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ASYMMETRIC DECOMPOSITIONS OF VECTORS IN JB^*-ALGEBRAS

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Abstract. By investigating the extent to which variation in the coefficients of a convex combination of unitaries in a unital JB^*-algebra permits that combination to be expressed as convex combination of fewer unitaries of the same algebra, we generalise various results of R. V. Kadison and G. K. Pedersen. In the sequel, we shall give a couple of characterisations of JB^*-algebras of *tsr* 1.

Introduction

The class of JB^*-algebras was introduced by Kaplansky in 1976 (see [7]). In [4, 5], we presented a theory of unitary isotopes of JB^*-algebras and by applying this theory some interesting results on convex combinations of unitaries were obtained. With these results now to hand, we in this article generalise results on asymmetric decompositions of elements in C^*-algebras from [2] for JB^*-algebras. We investigate the extent to which variation in the coefficient of a convex combination of unitaries in a unital JB^*-algebra permits that combination to be expressed as convex combination of fewer unitaries of the same algebra. In the sequel, we shall give a couple of characterisations of JB^*-algebras of *tsr* 1 [6].

Jordan algebras and their homotopes

We begin by recalling (from [1], for instance) that *a commutative (not necessarily associative) algebra* $(\mathcal{J}, \circ)$ *is called a Jordan algebra if for all* $x, y \in \mathcal{J}$,

$$x^2 \circ (x \circ y) = (x^2 \circ y) \circ x\,.$$

Let $\mathcal{J}$ be a Jordan algebra and $x \in \mathcal{J}$. The *x-homotope* of $\mathcal{J}$, denoted by $\mathcal{J}_{[x]}$, is the Jordan algebra consisting of the same elements and linear algebra structure as $\mathcal{J}$ but a different product, denoted by "$\cdot_x$", defined by

$$a \cdot_x b = \{axb\}$$

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for all a, b in $\mathcal{J}_{[x]}$. $\{pqr\}$ will always denote the Jordan triple product of p, q, r defined in the Jordan algebra $\mathcal{J}$ as below: $\{pqr\} = (p\circ q)\circ r - (p\circ r)\circ q + (q\circ r)\circ p$.

An element x of a Jordan algebra $\mathcal{J}$ with unit e is said to be invertible if there exists $x^{-1} \in \mathcal{J}$, called the inverse of x, such that $x \circ x^{-1} = e$ and $x^2 \circ x^{-1} = x$. The set of all invertible elements of $\mathcal{J}$ will be denoted by $\mathcal{J}_{\text{inv}}$. In this case, x acts as the unit for the homotope $\mathcal{J}_{[x^{-1}]}$ of $\mathcal{J}$.

If $\mathcal{J}$ is a unital Jordan algebra and $x \in \mathcal{J}_{\text{inv}}$ then by x-isotope of $\mathcal{J}$, denoted by $\mathcal{J}^{[x]}$, we mean the x^{-1}-homotope $\mathcal{J}_{[x^{-1}]}$ of $\mathcal{J}$. The following lemma gives the invariance of the set of invertible elements in a unital Jordan algebra on passage to any of its isotopes:

Lemma 1. *For any invertible element a in unital Jordan algebra $\mathcal{J}$, $\mathcal{J}_{\text{inv}} = \mathcal{J}^{[a]}_{\text{inv}}$.*

Proof. See from [4]. □

A Jordan algebra $\mathcal{J}$ with product $\circ$ is called a Banach Jordan algebra if there is a norm $\|\cdot\|$ on $\mathcal{J}$ such that $(\mathcal{J}, \|\cdot\|)$ is a Banach space and $\|a\circ b\| \le \|a\|\,\|b\|$. If, in addition, $\mathcal{J}$ has a unit e with $\|e\| = 1$ then $\mathcal{J}$ is called a unital Banach Jordan algebra. Throughout the sequel, we will only be considering unital Banach Jordan algebras.

