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SOME PROPERTIES ON THE CLOSED SUBSETS IN BANACH SPACES

ABDELHAKIM MAADEN AND ABDELKADER STOUTI

Abstract. It is shown that under natural assumptions, there exists a linear functional does not have supremum on a closed bounded subset. That is the James Theorem for non-convex bodies. Also, a non-linear version of the Bishop-Phelps Theorem and a geometrical version of the formula of the subdifferential of the sum of two functions are obtained.

1. Introduction

This work is concerned to the study of the James theorem [10] and the Bishop-Phelps theorem [2], [6], [15]. Recall that, R. C. James says that if X is non-reflexive then there exists an element of X^* which does not attains its supremum on the unit ball. We give an obvious generalization of this result for bounded closed subsets (see Proposition 2.2). Moreover, it is an easy consequence of the Hahn-Banach theorem that if a closed convex set C has non-empty interior, then every boundary point of C is a support point of C. But it is not obvious that a non-empty closed convex set with empty interior, has any support points. Even if it does, how many support functionals does it admit? For this, Bishop and Phelps [2], [15] have shown that: both the support points and support functional of C are necessarily dense. A new problem now is: what is the situation in the case when C is not necessarily convex? In this direction we give two positive answers in some particular cases. More precisely we prove that: Let A be a closed bounded subset of a Banach space X, such that $\mathrm{conv}(A)$ is closed, then the set A satisfies the Bishop-Phelps theorem. On the other hand, if we assume that the space X admits a C^1-Fréchet smooth and Lipschitz bump function or has the Radon-Nikodým property, there is a non-linear version of the Bishop-Phelps theorem, that is the set

$$\bigcup_{f\in C^1(X)} \{f'(x); f \text{ attains its supremum at some } x \in S\}$$

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is norm dense in X^* where S is a closed and bounded subset of X and $C^1(X) := \{f : X \longrightarrow \mathbb{R}; f \text{ is a } C^1\text{-Fréchet smooth function}\}$. This is a consequence of the smooth variational principles [4], [5]. As we shall study an application of the smooth variational principle of Deville [4], we give a geometrical version of the formula of the sum of the subdifferential of two functions (see Theorem 2.7).

2. Main results

Recall that James theorem [10] states that for every convex body B in a non-reflexive Banach space X there exists a continuous linear functional $f \in X^*$ such that f does not attains its supremum on B. We give the same result in the case when B is just a closed bounded subset.

For this, we us the same techniques as in [1] and we give the following lemma which is the key of the proof of our results:

Lemma 2.1. *Let $(X, \|\cdot\|)$ be a Banach space. Let A be a closed subset of X. Let $f : X \to R$ be a convex and lower semi-continuous function. Let $C = \overline{\operatorname{conv}(A)}$. Then* $\sup\{f(x); x \in C\} = \sup\{f(x); x \in A\}$.

Proof. We have $A \subset C$, then $\sup_A f \leq \sup_C f =: \alpha$.

Let $\varepsilon > 0$ so, $\alpha > \alpha - (\varepsilon/2)$. Then there exists $c \in C$ such that

$$f(c) \geq \alpha - (\varepsilon/2) \tag{1}$$

We have $c \in \overline{\operatorname{conv}(A)}$ and f is lower semi-continuous. Then there exists a neighborhood U of c and $x \in U \cap \operatorname{conv}(A)$ such that

$$f(c) \leq f(x) + (\varepsilon/2)\ . \tag{2}$$

Combining (1) and (2) we obtain that:

$$\alpha - \varepsilon \leq f(x) \tag{3}$$

In other hand, $x \in \operatorname{conv}(A)$, there exists $(a_i)_{i\leq n}$ in A and $(t_i)_{i\leq n}$ in $[0,1]$ such that $\sum_{i=1}^{n=1} t_i = 1$ and $x = \sum_{i=1}^{n} t_i a_i$. Since f is convex, $f(x) \leq \sum_{i=1}^{n} t_i f(a_i)$.

