be the linear r-tangent bundle functor. Let $F^{(r)} = F^{T^{(r)}} : \mathcal{VB}_m \to \mathcal{FM}$ be the corresponding vertical fiber product preserving gauge bundle functor.

Suppose that $C : TM \times_M T^{(r)}M \to T^{(r)}M$ is a $\mathcal{M}f_m$-natural fiber bilinear transformation. By the rank theorem, C is determined by the contraction $\langle C, j^r_0 x^1\rangle : T^{(r)}_0\mathbf{R}^m \to \mathbf{R}$. Then using the invariance of C with respect to the base homotheties one can easily show that this contraction is zero. Then $C = 0$. Thus we have

Corollary 3. *Any $\mathcal{VB}_{m,n}$-natural affinor on $F^{(r)}$ as above is a constant multiple of the identity one.*

Let $T^{r*} = J^r(\cdot, \mathbf{R})_0 : \mathcal{M}f_m \to \mathcal{VB}$ be the r-cotangent bundle functor. Let $F^{r*} = F^{T^{r*}} : \mathcal{VB}_m \to \mathcal{FM}$ be the corresponding vertical fiber product preserving gauge bundle functor.

Suppose that $C : TM \times_M T^{r*}M \to T^{r*}M$ is a $\mathcal{M}f_m$-natural fiber bilinear transformation. By the rank theorem, C is determined by the evaluations $C(v, j^r_0 x^1) \in T^{r*}_0\mathbf{R}^m$, where $v \in T_0\mathbf{R}^m$. Then using the invariance of C with respect to the base homotheties one can easily show that these evaluations are zero. Then $C = 0$. Thus we have

Corollary 4. *Any $\mathcal{VB}_{m,n}$-natural affinor on F^{r*} as above is a constant multiple of the identity one.*

Let $E^{r*} = J^r(\cdot, \mathbf{R}) : \mathcal{M}f_m \to \mathcal{VB}$ be the extended r-cotangent bundle functor. As we know the corresponding vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$ is equivalent to the vertical r-jet functor J^r_v (see Introduction).

Suppose that $C : TM \times_M E^{r*}M \to E^{r*}M$ is a $\mathcal{M}f_m$-natural fiber bilinear transformation. By the rank theorem, C is determined by the evaluations $C(\frac{\partial}{\partial x^1}{}_0, j^r_0 1) \in E^{r*}_0\mathbf{R}^m$ and $C(\frac{\partial}{\partial x^1}{}_0, j^r_0 x^1) \in E^{r*}_0\mathbf{R}^m$. Then using the invariance of C with respect to the base homotheties one can easily show that the second evaluation is a constant multiple of $j^r_0 1$ and the first one is zero. Then the vector space of all C in question is of dimension less or equal to 1. Thus we reobtain

Corollary 5 ([15])**.** *Any $\mathcal{VB}_{m,n}$-natural affinor on J^r_v is a linear combination with real coefficients of the identity affinor and the affinor* $\mathrm{Mod}(A^r_v)$.

Corollary 6. *Let F be a vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$. Any $\mathcal{VB}_{m,n}$-natural 1-form ω on F is zero.*

Proof. Let L be the Liouville vector field on the vector bundle FE. Then $\omega \otimes L$ is a $\mathcal{VB}_{m,n}$-natural affinor. Since it is not isomorphic, it is of the form $\omega \otimes L = \mathrm{Mod}(A)$ for some bilinear $\mathcal{VB}_{m,n}$-natural transformation $A : TM \times_M FE \to FE$. Then A is of the form $A(v, y) = \lambda(v)y$ for some uniquely (and then $\mathcal{M}f_m$-natural) 1-form $\lambda : TM \to \mathbf{R}$ on M. But any such 1-form is zero. Then $A = 0$. Then $\omega = 0$.

Corollary 7. *Let F be a vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$. There is no $\mathcal{VB}_{m,n}$-natural symplectic structure ω on F.*

Proof. Suppose that such ω exists. Then $\omega(L, \cdot)$ is a $\mathcal{VB}_{m,n}$-natural 1-form on F. Then $\omega(L, \cdot) = 0$ because of Corollary 6. Then ω is degenerate. Contradiction.