Lemma 2. *Let $\mathcal{J}$ be a Banach Jordan algebra with unit e. If $x \in \mathcal{J}$ and $\|x\| < 1$ then $e - x$ is invertible and $(e-x)^{-1} = \sum_{n=0}^{\infty} x^n$.*

Proof. See from [4]. □

JB^*-algebras and their unitary isotopes

We are interested in a special class of Banach Jordan algebras, called JB^*--algebras. These include all C^*-algebras as a proper subclass (see [7, 8]):

A complex Banach Jordan algebra $\mathcal{J}$ with involution $*$ (see [3], for instance) is called a JB^*-algebras if $\|\{xx^*x\}\| = \|x\|^3$ for all $x \in \mathcal{J}$. A JB^*-algebra $\mathcal{J}$ is said to be of tsr 1 if $\mathcal{J}_{\text{inv}}$ is norm dense in $\mathcal{J}$ (for some interesting properties of such algebras see [6]).

Let $\mathcal{J}$ be a JB^*-algebra. $u \in \mathcal{J}$ is called *unitary* if $u^* = u^{-1}$, the inverse of u. The set of all unitary elements of $\mathcal{J}$ will be denoted by $\mathcal{U}(\mathcal{J})$. If u is a unitary element of JB^*-algebra $\mathcal{J}$ then the isotope $\mathcal{J}^{[u]}$ is called a *unitary isotope* of $\mathcal{J}$.

Theorem 3. *Let u be a unitary element of the JB^*-algebra $\mathcal{J}$. Then the isotope $\mathcal{J}^{[u]}$ is a JB^*-algebra having u as its unit with respect to the original norm and the involution $*_u$ defined as $x^{*_u} = \{ux^*u\}$.*

Proof. See Theorem 2.4 of [4]. □

Convex combinations of unitaries

In [4], we presented several applications of the theory of unitary isotopes of JB^*-algebras; these include a new proof of the famous Russo-Dye theorem for JB^*-algebras and various results on means and convex combinations of unitaries. Here, we need for the sequel to recall some results from [4]:

Lemma 4. *For any JB^*-algebra $\mathcal{J}$, $\mathcal{J}_{\text{inv}} \cap (\mathcal{J})_1 \subseteq \frac{1}{2}(\mathcal{U}(\mathcal{J}) + \mathcal{U}(\mathcal{J}))$. Here, $(\mathcal{J})_1$ stands for the closed unit ball of $\mathcal{J}$.*

Lemma 5. *Let $\mathcal{J}$ be a JB^*-algebra with identity element e. Let $x \in (\mathcal{J})_1$ be such that* $\operatorname{dist}(x, \mathcal{U}(\mathcal{J})) \leq 2\alpha$ *with $\alpha < \frac{1}{2}$. Then*

$$x \in \alpha\,\mathcal{U}(\mathcal{J}) + (1-\alpha)\,\mathcal{U}(\mathcal{J})\,.$$

Let $\mathcal{J}$ be a unital JB^*-algebra and let $x \in \mathcal{J}$. We define two numbers $u_c(x)$ and $u_m(x)$ by

$$u_c(x) = \min\Big\{n : x = \sum_{j=1}^{n} \alpha_j u_j \text{ with } u_j \in \mathcal{U}(\mathcal{J})\,,\ \alpha_j \geq 0,\ \sum_{j=1}^{n} \alpha_j = 1\Big\}\,,$$

$$u_m(x) = \min\{n : x = \frac{1}{n}\sum_{j=1}^{n} u_j,\ u_j \in \mathcal{U}(\mathcal{J})\}\,.$$

If x has no decomposition as a convex combination of elements of $\mathcal{U}(\mathcal{J})$, we define $u_c(x)$ to be ∞.

Lemma 6. *Each convex combination of unitaries in a unital JB^*-algebra $\mathcal{J}$ is the mean of the same number of unitaries in the algebra. Hence $u_m(x) = u_c(x)$.*

In the sequel, the number $u_m(x) = u_c(x)$ will be called the *unitary rank* of x and denoted by $u(x)$.

