ARCHIVUM MATHEMATICUM (BRNO)
Tomus 42 (2006), 141 – 149

OSCILLATION CRITERIA FOR SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

BLANKA BACULÍKOVÁ

Abstract. Our aim in this paper is to present criteria for oscillation of the nonlinear differential equation

$$u''(t) + p(t)f\big(u(g(t))\big) = 0\,.$$

The obtained oscillatory criteria improve existing ones.

1. Introduction

We consider the second order nonlinear differential equation with delayed argument

$$u''(t) + p(t)f\big(u(g(t))\big) = 0 \tag{1}$$

We suppose throughout the paper that the following conditions hold:

(i) $p(t) \in C((t_0,\infty))$, $p(t) > 0$;
(ii) $f(x) \in C((-\infty,\infty))$, $xf(x) > 0$ for $x \neq 0$, $f \in C^1(R_D)$, where $R_D = (-\infty,-D) \cup (D,\infty)$, $D > 0$;
(iii) $g(t) \in C^1((t_0,\infty))$, where $t_0 \in R^+$, $g'(t) > 0$, $g(t) \to \infty$ as $t \to \infty$, $g(t) \le t$ for all large t.

We make standing hypotesis that (1) possesses solutions on (t_0,∞) only and they are nontrivial in any neighbourhood of ∞. Such solution is called oscillatory if it has a sequence of zeros tending to infinity, otherwise it called nonoscillatory. An equation is said to be oscillatory if all its solutions are oscillatory. All functional inequalities are supposed to hold eventually, that is they are assumed to hold for all t large enough.

2. Main results

The following lemma is a partial case of well-known lemma of Kiguradze [1].

2000 *Mathematics Subject Classification*: Primary: 34C10,34K11.
Key words and phrases: Oscillatory solution.
This work was supported by Slovak Scientific Grant Agency, No.1/0426/03.
Received October 25, 2004, revised 2005.

Lemma 2.1. *Let $u(t)$ be nonoscillatory solution of* (1). *Then there exists a $\tau \geq t_0$ such that $u(t)u'(t) > 0$, $u(t)u''(t) < 0$ for $t \geq \tau$.*

The following theorem presents oscillatory criterion for (1).

Theorem 2.1. *Let there exist constant $k > 0$ such that $f'(x) \geq k$ for all $x \in R_D$. If*

$$\int_{t_0}^{\infty} \Big[\int_{s_1}^{\infty} p(s)\, ds \Big]\, ds_1 = \infty \tag{2}$$

and

$$\int_{t_0}^{\infty} \Big(tp(t) - \frac{1}{4tg'(t)k} \Big)\, dt = \infty\,. \tag{3}$$

Then equation (1) *is oscillatory.*

Proof. Assume that $u(t)$ is a nonoscillatory solution of (1).

1. Let $u(t) > 0$. Then by Lemma 2.1, we obtain that $u'(t) > 0, u''(t) < 0$ for $t \in (\tau, \infty)$, $\tau \geq t_0$.

Define

$$W(t) = \frac{tu'(t)}{f(u(g(t)))}\,, \qquad t \in (\tau, \infty)\,. \tag{4}$$

Differentiating $W(t)$ and using (1), we have

$$\frac{dW(t)}{dt} = \frac{W(t)}{t} - tp(t) - W(t)\frac{f'(u(g(t)))u'(g(t))g'(t)}{f(u(g(t)))}\,.$$

Since $u'(t)$ is decreasing, we see that

$$u'\big(g(t)\big) \geq u'(t)\,.$$

Consequently,

$$\begin{aligned} \frac{dW(t)}{dt} &\leq \frac{W(t)}{t} - tp(t) - \frac{W(t)f'(u(g(t)))u'(t)g'(t)}{f(u(g(t)))} \\ &= \frac{W(t)}{t} - tp(t) - \frac{W^2(t)}{t}f'(u(g(t)))g'(t)\,. \end{aligned}$$

