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SPACES WITH σ-LOCALLY COUNTABLE WEAK-BASES

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Abstract. In this paper, spaces with σ-locally countable weak-bases are characterized as the weakly open msss-images of metric spaces (or g-first countable spaces with σ-locally countable cs-networks).

To find the internal characterizations of certain images of metric spaces is an interesting research topic on general topology. Recently, S. Xia[12] introduced the concept of weakly open mappings, by using it, certain g-first countable spaces are characterized as images of metric spaces under various weakly open mappings. The present paper establish the relationships spaces with σ-locally countable weak-bases and metric spaces by means of weakly pen mappings and msss-mappings, and give a characterization of spaces with σ-locally countable weak-bases.

In this paper, all spaces are regular and T_1, all mappings are continuous and surjective. N denotes the set of all natural numbers. ω denotes $N \cup \{0\}$. For a family $\mathcal{P}$ of subsets of a space X and a mapping $f : X \to Y$, denote $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$. For the usual product space $\prod_{i\in N} X_i, p_i$ denotes the projection from $\prod_{i\in N} X_i$ onto X_i.

Definition 1. Let $\mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}$ be a family of subsets of a space X satisfying that for each $x \in X$,

(1) $\mathcal{P}_x$ is a network of x in X,

(2) If $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

$\mathcal{P}$ is called a weak-base for X[1] if $G \subset X$ is open in X if and only if for each $x \in G$, there exists $P \in \mathcal{P}_x$ such that $P \subset G$.

A space X is called g-first countable[1] if X has a weak-base $\mathcal{P}$ such that each $\mathcal{P}_x$ is countable.

A space X is called a g-metrizable space[4] if X has a σ-locally finite weak-base.

Definition 2. Let $\mathcal{P}$ be a cover of a space X.

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(1) $\mathcal{P}$ is called a k-network for X if for each compact subset K of X and its open neighbourhood V, there exists a finite subfamily $\mathcal{P}'$ of $\mathcal{P}$ such that $K \subset \cup\mathcal{P}' \subset V$.

(2) $\mathcal{P}$ is called a cs-network for X if for each $x \in X$, its open neighbourhood V and a sequence $\{x_n\}$ converging to x, there exists $P \in \mathcal{P}$ such that $\{x_n : n \geq m\} \cup \{x\} \subset P \subset V$ for some $m \in N$.

A space X is called an $\aleph$-space if X has a σ-locally finite k-network.

Definition 3. Let $f : X \to Y$ be a mapping.

(1) f is called a weakly open mapping[12] if there exists a weak-base $\mathcal{B} = \cup\{\mathcal{B}_y : y \in Y\}$ for Y and for $y \in Y$, there exists $x(y) \in f^{-1}(y)$ satisfying condition $(*)$: for each open neighbourhood U of $x(y)$, $B_y \subset f(U)$ for some $B_y \in \mathcal{B}_y$.

(2) f is called a msss-mapping[7] (i.e., metrizably stratified strong s-mapping) if there exists a subspace X of the usual product space $\prod_{i\in N} X_i$ of the family $\{X_i : i \in N\}$ of metric spaces satisfying the following condition: for each $y \in Y$, there exists an open neighbourhood sequence $\{V_i\}$ of y in Y such that each $p_i f^{-1}(V_i)$ is separable in X_i.

Theorem 4. *A space Y has a σ-locally countable weak-base if and only if Y is the weakly open msss-image of a metric space.*

Proof. Sufficiency. Suppose Y is the image of a metric space X under a weakly open msss-mapping f. Since f is a msss-mapping, then exists a family $\{X_i : i \in N\}$ of metric spaces satisfying the condition of Definition 3 (2).

For each $i \in N$, let $\mathcal{P}_i$ be a σ-locally finite base for X_i, put

$$\mathcal{B}_i = \Big\{X \cap \Big(\bigcap_{j\leq i} p_j^{-1}(P_j)\Big) : P_j \in \mathcal{P}_j \text{ and } j \leq i\Big\}\,,$$

$$\mathcal{B} = \cup\{\mathcal{B}_i : i \in N\}\,.$$

Then $\mathcal{B}$ is a base for X. For each $n \in N$, put

$$V = \bigcap_{j\leq i} V_i\,,$$

then $\{Q \in f(\mathcal{B}_i) : V \cap Q \neq \Phi\}$ is countable. Thus $f(\mathcal{B}_i)$ is locally countable in Y. Hence $f(\mathcal{B})$ is σ-locally countable in Y.