Asymmetric decompositions

We now prove JB^*-algebra analogue of various results on asymmetric decompositions of elements in C^*-algebras from [2]. We investigate the extent to which variation in the coefficients of a convex combination of unitaries in a unital JB^*--algebra permits that combination to be expressed as convex combination of fewer unitaries of the same algebra. As a generalisation of [2, Proposition 18] we shall give two characterisations of JB^*-algebras of *tsr* 1.

Definition 7. Let $\mathcal{J}$ be a unital JB^*-algebra. For every positive integer n, we define $co_n\,\mathcal{U}(\mathcal{J})$ as the set given by

$$co_n\,\mathcal{U}(\mathcal{J}) = \Big\{\sum_{i=1}^{n} \alpha_i u_i :\ u_i \in \mathcal{U}(\mathcal{J}),\ \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1\Big\}\,.$$

Hence

$$co_n\,\mathcal{U}(\mathcal{J}) = \{x \in \mathcal{J} :\ u(x) \leq n\}\,.$$

Lemma 8. *Let $\mathcal{J}$ be a unital JB^*-algebra and let $x \in \mathcal{J}$ be such that*

(i) $$\|x\| \leq 1 - \epsilon \quad \textit{for some} \quad \epsilon \in (0, (n+1)^{-1})\,.$$

Let x have the distance to $co_n\,\mathcal{U}(\mathcal{J})$ *less than* $\frac{\epsilon^2}{1-\epsilon}$. *Then there exist unitaries* $u_i \in \mathcal{U}(\mathcal{J})$, $i = 1, \ldots, n+1$, *such that*

$$x = \sum_{i=1}^{n} \alpha_i u_i + \epsilon u_{n+1}$$

where $\alpha_k \geq 0$ *and* $\sum_{i=1}^{n} \alpha_i + \epsilon = 1$.

Proof. Since $\operatorname{dist}(x, co_n\,\mathcal{U}(\mathcal{J})) < \epsilon^2(1-\epsilon)^{-1}$, there exist (by definition of $co_n\,\mathcal{U}(\mathcal{J})$) unitaries $v_1, \ldots, v_n$ in $\mathcal{J}$ such that

$$\text{(ii)} \qquad \Big\| x - \sum_{j=1}^{n} \beta_j v_j \Big\| < \epsilon^2(1-\epsilon)^{-1}$$

for some $\beta_k \geq 0$ with $\sum_{j=1}^{n} \beta_j = 1$. Without loss of generality we can assume that $\beta_j = \frac{1}{n}$ for all j (by Lemma 6). Let w be defined by

$$\text{(iii)} \qquad w = \beta^{-1}\Big(x - (1-\epsilon)\sum_{j=1}^{n-1} \beta_j v_j\Big)$$

where β is given by

$$\text{(iv)} \qquad \beta = \epsilon + (1-\epsilon)\beta_n \,.$$

Then $0 < \beta \leq 1$ since $\beta_n + \epsilon(1-\beta_n) \leq 1$.

Now, we observe that

$$\begin{aligned}
\|w\| &= \Big\|\beta^{-1}\Big(x - (1-\epsilon)\sum_{j=1}^{n-1} \beta_j v_j\Big)\Big\| \\
&= \beta^{-1}]\Big\| x - \epsilon x + \epsilon x - (1-\epsilon)\sum_{j=1}^{n} \beta_j v_j + (1-\epsilon)\beta_n v_n \Big\| \\
&= \beta^{-1}\Big\|(1-\epsilon)(x - \sum_{j=1}^{n} \beta_j v_j) + (1-\epsilon)\beta_n v_n + \epsilon x\| \\
&\leq \beta^{-1}((1-\epsilon)\|x - \sum_{j=1}^{n} \beta_j v_j\| + (1-\epsilon)\beta_n \|v_n\| + \epsilon\|x\|) \\
&< \ \beta^{-1}(\epsilon^2 + (1-\epsilon)\beta_n + \epsilon(1-\epsilon)) = 1
\end{aligned}$$