We confirm that there is $a_i \in A$ such that $f(a_i) \geq \alpha - \varepsilon$. Assume the contrary, then $f(a_i) < \alpha - \varepsilon$ for all i. Thus $f(x) \leq \sum_{i=1}^{n} t_i f(a_i) < \sum_{i=1}^{n} t_i (\alpha - \varepsilon) = (\alpha - \varepsilon)$. Therefore, $f(x) < \alpha - \varepsilon$, a contradiction with (3). Then for all $\varepsilon > 0$ there exists $a \in A$ such that $f(a) \geq \alpha - \varepsilon$, which means that $\sup_A f \geq \alpha$ and the proof of our lemma is complete. □

Now, we are ready to prove a James like theorem for non-convex bodies:

Proposition 2.2. *Let $(X, \|\cdot\|)$ be a non-reflexive Banach space. Let A be a closed and bounded subset of X. Then, there exists a linear functional x^* in X^* such that x^* has no supremum on A.*

Proof. Let $C = \overline{\operatorname{conv}(A)}$. By hypothesis X is non-reflexive then, by James theorem there exists a linear functional x^* in X^* such that x^* has no supremum on C. By Lemma 2.1, we have $\sup_A x^* = \sup_C x^*$. So that, x^* has no supremum on A and the proof is complete. $\square$

Let A be a subset of a Banach space X. Put:

$$M_A = \left\{ x^* \in X^*; \sup_A x^* = \langle x^*, x_0 \rangle \text{ for some } x_0 \in A \right\}.$$

Recall that the Bishop-Phelps theorem [2], [6], [15] says that the set M_A is norm dense in X^* whenever A is a closed, bounded and convex subset of X.

In the case when A is not a convex subset we give the following generalization of the Bishop-Phelps theorem:

Proposition 2.3. *Let $(X, \|\cdot\|)$ be a Banach space. Let A be a closed bounded subset of X such that* $\operatorname{conv}(A)$ *is closed. Then, the set M_A is norm dense in X^*.*

Proof. Let $C = \overline{\operatorname{conv}(A)}$. By the Bishop-Phelps theorem the set M_C is norm dense in X^*. Let $\varepsilon > 0$ and $y^* \in X^*$, then there exist $x_0 \in C$ and $x^* \in M_C$ such that $\langle x^*, x_0 \rangle = \sup_C x^*$ and $\|x^* - y^*\| < \varepsilon$. By Lemma 2.1, we have $\sup_C x^* = \sup_A x^*$.

Let $(t_i)_{1 \le i \le n}$ in $[0,1]$ be such that $\sum_{i=1}^{n} t_i = 1$ and let $(a_i)_{1 \le i \le n}$ in A be such that $x_0 = \sum_{i=1}^{n} t_i a_i$. Then,

$$\langle x^*, x_0 \rangle = \sum_{i=1}^{n} t_i \langle x^*, a_i \rangle \tag{4}$$

Assume that for all $1 \le i \le n$, $\langle x^*, a_i \rangle < \langle x^*, x_0 \rangle$. Then, $\sum_{i=1}^{n} t_i \langle x^*, a_i \rangle < \sum_{i=1}^{n} t_i \langle x^*, x_0 \rangle = \langle x^*, x_0 \rangle$, a contradiction with (4). Hence, there exists $i \in \{1, \ldots, n\}$ such that $\langle x^*, x_i \rangle = \langle x^*, x_0 \rangle$ and x_i is in A. Therefore x^* is in M_A and the proof is complete. $\square$

Now from Proposition 2.3, one can ask that, what happens if conv(A) is not closed. In this direction, we give an other type of the Bishop-Phelps theorem in Banach spaces satisfying some properties. The first result, is in Banach spaces which admit a Fréchet smooth Lipschitzian bump function. This is a consequence of the smooth variational principle of Deville-Godefroy-Zizler [4]. The second result, is in Banach spaces with the Radon-Nikodým property, in this case we us the smooth variational principle of Deville-Maaden [5].

Recall that, a function which is not identically equal to zero and with bounded support is called a bump function.

The smooth variational principle of Deville-Godefroy-Zizler says:

Theorem 2.4. *Let X be a Banach space which admits a C^1-Fréchet smooth, Lipschitz bump function. Then for each $\varepsilon > 0$ and for each lower semi-continuous and bounded below function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ such that f is not identically* *equal to* $+\infty$ *and for each* $x_1 \in X$ *such that* $f(x_1) < \inf_X f + \varepsilon$*, there exists a* C^1*-Fréchet smooth, Lipschitz function* g *such that*

1) $\|g\|_\infty = \sup\{|g(x)|; x \in X\} < \varepsilon$,
2) $\|g'\|_\infty = \sup\{\|g'(x)\|_{X^*}; x \in X\} < \varepsilon$,
3) $f + g$ *has its minimum at some point* $x_0 \in X$,
4) $\|x_1 - x_0\| < \varepsilon$.