Now, we shall show that (2) implies $u(t) \to \infty$ as $t \to \infty$. On the contrary, assume that u(t) is bounded above, that is $u(t) \in \langle \alpha, \beta \rangle$, where $\alpha > 0$. Using properties of $g(t)$, we may assume that $u(g(t)) \in \langle \alpha, \beta \rangle$. Since $u'(t)$ is positive and decreasing $\lim_{t\to\infty} u'(t)$ exists and it is finite. Integrating equation (1) from t to ∞, we obtain

$$u'(\infty) - u'(t) = -\int_t^{\infty} p(s)f\big(u(g(s))\big)\, ds\,.$$

Using properties of u(t), we have

$$u'(t) \geq \int_t^{\infty} p(s)f\big(u(g(s))\big)\, ds\,.$$

Let $f_0 = min_{u\in\langle\alpha,\beta\rangle} f(u)$, $f_0 > 0$. Then

$$u'(t) \geq f_0 \int_t^\infty p(s)\,ds\,.$$

Integrating this inequality from t_0 to t, we have

$$\beta \geq u(t) \geq f_0 \int_{t_0}^t \Big(\int_{s_1}^\infty p(s)\,ds\Big)\,ds_1\,.$$

Letting $t \to \infty$ the last inequality contradicts to (2). Therefore we conclude $u(t) \to \infty$ as $t \to \infty$. Thus $u\big(g(t)\big) \in R_D$ for all t large enough. Now it is easy to see that condition $f'\big(u(g(t))\big) \geq k$ implies

$$\frac{dW(t)}{dt} \leq \frac{W(t)}{t} - tp(t) - \frac{W^2(t)}{t} g'(t)k\,, \tag{5}$$

$$\frac{dW(t)}{dt} \leq -tp(t) + \frac{1}{t} g'(t)k\Big(-\Big(W(t) - \frac{1}{2g'(t)k}\Big)^2 + \frac{1}{4\big(g'(t)\big)^2 k^2}\Big)\,.$$

Thus

$$\frac{dW(t)}{dt} \leq -tp(t) + \frac{1}{4tg'(t)k}\,. \tag{6}$$

Integrating this inequality from t_1 to t, we obtain

$$W(t) \leq W(t_1) - \int_{t_1}^t \Big(sp(s) - \frac{1}{4sg'(s)k}\Big)\,ds\,.$$

Hence for $t \to \infty, W(t) \to -\infty$ and we have contradiction, because $W(t) > 0$.

2. Let $u(t) < 0$. This case can be treated similarly as the case $u(t) > 0$ and so it is omitted. □

Now we provide an easily verifiable oscillatory criteria for (1).

Corollary 2.1. *Let there exist constant $k > 0$ such that $f'(x) \geq k$ for all $x \in R_D$. Assume that* (2) *is satisfied and*

$$\liminf_{t\to\infty} (t^2 p(t) g'(t)) > \frac{1}{4k}\,. \tag{7}$$

Then equation (1) *is oscillatory.*

Proof. Simple calculation shows that (7) implies (3). □

Remark 1. We do not require boundedness of $f'(x)$ around zero, therefore our results can applied to sublinear and superlinear equations.

Corollary 2.2. *Assume that* (2) *holds and $\alpha > 1$. If*

$$\liminf_{t\to\infty} (t^2 p(t) g'(t)) > 0\,, \tag{8}$$

then equation

$$u''(t) + p(t)\big[|u(g(t))|\big]^\alpha \operatorname{sgn}\big(u(g(t))\big) = 0 \tag{9}$$

is oscillatory.

Proof. Inequality (8) implies

$$\liminf_{t\to\infty}\big(t^2p(t)g'(t)\big) > \frac{1}{4k}$$

and for $f(x)=|x|^{\alpha}\operatorname{sgn} x$, $f'(x)\ge k$ for all $x\in R_D$, k large enough. Hence (7) holds and the statement follows from Corollary 2.1. □

Corollary 2.3. *Let* (2) *hold. If*

$$\int_{t_0}^{\infty}\Big(tp(t)-\frac{1}{4tg'(t)}\Big)\,dt=\infty\,, \tag{10}$$

then equation

$$u''(t)+p(t)u\big(g(t)\big)=0 \tag{11}$$

is oscillatory.

Proof. It is easy to see that (3) reduces to (10) for $f(u)=u$. □

Corollary 2.4. *Let* (2) *hold. If*

$$\liminf_{t\to\infty}\big(t^2p(t)g'(t)\big) > \frac{1}{4}\,, \tag{12}$$

then (11) *is oscillatory.*

Proof. The result follows from Corollaries 2.1 and 2.3. □

Example 1. Let us consider the second order differential equation

$$u''(t)+\frac{1}{t^2}\,\frac{4|u(\frac{t}{2})|}{|u(\frac{t}{2})|+1}\,u\Big(\frac{t}{2}\Big)=0\,. \tag{13}$$

For this equation

- $p(t)=\frac{1}{t^2}$,
- $g(t)=\frac{t}{2}$,
- $f(x)=\frac{4|x|}{|x|+1}x$.