by (i)–(iv). That is, $\|w\| < 1$. Hence, as $n^{-1} = \beta_n$, we have that

$$\begin{aligned}
\|w - v_n\| &\le \|w - \beta^{-1}(1-\epsilon)\beta_n v_n\| + \|\beta^{-1}(1-\epsilon)\beta_n v_n - v_n\| \\
&\le \beta^{-1}\Big((1-\epsilon)\Big\|x - \sum_{j=1}^{n}\beta_j v_j\Big\| + \epsilon\|x\|\Big) + (1 - \beta^{-1}(1-\epsilon)\beta_n)\|v_n\| \quad \text{(by (iii))} \\
&\le \beta^{-1}\big(\epsilon^2 + \epsilon(1-\epsilon)\big) + 1 - \beta^{-1}(1-\epsilon)\beta_n \quad \text{(by (i) and (ii))} \\
&= \beta^{-1}\big(\epsilon + 1 - (1-\epsilon)\big) = 2\epsilon\beta^{-1} \\
&\le 2n(\epsilon^{-1} + n - 1)^{-1} \quad \text{(by (iv))} < 1
\end{aligned}$$

since $\epsilon < (n+1)^{-1}$ by (i).

Now, since $\|w - v_n\| \le 2\epsilon\beta^{-1} < 1$ and since v_n is a unitary, we get from Lemma 5 the existence of two unitaries u_n, u_{n+1} in $\mathcal{J}$ such that

$$w = (1 - \epsilon\beta^{-1})u_n + \epsilon\beta^{-1}u_{n+1}\,.$$

Hence, by (iii),

$$x = \beta w + (1-\epsilon)\sum_{j=1}^{n-1}\beta_j v_j = (1-\epsilon)\sum_{j=1}^{n-1}\beta_j v_j + (\beta - \epsilon)u_n + \epsilon u_{n+1}\,.$$

But $\beta - \epsilon = (1-\epsilon)\beta_n$. Therefore, $x = (1-\epsilon)\sum_{j=1}^{n-1}\beta_j v_j + (1-\epsilon)\beta_n u_n + \epsilon u_{n+1}$. Thus

$$x = \sum_{i=1}^{n}\alpha_i u_i + \epsilon u_{n+1} \quad \text{with} \quad \alpha_i = (1-\epsilon)\beta_i$$

for $i = 1, \ldots, n$ and $u_i = v_i$ for $i = 1, \ldots, n-1$. Clearly, each $\alpha_k \ge 0$ and $\sum_{i=1}^{n}\alpha_i + \epsilon = (1-\epsilon)\sum_{i=1}^{n}\beta_i + \epsilon = 1$. $\square$

Definition 9. For any unital JB^*-algebra $\mathcal{J}$, we define $co_{n+}\mathcal{U}(\mathcal{J})$ as the set of elements x in $\mathcal{J}$ with the property that for each real number $\epsilon > 0$ there is a convex decomposition $\sum_{i=1}^{n+1}\alpha_i u_i$ of x with $u_i \in \mathcal{U}(\mathcal{J})$ and $\alpha_{n+1} < \epsilon$.

Lemma 10. *Let $\mathcal{J}$ be a unital JB^*-algebra, $(\mathcal{J})_1^\circ$ and $\overline{co}_n\,\mathcal{U}(\mathcal{J})$ denote the open unit ball and norm closure of the set $co_n\,\mathcal{U}(\mathcal{J})$ in $\mathcal{J}$, respectively. Then*

$$(\mathcal{J})_1^\circ \cap \overline{co}_n\,\mathcal{U}(\mathcal{J}) = (\mathcal{J})_1^\circ \cap co_{n+}\,\mathcal{U}(\mathcal{J})\,.$$