A few years ago, it was given in [5] the following smooth variational principle in Radon-Nikodým spaces.

Theorem 2.5. *Let* X *be a Banach space with the Radon-Nikodým property. Then for each* $\varepsilon > 0$ *and for each lower semi-continuous and bounded below function* $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ *not identically equal to* $+\infty$*, for which there exist* $a > 0$ *and* $b \in \mathbb{R}$ *such that* $f(x) \geq 2a\|x\| + b$ *for all* $x \in X$*, there exists a* C^1*-Fréchet smooth function* g *such that*

1) $\|g\|_\infty = \sup\{|g(x)|; x \in X\} < \varepsilon$,
2) $\|g'\|_\infty = \sup\{\|g'(x)\|_{X^*}; x \in X\} < \varepsilon$,
3) g' *is weakly continuous,*
4) $f + g$ *has its minimum at some point* $x_0 \in X$.

These theorems have many consequences in diverse areas of optimization and non-linear analysis. As an example, we prove the existence of many non-trivial normal vectors at many points on the boundary of a closed subset of a Banach space. This is a non-linear version of the Bishop-Phelps theorem:

Theorem 2.6. *Let* X *be a Banach space admitting either* C^1*-Fréchet smooth Lipschitzian bump function or having the Radon-Nikodým property. Let* S *be a closed non-void bounded subset of* X*. Then the set*

$$\bigcup_{f \in C^1(X)} \{f'(x); f \text{ attains its supremum at some } x \in S\}$$

is norm dense in X^*.

Proof. Let $x^* \in X^*$ and let $\varepsilon > 0$. Let

$$g : X \longrightarrow \mathbb{R} \cup \{+\infty\}$$
$$x \longmapsto \begin{cases} < -x^*, x > & \text{if } \ x \in S \\ +\infty & \text{otherwise} \end{cases}$$

It is clear that g is lower semi-continuous, and since S is bounded, then g is bounded below.

Case 1. Suppose that the space X admits a C^1-Fréchet smooth Lipschitzian bump function.

Therefore, by Theorem 2.4, there exists a C^1-Fréchet smooth Lipschitz function $\varphi : X \longrightarrow \mathbb{R}$ such that $\|\varphi\| := \|\varphi\|_\infty + \|\varphi'\|_\infty < \varepsilon$ and $g + \varphi$ has its minimum at some x_0 in S.

Let $h := x^* - \varphi$. Then h attains its supremum at x_0 on S, and we have:

$$\|h'(x) - x^*\|_{X^*} = \| - \varphi'(x)\|_{X^*} < \varepsilon$$

and the proof is complete in this case.

Case 2. Suppose that the space X satisfies the Radon-Nikodým property.

Since S is bounded, there exists $\alpha > 0$ such that $\|x\| \leq \alpha$ for all $x \in S$. Let $\beta = \inf\{< -x^*, x >; x \in S\}$. Therefore, for all $x \in S$, we have

$$\|x\| - \alpha + \beta \leq \beta \leq \langle -x^*, x\rangle$$

which means that

$$\|x\| - \alpha + \beta \leq g(x) \quad \text{for all} \quad x \in X\,.$$

Applying now Theorem 2.5, there exists a C^1-Fréchet smooth function $\varphi : X \longrightarrow \mathbb{R}$ such that $\|\varphi\| := \|\varphi\|_\infty + \|\varphi'\|_\infty < \varepsilon$ and $g + \varphi$ attains its minimum at some x_0 in S.

Let $h := x^* - \varphi$. Then h attains its supremum at x_0 on S, and we have:

$$\|h'(x) - x^*\|_{X^*} = \| - \varphi'(x)\|_{X^*} < \varepsilon\,.$$

Then the proof is complete in this case. □

Another consequence of the smooth variational principle of Deville-Godefroy-Zizler [4] is the following theorem, which can be seen as a geometrical version of the formula of the subdifferential of the sum of two functions [3], [9].