The function $f(x)$ satisfies
1. $xf(x)>0$ for $x\ne 0$,
2. $f'(x)\ge 3=k$ for $x\in(-\infty,-1)\cup(1,\infty)$.

Note that we do not require this condition to hold on R.
Moreover it holds that

$$\int_{t_0}^{\infty}\Big[\int_{s_1}^{\infty}p(s)\,ds\Big]\,ds_1=\int_{t_0}^{\infty}\Big[\int_{s_1}^{\infty}\frac{1}{s^2}\,ds\Big]\,ds_1=\infty\,.$$

Since

$$\int_{t_0}^{\infty}\Big(tp(t)-\frac{1}{4tg'(t)k}\Big)\,dt=\int_{t_0}^{\infty}\Big(t\frac{1}{t^2}-\frac{1}{6t}\Big)dt=\infty\,,$$

then condition (3) holds and by Theorem 2.1 equation (13) is oscillatory.

In the following theorems we shall show further oscillatory criteria for Eq. (1).

Theorem 2.2. *Let there exist constant $k > 0$ such that $f'(x) \geq k$ for all $x \in R_D$. If*

$$\int_{t_0}^{\infty} \Big[\int_{s_1}^{\infty} p(s)\, ds \Big]\, ds_1 = \infty$$

and

$$\int_{t_0}^{\infty} \Big(g(t)p(t) - \frac{g'(t)}{4g(t)k} \Big)\, dt = \infty\,. \tag{14}$$

Then equation (1) *is oscillatory.*

Proof. Define

$$W(t) = \frac{g(t)u'(t)}{f(u(g(t)))}\,, \qquad t \in (t_0\,, \infty)\,. \tag{15}$$

The proof is similar as the proof of Theorem 2.1 and we can it to omit. □

Corollary 2.5. *Let there exist constant $k > 0$ such that $f'(x) \geq k$ for all $x \in R_D$. Assume that* (2) *is satisfied and*

$$\liminf_{t\to\infty} \frac{g^2(t)p(t)}{g'(t)} > \frac{1}{4k}\,. \tag{16}$$

Then equation (1) *is oscillatory.*

Proof. A simple calculation shows that (16) implies (14). □

Corollary 2.6. *Assume that* (2) *holds and $\alpha > 1$. If*

$$\liminf_{t\to\infty} \frac{g^2(t)p(t)}{g'(t)} > 0\,, \tag{17}$$

then equation (9) *is oscillatory.*

Proof. Inequality (17) implies

$$\liminf_{t\to\infty} \frac{g^2(t)p(t)}{g'(t)} > \frac{1}{k}$$

and for $f(x) = |x|^{\alpha} \operatorname{sgn} x$, $f'(x) \geq k$ for all $x \in R_D$, k large enough. □

Corollary 2.7. *Let* (2) *holds. If*

$$\int_{t_0}^{\infty} \Big(g(t)p(t) - \frac{g'(t)}{4g(t)} \Big)\, dt = \infty, \tag{18}$$

then equation (11) *is oscillatory.*

Proof. It is easy to see that (14) reduces to (18) for $f(u) = u$. □

Corollary 2.8. *Let* (2) *hold. If*

$$\liminf_{t\to\infty} \frac{g^2(t)p(t)}{g'(t)} > \frac{1}{4}\,, \tag{18}$$

then (11) *is oscillatory.*

Proof. The result follows from Corollaries 2.5 and 2.7. □

Example 2. Let us consider the equation (1). For $g(t) = \lambda t$, $\lambda \in (0,1\rangle$ the condition (7) is equivalent to (16). For $g(t) = \sqrt{t}$ the condition (16) takes the form

$$\liminf_{t\to\infty}(2t\sqrt{t}p(t)) > \frac{1}{4k}\,,$$

on the other hand (7) takes the form

$$\liminf_{t\to\infty}\frac{t\sqrt{t}p(t)}{2} > \frac{1}{4k}\,.$$

For this partial case (16) provides evidently better criterion.

Remark 2. Corollaries 2.2 and 2.6 complement Theorem 2 in [2], Corollary 2.5.3 and Theorem 4.5.5 in [3].

Remark 3. Corollaries 2.3 and 2.8 generalize Theorem 11 in [4], Theorem 1 in [5] and the results of Kiguradze and Chanturia [6].