Proof. If $x \in (\mathcal{J})_1^\circ \cap co_{n+}\mathcal{U}(\mathcal{J})$, then for arbitrary but fixed $\epsilon > 0$, there exist $u_1, \ldots, u_{n+1}$ in $\mathcal{U}(\mathcal{J})$ and non-negative real numbers $\alpha_1, \ldots, \alpha_{n+1}$ with $\sum_{i=1}^{n+1}\alpha_i = 1$ such that $\alpha_{n+1} < \frac{\epsilon}{2}$ and $x = \sum_{i=1}^{n+1}\alpha_i u_i$. We observe

$$\|x - \sum_{i=1}^{n-1}\alpha_i u_i - (\alpha_n + \alpha_{n+1})u_n\| = \| - \alpha_{n+1}u_n + \alpha_{n+1}u_{n+1}\| \le 2\alpha_{n+1} < \epsilon\,.$$

But ϵ is an arbitrary positive real number. It follows that $x \in (\mathcal{J})_1^\circ \cap \overline{co}_n\,\mathcal{U}(\mathcal{J})$.

Conversely, let $x \in (\mathcal{J})_1^\circ \cap \overline{co}_n\,\mathcal{U}(\mathcal{J})$. Let $\epsilon > 0$. Reducing ϵ if necessary we may assume that $\epsilon < \frac{1}{n+1}$ and $\|x\| < 1-\epsilon$. Now, since $\mathrm{dist}(x, co_n\,\mathcal{U}(\mathcal{J})) = 0 < \frac{\epsilon^2}{1-\epsilon}$, the previous Lemma 8 is applicable so that $x \in co_{n+}\,\mathcal{U}(\mathcal{J})$. Hence, $x \in (\mathcal{J})_1^\circ \cap co_{n+}\,\mathcal{U}(\mathcal{J})$. □

Now, by using above Lemma 10 we get the following characterisations of JB^*--algebras that are the norm closures of their invertible elements:

Theorem 11. *Let $\mathcal{J}$ be a unital JB^*-algebra. The following statements are equivalent:*

(i) *$\mathcal{J}$ is of tsr 1;*
(ii) *$\frac{1}{2}\mathcal{U}(\mathcal{J}) + \frac{1}{2}\mathcal{U}(\mathcal{J})$ is norm dense in $(\mathcal{J})_1$;*
(iii) *$(\mathcal{J})_1^\circ \subseteq co_{2+}\,\mathcal{U}(\mathcal{J})$.*

Proof. (i)⇒(ii): Let $x \in (\mathcal{J})_1$. By (i), there exists a sequence (x_n) in $\mathcal{J}_{\mathrm{inv}}$ which converges uniformly to x. Putting $\alpha_n = (max\{1, \|x_n\|\})^{-1}$ we see that

$$\|x - \alpha_n x_n\| \leq \|x - x_n\| + \|x_n - \alpha_n x_n\|$$

where we note that

$$\|x_n - \alpha_n x_n\| = (1-\alpha_n)\|x_n\| \to 0 \quad \text{as } n \to \infty$$

since $\alpha_n \to 1$ as $\|x_n\| \to \|x\| \leq 1$ when $n \to \infty$. Therefore,

$$\text{(I)} \qquad \|x - \alpha_n x_n\| \to 0 \quad \text{as} \quad n \to \infty\,.$$

Further, we note that for each n, $\alpha_n x_n \in (\mathcal{J})_1 \cap \mathcal{J}_{\mathrm{inv}}$ because $x_n \in \mathcal{J}_{\mathrm{inv}}$ and

$$\|\alpha_n x_n\| = \begin{cases} \|x_n\| < 1 & \text{if} \quad \alpha_n = 1\,; \\ \|\,\|x_n\|^{-1} x_n\| = 1 & \text{otherwise}\,. \end{cases}$$

Since each $\alpha_n x_n \in \mathcal{J}_{\mathrm{inv}} \cap (\mathcal{J})_1$, it follows from Lemma 4 that

$$\alpha_n x_n \in \Big(\frac{1}{2}\mathcal{U}(\mathcal{J}) + \frac{1}{2}\mathcal{U}(\mathcal{J})\Big)\,.$$

This together with (I) gives the norm density of $\frac{1}{2}\mathcal{U}(\mathcal{J}) + \frac{1}{2}\mathcal{U}(\mathcal{J})$ in $(\mathcal{J})_1$.