Quite similarly one can prove

Corollary 8. *Let F be a vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$. Then there is no $\mathcal{VB}_{m,n}$-natural non-degenerate Riemannian tensor field g on F.*

4. A dual version of the main result

Let F be a vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$. Let F^* be the dual gauge bundle functor on $\mathcal{VB}_{m,n}$, $F^*E = (FE)^*$ and $F^*f = (Ff^{-1})^*$. Replacing in the proof of Theorem 1 F by F^* we obtain

Theorem 1'. *Let F be a vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$. Let F^* be the dual gauge bundle functor. Any $\mathcal{VB}_{m,n}$-natural affinor B on F^* is of the form*

$$B = \lambda\,\mathrm{Id} + \mathrm{Mod}(A^*)$$

for some $\lambda \in \mathbf{R}$ and some $\mathcal{VB}_{m,n}$-natural fiber bilinear transformation $A : TM \times_M FM \to FM$, where $A^* : TM \times_M F^*E \to F^*E$ is the $\mathcal{VB}_{m,n}$-natural fiber bilinear transformation given by $A^*(v, \cdot) = \big(A(v, \cdot)\big)^*$ for any $v \in TM$.

5. The not necessarily vertical case

In our opinion, it is very probably that Theorem 1 holds for (not necessarily vertical) fiber product preserving gauge bundle functors on $\mathcal{VB}_m$. For example, in [15] we proved.

Fact 1 ([15]). *Any $\mathcal{VB}_{m,n}$-natural affinor on the r-jet prolongation functor J^r, which is a not vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$, is a constant multiple of the identity affinor.*

The crucial property of J^r which we used to prove Fact 1 is that any $\mathcal{VB}_{m,n}$-natural linear operator lifting linear vector fields from E to vector fields on J^rE is a constant multiple of the flow operator.

Replacing in [15] J^r be an arbitrary fiber product preserving gauge bundle functor F on $\mathcal{VB}_m$ we can obtain

Proposition 2. *Let F be a (not necessarily vertical) fiber product preserving gauge bundle functor on $\mathcal{VB}_m$ such that any $\mathcal{VB}_{m,n}$-natural linear operator lifting linear vector fields from E into vector fields on FE is a constant multiple of the flow operator $\mathcal{F}$. Then any $\mathcal{VB}_{m,n}$-natural affinor B on F is a constant multiple of the identity affinor.*

Proof. Clearly, $B \circ \mathcal{F}$ is a $\mathcal{VB}_{m,n}$-natural linear operator lifting linear vector fields to F. By the assumption, there is $\lambda \in \mathbf{R}$ such that $B \circ \mathcal{F} = \lambda\mathcal{F}$. Next we use the same proof as the one of Theorem 1 up to the formula (*). Obviously, after Step 1, B satisfies $B(\mathcal{F}X) = 0$ for any linear vector field on $\mathbf{R}^{m,n}$. Putting in (*) $X = a\frac{\partial}{\partial x^1}$ (i.e. $(a, u, v) = (a, u, 0)$) we get $G(a, u) = 0$. Putting $X = L$, the Liouville vector field on $\mathbf{R}^{m,n}$ (i.e. $(a, u, v) = (0, v, v)$) we get $H(v) = 0$.

In [7], we proved that the assumption of Proposition 1 is satisfied for the vector r-tangent gauge bundle functor $T^{(r)\,\mathrm{fl}}$ on $\mathcal{VB}_m$ defined as follows. Given a $\mathcal{VB}_m$-object $p : E \to M$, $T^{(r)\,\mathrm{fl}}E = (J^r_{\mathrm{fl}}(E, \mathbf{R})_0)^*$ is the vector bundle over M dual to $J^r_{\mathrm{fl}}E = \{j^r_x\gamma \mid \gamma : E \to \mathbf{R}$ is fiber linear, $\gamma_x = 0$, $x \in M\}$. For every $\mathcal{VB}_m$-map $f : E_1 \to E_2$ covering $\underline{f} : M_1 \to M_2$, $T^{(r)\,\mathrm{fl}}f : T^{(r)\,\mathrm{fl}}E_1 \to T^{(r)\,\mathrm{fl}}E_2$

is a vector bundle map covering $\underline{f}$ such that $\langle T^{(r)\,\mathrm{fl}} f(\omega), j^r_{\underline{f}(x)}\xi\rangle = \langle \omega, j^r_x(\xi\circ f)\rangle$, $\omega \in T^{(r)\,\mathrm{fl}}_x E_1$, $j^r_{\underline{f}(x)}\xi \in J^r_{\mathrm{fl}}(E_2,\mathbf{R})_0$, $x\in M$. (The correspondence $T^{(r)\,\mathrm{fl}}$ is a not vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$.) Thus we have

Fact 2. *Any $\mathcal{VB}_{m,n}$-natural affinor on $T^{(r)\,\mathrm{fl}}$ is a constant multiple of the identity affinor.*

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