Theorem 2.3. *Let there exist constant $k > 0$ such that, $f'(x) \ge k$ for all $x \in R_D$. Assume that* (2) *holds. If for some n integer*

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_0}^{t}(t-s)^n\Big(sp(s)-\frac{1}{4sg'(s)k}\Big)\,ds=\infty\,. \tag{19}$$

Then (1) *is oscillatory.*

Proof. Using the function $W(t)$ defined in (4) and proceeding similarly as in the proof of Theorem 2.1, we have inequality (6)

$$\frac{dW(t)}{dt} \le -tp(t) + \frac{1}{4tg'(t)k}.$$

We use the following notation

$$P(t) = tp(t) - \frac{1}{4tg'(t)k}\,.$$

Then

$$W'(t) + P(t) \le 0\,.$$

Multiplying this inequality by $(t-s)^n, t > s$, we obtain

$$(t-s)^n W'(s) + (t-s)^n P(s) \le 0\,.$$

Integrating this inequality from t_0 to t and after simple computation, we have

$$\begin{aligned}\frac{1}{t^n}\int_{t_0}^{t}(t-s)^n P(s)\,ds &\le -\frac{n}{t^n}\int_{t_0}^{t}(t-s)^{n-1}W(s)\,ds + \frac{1}{t^n}(t-t_0)^n W(t_0)\\ &\le (1-\frac{t_0}{t})^n W(t_0)\,.\end{aligned}$$

This contradicts with (20) and the proof is complete. □

Theorem 2.4. *Let there exist constant $k > 0$ such that, $f'(x) \geq k$ for all $x \in R_D$. Assume that* (2) *holds. If for some n integer*

$$\limsup_{t\to\infty} \frac{1}{t^n}\int_{t_0}^{t} (t-s)^n \Big(g(s)p(s) - \frac{g'(s)}{4g(s)k}\Big)\,ds = \infty\,. \tag{20}$$

Then (1) *is oscillatory.*

Proof. Using the function $W(t)$ defined in (15) and proceeding exactly as in the proof of Theorem 2.3, we obtain that (21) holds. □

Now we use the integral averaging technique similar to that exploited by Grace [7], Philos [8], Rogovchenko [9-10] and Yan [12]. In contrast to the know theorems, we do not require the funtion f to be nondecreasing on R but only on R_D.
Let us consider a function $H(t,s)$ satisfying $H(t,s) > 0$ for $t > s \geq t_0$, $H(t,t) = 0$ and $h(t,s) = \frac{-\frac{\partial H(t,s)}{\partial s}}{\sqrt{H(t,s)}}$.

Theorem 2.5. *Let there exist constant $k > 0$ such that, $f'(x) \geq k$ for all $x \in R_D$. Assume that* (2) *holds. Then Eq.* (1) *is oscillatory if*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^{t} \Big[H(t,s)sp(s) - \frac{s}{4g'(s)k}\Big(h(t,s) - \frac{\sqrt{H(t,s)}}{s}\Big)^2\Big]\,ds = \infty\,. \tag{21}$$

Proof. Using the function $W(t)$ defined in (4) and proceeding similarly as in the proof of Theorem 2.1, we obtain inequality (5)

$$\frac{dW(t)}{dt} \leq \frac{W(t)}{t} - tp(t) - \frac{W^2(t)}{t}g'(t)k\,.$$

We introduce the notation $tp(t) = \tilde{p}(t)$. Then

$$\tilde{p}(t) \leq -W'(t) + \frac{1}{t}W(t) - \frac{1}{t}W^2(t)g'(t)k\,.$$

Multiplying this inequality with $H(t,s) > 0$ and next integrating from t_0 to t we have

$$\int_{t_0}^{t} H(t,s)\tilde{p}(s)\,ds \leq H(t,t_0)W(t_0) - \int_{t_0}^{t}\Big[\frac{H(t,s)}{s}W^2(s)g'(s)k + \sqrt{H(t,s)}W(s)\Big(h(t,s) - \frac{\sqrt{H(t,s)}}{s}\Big)\Big]\,ds\,.$$

Using the following notation $h(t,s) - \frac{\sqrt{H(t,s)}}{s} = Q(t,s)$, then we have

$$\int_{t_0}^{t} H(t,s)\tilde{p}(s)\,ds \leq H(t,t_0)W(t_0) - \int_{t_0}^{t}\Big[\frac{\sqrt{H(t,s)}}{\sqrt{s}}W(s)\sqrt{g'(s)}\sqrt{k} + \frac{1}{2}\frac{Q(t,s)\sqrt{s}}{\sqrt{g'(s)}\sqrt{k}}\Big]^2\,ds + \int_{t_0}^{t}\frac{sQ^2(t,s)}{4g'(s)k}\,ds\,.$$