Since f is a weakly open mapping, then exists a weak-base $\mathcal{P} = \cup\{\mathcal{P}_y : y \in Y\}$ for Y such that for each $y \in Y$, there exists $x(y) \in f^{-1}(y)$ satisfying the condition $(*)$ of Definition 3 (1). For each $y \in Y$, put

$$\mathcal{F}_{i,y} = \{f(B) : x(y) \in B \in \mathcal{B}_i\}\,,$$

$$\mathcal{F}_y = \cup\{\mathcal{F}_{i,y} : i \in N\}\,,$$

$$\mathcal{F}_i = \cup\{\mathcal{F}_{i,y} : y \in Y\}\,,$$

$$\mathcal{F} = \cup\{\mathcal{F}_y : y \in Y\}\,.$$

Obviously, $\mathcal{F}_i \in f(\mathcal{B}_i)$ for each $i \in N$, then $\mathcal{F}_i$ is locally countable in Y. Thus $\mathcal{F} = \cup\{\mathcal{F}_i : i \in N\}$ is σ-locally countable in Y. We will prove that $\mathcal{F}$ is a weak-base for Y.

It is obvious that $\mathcal{F}$ satisfies the condition (1) of Definition 1. For each $y \in Y$, suppose $U, V \in \mathcal{F}_y$, then $U \in \mathcal{F}_{m,y}, V \in \mathcal{F}_{n,y}$ for some $m, n \in N$. Thus there exist $B_1 \in \mathcal{B}_m$ and $B_2 \in \mathcal{B}_n$ such that $x(y) \in B_1 \cap B_2$, $f(B_1) = U$ and $f(B_2) = V$. Since $B_1, B_2 \in \mathcal{B}$ and $\mathcal{B}$ is a base for X, then there exist $l \in N$ and $B \in \mathcal{B}_l$ such that $x(y) \in B \subset B_1 \cap B_2$. Thus $f(B) \in \mathcal{F}_{l,y} \subset \mathcal{F}_y$ and $f(B) \subset f(B_1 \cap B_2) \subset U \cap V$. Hence $\mathcal{F}$ satisfies the condition (2) of Definition 1.

Suppose $G \subset Y$ and for $y \in G$, there exists $F \in \mathcal{F}_y$ such that $F \subset G$, then there exists $B \in \mathcal{B}$ such that $x(y) \in B$ and $F = f(B)$. Since B is an open neighbourhood of $x(y)$ and f is a weakly open mapping, then exists $P_y \in \mathcal{P}_y$ such that $P_y \subset f(B)$. Thus for each $y \in G$, there exists $P_y \in \mathcal{P}_y$ such that $P_y \subset G$. Hence G is open in Y because $\mathcal{P}$ is a weak-base for Y. On the other hard. Suppose $G \subset Y$ is open in Y, then for each $y \in G$, $x(y) \in f^{-1}(G)$. Since $\mathcal{B}$ is a base for X, then $x(y) \in B \subset f^{-1}(G)$ for some $B \in \mathcal{B}$. Thus $f(B) \in \mathcal{F}_y$ and $f(B) \subset G$.

Therefore $\mathcal{F}$ is a weak-base for Y.

Necessity. Suppose Y has a σ-locally countable weak-base. Let $\mathcal{P} = \cup\{\mathcal{P}_i : i \in N\}$ be a σ-locally countable weak-base for Y, where each $\mathcal{P}_i = \{P_\alpha : \alpha \in A_i\}$ is a locally countable of subsets of Y which is closed under finite intersections and $Y \in \mathcal{P}_i \subset \mathcal{P}_{i+1}$. For each $i \in N$, endow A_i with discrete topology, then A_i is a metric space. Put

$$X = \Big\{\alpha = (\alpha_i) \in \prod_{i \in N} A_i : \{P_{\alpha_i} : i \in N\} \subset P \text{ forms a network at some point } x(\alpha) \in X\Big\},$$

and endow X with the subspace topology induced from the usual product topology of the family $\{A_i : i \in N\}$ of metric spaces, then X is a metric space. Since Y is Hausdroff, $x(\alpha)$ is unique in Y for each $\alpha \in X$. We define $f : X \to Y$ by $f(\alpha) = x(\alpha)$ for each $\alpha \in X$. Because $\mathcal{P}$ is a σ-locally countable weak-base for Y, then f is surjective. For each $\alpha = (\alpha_i) \in M$, $f(\alpha) = x(\alpha)$. Suppose V is an open neighbourhood of $x(\alpha)$ in Y, there exists $n \in N$ such that $x(\alpha) \in P_{\alpha_n} \subset V$, set $W = \{c \in X :$ the n-the coordinate of c is $\alpha_n\}$, then W is an open neighbourhood of α in X, and $f(W) \subset P_{\alpha_n} \subset V$. Hence f is continuous. We will show that f is a weakly open msss-mapping.