(ii)⇒(iii): By the hypothesis, $\overline{co}_2\,\mathcal{U}(\mathcal{J}) = (\mathcal{J})_1$ so that

$$(\mathcal{J})_1^\circ = (\mathcal{J})_1^\circ \cap (\mathcal{J})_1 = (\mathcal{J})_1^\circ \cap \overline{co}_2\,\mathcal{U}(\mathcal{J})\,.$$

And, by Lemma 10,

$$(\mathcal{J})_1^\circ \cap \overline{co}_2\,\mathcal{U}(\mathcal{J}) \;=\; (\mathcal{J})_1^\circ \cap co_{2+}\,\mathcal{U}(\mathcal{J})\,.$$

Thus

$$(\mathcal{J})_1^\circ \subseteq co_{2+}\,\mathcal{U}(\mathcal{J})\,.$$

(iii)⇒(i): It is sufficient to show that $(\mathcal{J})_1^\circ \subseteq \bar{\mathcal{J}}_{\mathrm{inv}}$. Choose any positive $\epsilon \leq \frac{1}{3}$. Under the hypothesis, each $x \in (\mathcal{J})_1^\circ$ has the form $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3$ with $u_1, u_2, u_3 \in \mathcal{U}(\mathcal{J})$, $\alpha_1, \alpha_2, \alpha_3 \geq 0$ such that $\alpha_3 < \epsilon\ \big(\leq \frac{1}{3}\big)$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Without any loss of generality, we assume that $\alpha_1 \leq \alpha_2$.

Case 1. If $\alpha_1 = 0$, then $\alpha_2 + \alpha_3 = 1$ together with $\alpha_3 \leq \frac{1}{3}$ gives that $\alpha_2 > \frac{1}{3} \geq \alpha_3$, hence $\|\alpha_2^{-1}\alpha_3 u_3\| = \alpha_2^{-1}\alpha_3 < 1$. Then, by Lemma 2 and Theorem 3, $u_2 + \alpha_2^{-1}\alpha_3 u_3$ is invertible in the isotope $\mathcal{J}^{[u_2]}$. Therefore, by Lemma 1, $u_2 + \alpha_2^{-1}\alpha_3 u_3 \in \mathcal{J}_{\text{inv}}$ and hence $x \in \mathcal{J}_{\text{inv}}$, in this case.

Case 2. If $\alpha_1 > 0$, then we put $\delta = \min\{\epsilon, \frac{1}{2}\alpha_1\}$ and let $y = (\alpha_1 - \delta)u_1 + (\alpha_2 + \alpha_3 + \delta)u_2$. Then

$$\text{(II)} \qquad \|x - y\| = \|\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 - ((\alpha_1 - \delta)u_1 + (\alpha_2 + \alpha_3 + \delta)u_2)\|$$

$$\|\delta u_1 - (\alpha_3 + \delta)u_2 + \alpha_3 u_3\| \leq 2(\alpha_3 + \delta) < 4\epsilon$$

since $\alpha_3 < \epsilon$.

Now, we observe that $\alpha_1 > 0$ together with the non-negativity of α_3, positivity of ϵ, the construction of δ and the assumption $\alpha_1 \leq \alpha_2$ gives that $\frac{\alpha_1 - \delta}{\alpha_2 + \alpha_3 + \delta} < 1$. So that $\|\frac{\alpha_1 - \delta}{\alpha_2 + \alpha_3 + \delta} u_1\| = \frac{\alpha_1 - \delta}{\alpha_2 + \alpha_3 + \delta} < 1$. We deduce (as we did in the Case 1), by Lemmas 1, 2 and Theorem 3, that

$$u_2 + \frac{\alpha_1 - \delta}{\alpha_2 + \alpha_3 + \delta} u_1 \in \mathcal{J}_{\text{inv}}\,.$$

Hence, $y \in \mathcal{J}_{\text{inv}}$. This together with (II) implies that $x \in \bar{\mathcal{J}}_{\text{inv}}$. $\square$

Remark 12. Generally, it is not possible to replace $co_{2+}\mathcal{U}(\mathcal{J})$ by $co_2\mathcal{U}(\mathcal{J})$ in the statement (iii) of above Theorem 11. This follows from the fact that any C^*-algebra can be considered as a JB^*-algebra and the illustration given with the C^*-algebra of convergent complex sequences, by Kadison and Pedersen in [2].