Consequently

$$\int_{t_0}^{t} H(t,s)\tilde{p}(s)\,ds - \int_{t_0}^{t}\frac{sQ^2(t,s)}{4g'(s)k}\,ds \leq H(t,t_0)W(t_0)\,.$$

Multiplying this inequality with $\frac{1}{H(t,t_0)}$, we obtain

$$\frac{1}{H(t,t_0)}\int_{t_0}^{t}\Big(H(t,s)\tilde{p}(s)\,ds-\frac{sQ^2(t,s)}{4g'(s)k}\Big)\,ds\le W(t_0)\,.$$

Since, we have assumed (22) for $t\to\infty$, we have contradiction $\infty\le W(t_0)$. $\square$

Theorem 2.6. *Let there exist constant $k>0$ such that, $f'(x)\ge k$ for all $x\in R_D$. Assume that* (2) *holds. Then Eq.* (1) *is oscillatory if*
(22)

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\Big[H(t,s)g(s)p(s)-\frac{g(s)}{4g'(s)k}\Big(h(t,s)-\frac{\sqrt{H(t,s)}g'(s)}{g(s)}\Big)^2\Big]\,ds=\infty\,.$$

Proof. We proceed similarly in the proof of Theorem 2.5 for $W(t)$ defined in (15). $\square$

For more results on "H-function averagin technique" we refer for example to [7-12] and to the monograph [13].

Example 3. Let us consider the second order differential equation

$$u''(t)+\frac{a}{t^2}u\Big(\frac{t}{2}\Big)=0\,, \tag{23}$$

where $f(x)=x$, $p(t)=\frac{a}{t^2}, a\in R$, $g(t)=\frac{t}{2}$. Then Eq. (24) is oscillatory by Theorem 2.3 if constant $a>\frac{1}{2}$.

Our results complement the results in [14], where equation without deviating argument is studied.

References

[1] Kiguradze, I. T., *On the oscillation of solutions of the equation $d^m u/dt^m + a(t)|u|^n \operatorname{sign} u = 0$*, Mat. Sb. **65** (1964), 172–187. (Russian)

[2] Džurina, J., *The oscillation of a differential equation of second order with deviating argument*, Math. Slovaca **42**, No. 3 (1992), 317–324.

[3] Ladde, G. S., Lakshmikhantam, V., Zhang, B. G., *Oscillation theory of differential equations with deviating arguments*, Dekker, New York, 1987.

[4] Džurina, J., *Comparison theorems for nonlinear ODEs*, Math. Slovaca **42**, No. 3 (1992), 299–315.

[5] Džurina, J., *Oscillation of a second order delay differential equations*, Arch. Math. (Brno) **33**, No. 4 (1997), 309–314.

[6] Kiguradze, I. T., Chanturia, T. A., *Asymtotic properties of solutions of nonautonomous ordinary differential equations*, Nauka, Moscow, 1990.

[7] Grace, S. R., *Oscillation theorems for nonlinear differential equations of second order*, J. Math. Anal. Appl. **171** (1992), 220–241.

[8] Philos, Ch. G., *Oscillation theorems for linear differential equations of second order*, Arch. Math. (Basel) **53**(5) (1989), 482–492.

[9] Rogovchenko, Yu. V., *Oscillation theorems for second order equations with damping*, Nonlinear Anal. (1999).

[10] Rogovchenko, Yu. V., *Oscillation criteria for second order nonlinear perturbed differential equations*, J. Math. Anal. Appl. **215** (1997), 334–357.

[11] Rogovchenko, Yu. V., *Oscillation criteria for certain nonlinear differential equations*, J. Math. Anal. Appl. **229** (1999), 399–416.

[12] Yan, J., *Oscillation theorems for second order linear differential equations with damping*, Proc. Amer. Math. Soc. **98** (1986), 276–282.

[13] Agarwal, R., Grace, S. R., O'Regan, D., *Oscillation theory for second order dynamic equations*, Taylor & Francis, 2003.

[14] Kirane, M., Rogovchenko, Yu. V., *On oscillation of nonlinear second order differential equation with damping term*, Appl. Math. Comput. **117** (2001), 177–192.

[15] Shaker, S. H., *Oscillation criteria of hyperbolic equations with deviating argument*, Publ. Math. Debrecen, (2003), 165–185.

Department of Mathematics FEI Technical University
B. Němcovej 32, 042 00 Košice, Slovak Republic
E-mail: blanka.baculikova@tuke.sk