(i) f is a msss-mapping. For each $x \in X$ and each $i \in N$, there exists an open neighbourhood V_i of x in X such that $\{\alpha \in A_i : P_\alpha \cap V_i \neq \Phi\}$ is countable. Put

$$B_i = \{\alpha \in A_i : P_\alpha \cap V_i \neq \Phi\},$$

then $p_i f^{-1}(V_i) \subset B_i$. Thus $p_i f^{-1}(V_i)$ is separable in A_i, Hence f is a msss-mapping.

(ii) f is a weakly open mapping

For each $n \in N$ and $\alpha_n \in A_n$, put

$$V(\alpha_1, \cdots, \alpha_n) = \{\beta \in X : \text{ for each } i \leq n, \text{ the i-th coordinate of } \beta \text{ is } \alpha_i\}.$$

It is easy to check that $\{V(\alpha_1, \cdots, \alpha_n) : n \in N\}$ is a locally neighbourhood base of α in X.

□

Claim. $f\big(V(\alpha_1,\cdots,\alpha_n)\big)=\bigcap\limits_{i\le n}P_{\alpha_i}$ for each $n\in N$.

For each $i\le n$, $f\big(V(\alpha_1,\cdots,\alpha_n)\big)\subset P_{\alpha_i}$, then $f\big(V(\alpha_1,\cdots,\alpha_n)\big)\subset\bigcap\limits_{i\le n}P_{\alpha_i}$. On the other hand. For each $x\in\bigcap\limits_{i\le n}P_{\alpha_i}$, there is $\beta=(\beta_j)\in X$ such that $f(\beta)=x$. For each $j\in N$, $P_{\beta_j}\in\mathcal{P}_j\subset\mathcal{P}_{j+n}$, then there is $\alpha_{j+n}\in A_{j+n}$ such that $P_{\alpha_{j+n}}=P_{\beta_j}$. Set $\alpha=(\alpha_j)$, then $\alpha\in V(\alpha_1,\cdots,\alpha_n)$ and $f(\alpha)=x$. Thus $\bigcap\limits_{i\le n}P_{\alpha_i}\subset f\big(V(\alpha_1,\cdots,\alpha_n)\big)$. Hence $f\big(V(\alpha_1,\cdots,\alpha_n)\big)=\bigcap\limits_{i\le n}P_{\alpha_i}$.

Denote $\mathcal{P}_y=\{P\in\mathcal{P}:y\in P\}$, then $\mathcal{P}=\cup\{P_y:y\in Y\}$.

For each $y\in Y$, by the idea $\mathcal{P}$, there exists $(\alpha_i)\in\prod\limits_{i\in N}A_i$ such that $\{P_{\alpha_i}:i\in N\}\subset\mathcal{P}$ is a network of y in Y, then $\alpha=(\alpha_i)\in f^{-1}(y)$.

Suppose G is an open neighbourhood of α in X, then there exists $j\in N$ such that $V(\alpha_1,\cdots,\alpha_j)\subset G$. Thus $f\big(V(\alpha_1,\cdots,\alpha_j)\big)\subset f(G)$. By the Claim, $f\big(V(\alpha_1,\cdots,\alpha_j)\big)=\bigcap\limits_{i\le j}P_{\alpha_i}$. Since $P_y\subset\bigcap\limits_{i\le j}P_{\alpha_i}$ for some $P_y\in\mathcal{P}_y$. Hence $P_y\subset f(G)$.

Therefore there exists a weak-base $\mathcal{P}$ for Y and $\alpha\in f^{-1}(y)$ satisfying the condition $(*)$ of Definition 3 (1), and so f is a weakly open mapping.

Theorem 5. *For a space X, (1) $\iff$ (2) $\Rightarrow$ (3) below hold.*

(1) *X has a σ-locally countable weak-base.*

(2) *X is a g-first countable space with a σ-locally countable cs-network.*

(3) *X is a g-first countable space with a σ-locally countable k-network.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3). Suppose X is a g-first countable space with a σ-locally countable cs-network. Let $\mathcal{P}=\cup\{\mathcal{P}_n:n\in N\}$ be a σ-locally countable cs-network for X, where each $\mathcal{P}_n$ is locally countable in X. We will show that $\mathcal{P}$ is a k-network for X. Suppose $K\subset V$ with K non-empty compact and V open in X. For each $n\in N$, put