Theorem 13. *Let $\mathcal{J}$ be a unital JB^*-algebra and let $x \in \mathcal{J}$ be such that $u(x) = n \geq 3$. Suppose that $x = \sum_{i=1}^n \alpha_i u_i$, where $u_1, \ldots, u_n \in \mathcal{U}(\mathcal{J})$ and $\alpha_1, \ldots, \alpha_n$ are non-negative real numbers with sum equal to 1. Then*

(i) $\alpha_i \leq \alpha_j + \alpha_k$, *(for $j \neq k$);*
(ii) $\frac{1}{n-1} \leq \alpha_j + \alpha_k$, *(for $j \neq k$);*
(iii) $\alpha_j \leq \frac{2}{n+1}$, $\forall j$.

Proof. We may assume that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$. If $\alpha_n > \alpha_1 + \alpha_2$, then

$$\|\alpha_n^{-1}(\alpha_1 u_1 + \alpha_2 u_2\| \leq \alpha_n^{-1}(\|\alpha_1 u_1\| + \|\alpha_2 u_2\|) < 1\,.$$

So, by Lemmas 1, 2 and Theorem 3 (similarly as in the proof of previous Theorem 11), we see that $u_n + \alpha_n^{-1}(\alpha_1 u_1 + \alpha_2 u_2) \in \mathcal{J}_{\text{inv}}$ and hence

$$(\alpha_1 + \alpha_2 + \alpha_n)^{-1}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_n u_n) \in \mathcal{J}_{\text{inv}}$$

such that

$$\|(\alpha_1 + \alpha_2 + \alpha_n)^{-1}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_n u_n)\| \leq 1\,.$$

Therefore, by Lemma 4,

$$(\alpha_1 + \alpha_2 + \alpha_n)^{-1}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_n u_n) = \frac{1}{2}(v_1 + v_2)$$

for some $v_1, v_2 \in \mathcal{U}(\mathcal{J})$. Hence

$$\alpha_1 u_1 + \alpha_2 u_2 + \alpha_n u_n = \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_n)(v_1 + v_2)$$

provides a convex decomposition of x in terms of $n-1$ unitaries in $\mathcal{U}(\mathcal{J})$. This contradicts the hypothesis that $u(x) = n$. This gives (i) as

$$\alpha_i \leq \alpha_n \leq \alpha_1 + \alpha_2 \leq \alpha_j + \alpha_k \quad \text{for all} \quad j \neq k\,.$$

Now, for $j \neq k$, we get from (i) that

$$\begin{aligned} 1 = \sum_{i=1}^{n} \alpha_i &\leq \overbrace{(\alpha_1 + \alpha_2) + (\alpha_1 + \alpha_2) + \cdots + (\alpha_1 + \alpha_2)}^{(n-1)\text{ pairs}} \\ &= (n-1)(\alpha_1 + \alpha_2) \leq (n-1)(\alpha_j + \alpha_k) \end{aligned}$$

since $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$. This gives (ii).

Finally, we see from (i) that

$$\begin{aligned} (n-1)\alpha_n &\leq \overbrace{(\alpha_1 + \alpha_2) + (\alpha_2 + \alpha_3) + \cdots + (\alpha_{n-2} + \alpha_{n-1}) + (\alpha_{n-1} + \alpha_1)}^{(n-1)\text{ pairs}} \\ &= 2(\alpha_1 + \cdots + \alpha_{n-1}) = 2(1 - \alpha_n)\,. \end{aligned}$$

This gives that $\alpha_n \leq \frac{2}{n+1}$. Thus $\alpha_j \leq \alpha_n \leq \frac{2}{n+1}$ for all j. $\square$

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