$$\mathcal{A}_n=\{P\in\mathcal{P}_n:P\cap K\ne\Phi\text{ and }P\subset V\}\,,$$

then $\mathcal{A}_n$ is countable, and so $\mathcal{A}=\cup\{\mathcal{A}_n:n\in N\}$ is countable. Denote $\mathcal{A}=\{P_i:i\in N\}$, then $K\subset\bigcup\limits_{i\le n}P_i$ for some $n\in N$. Otherwise, $K\not\subset\bigcup\limits_{i\le n}P_i$ for each $n\in N$, so choose $x_n\in K\backslash\bigcup\limits_{i\le n}P_i$. Because $\{P\cap K:P\in\mathcal{P}\}$ is a countable cs-network for a subspace K and a compact space with a countable network is metrizable, then K is a compact metrizable space. Thus $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$, where $x_{n_k}\to x$. Obviously $x\in K$. Since $\mathcal{P}$ is a cs-network for X, then there exist $m\in N$ and $P\in\mathcal{P}$ such that $\{x_{n_k}:k\ge m\}\cup\{x\}\subset P\subset V$. Now, $P=P_j$ for some $j\in N$. Take $l\ge m$ such that $n_l\ge j$, then $x_{n_l}\in P_j$. This is a contradiction. Therefore, (2) $\Rightarrow$ (3) holds.

(2) $\Rightarrow$ (1). Suppose X is a g-first countable space with σ-locally countable cs-network. Let $\mathcal{P}=\cup\{\mathcal{P}_m:m\in N\}$ be a σ-locally countable cs-network for X, where each $\mathcal{P}_m$ is locally countable in X which is closed under finite intersections

and $X \in \mathcal{P}_m \subset \mathcal{P}_{m+1}$, and for each $x \in X$, let $\{B(n,x) : n \in N\}$ be a decreasing weak neighbourhood sequence of x in X. Put

$$\begin{aligned}\mathcal{F}_{m,x} &= \{P \in \mathcal{P}_m : B(n,x) \subset P \text{ for some } n \in N\},\\ \mathcal{F}_x &= \cup\{\mathcal{F}_{m,x} : m \in N\}\\ \mathcal{F}_m &= \cup\{\mathcal{F}_{m,x} : x \in X\}\\ \mathcal{F} &= \cup\{\mathcal{F}_x : x \in X\}\end{aligned}$$

we will show that $\mathcal{F}$ is a σ-locally countable weak-base for X.

It is easy to check that $\mathcal{F}$ satisfies the condition (1), (2) of Definition 1.

Suppose G be an open subset of X, then for each $x \in G$, there exists $P \in \mathcal{F}_x$ with $P \subset G$. Otherwise, denote $\{P \in \mathcal{P} : x \in P \subset G\} = \{P(m,x) : m \in N\}$. Then $B(n,x) \not\subset P(m,x)$ for each $n, m \in N$, so choose $x_{n,m} \in B(n,x) \backslash P(m,x)$. For $n \geq m$, let $x_{n,m} = y_k$, where $k = m + \frac{n(n-1)}{2}$. The the sequence $\{y_k : k \in N\}$ converges to the point x. Thus, there exist $m, i \in N$ such that $\{y_k : k \geq i\} \cup \{x\} \subset P(m,x) \subset G$ because $\mathcal{P}$ is a cs-network for X. Take $j \geq i$ with $y_j = x_{n,m}$ for some $n \geq m$. Then $x_{n,m} \in P(m,x)$. This is a contradiction. On the other hand. If $G \subset X$ satisfies that for each $x \in G$ there exists $P \in \mathcal{F}_x$ with $P \subset G$, then $B(n,x) \subset G$ for some $n \in N$. Thus G is open in X.

Hence $\mathcal{F}$ is a weak-base for X.

For each $m \in N$, $\mathcal{F}_m \subset \mathcal{P}_m$, then $\mathcal{F}_m$ is locally countable in X. Thus $\mathcal{F} = \cup\{\mathcal{F}_m : m \in N\}$ is σ-locally countable in X. Therefore, (2) $\Rightarrow$ (1) holds. $\square$

Corollary 6. *A paracompact space with a σ-locally countable weak-base is g-metrizable.*

Proof. Suppose X is a paracompact space with a σ-locally countable weak-base. By Theorem 5, X is a g-first countable space with a σ-locally countable k-network. Since a paracompact space with a σ-locally countable k-network is an $\aleph$-space ([9, Lemma 1]), then X is an $\aleph$-space. Thus X is g-metrizable by Theorem 2.4 in [6]. $\square$

In conclusion of this paper, we pose the following question in view of Theorem 5.

Question 7. Does (3) $\Rightarrow$ (1) in Theorem 6 hold?

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