In all the sequel we denote by $N(S, x)$ the set:

$$N(S,x) = \left\{x^* \in X^*; \limsup_{u \to_S x} \frac{\langle x^*, u - x\rangle}{\|u - x\|} \leq 0\right\}$$

the symbol $u \to_S x$ means that $u \to x$ and $u \in S$.

Theorem 2.7. *Let X be a Banach space which admits a C^1-Fréchet smooth, Lipschitz bump function b. Let S_1 and S_2 be two closed subsets of X such that $S_1 \cap S_2 = \emptyset$. Let $\varepsilon > 0$ and $(x_1, x_2) \in S_1 \times S_2$ be such that $\|x_1 - x_2\| < d\,(S_1, S_2) + \varepsilon$. Then there are x_i^*, x_i, $i = 1, 2$, such that:*

1) $x_i \in S_i, i = 1, 2$,
2) $\|x_1 - x_2\| < \varepsilon + d\,(S_1, S_2)$,
3) $x_i^* \in N\,(S_i, x_i)$, $i = 1, 2$,
4) $\|x_1^* + x_2^*\| < \varepsilon$,
5) $1 - (\varepsilon/2) < \|x_i^*\| < 1 + (\varepsilon/2)$, $i = 1, 2$.

Proof. We know that the space X admits a C^1-Fréchet differentiable, Lipschitz bump function. According to a construction of Leduc [11], there exists a Lipschitz function $d : X \longrightarrow \mathbb{R}$ which is C^1-Fréchet smooth on $X \setminus \{0\}$ and satisfies:

i) $d(\lambda x) = \lambda d(x)$ for all $\lambda > 0$ and for all $x \in X$,

ii) there exist $\alpha > 0$ and $\beta > 0$ such that $\alpha\|x\| \leq d(x) \leq \beta\|x\|$ for all $x \in X$.

Without loss of generality, we can assume that $\beta = 1$.

Consider the following function:

$$f : X \times X \longrightarrow \mathbb{R} \cup \{+\infty\}$$

$$(x, y) \longmapsto \begin{cases} d(x-y) & \text{if} \quad (x,y) \in S_1 \times S_2 \\ 0 & \text{otherwise} \end{cases}$$

where d is the Leduc function such that $d(x) \leq \|x\|$ for all $x \in X$. We assume that the norm $\|\cdot\|$ in $X \times X$ satisfies that $\|(x_1, x_2) - (y_1, y_2)\| = \|x_1 - y_1\|_X + \|x_2 - y_2\|_X$. Let $(y_1, y_2) \in S_1 \times S_2$ be such that $\|y_1 - y_2\| \leq d(S_1, S_2) + \varepsilon'$ where ε' is such that $0 < 2\varepsilon' < \varepsilon$. The function f is lower semi-continuous and positive. Thanks to the smooth variational principle of Deville-Godefroy-Zizler (Theorem 2.4), we find $(x_1, x_2) \in S_1 \times S_2$ such that $\|(x_1, x_2) - (y_1, y_2)\| < \varepsilon'$ and $\varphi \in C^1(X \times X)$ with $\|\varphi\| := \|\varphi\|_\infty + \|\varphi'\|_\infty < \varepsilon'$, such that the function $f + \varphi$ attains its minimum at (x_1, x_2), and we have

$$\begin{aligned} \|x_1 - x_2\| &\leq \|x_1 - y_1\| + \|y_1 - y_2\| + \|y_2 - x_2\| \\ &= \|(x_1, x_2) - (y_1, y_2)\|_{X \times X} + \|y_1 - y_2\| \\ &\leq \varepsilon' + d(S_1, S_2) + \varepsilon' \\ &< d(S_1, S_2) + \varepsilon \,. \end{aligned}$$

Let $\psi_2(x) = d(x_1 - x) + \varphi(x_1, x)$. We know that for all $(x, y) \in S_1 \times S_2$:

$$f(x,y) + \varphi(x,y) \geq f(x_1, x_2) + \varphi(x_1, x_2)$$

in particular for $x = x_1 \in S_1$, we obtain that, for all $y \in S_2$

$$\begin{aligned} f(x_1, y) + \varphi(x_1, y) &= d(x_1 - y) + \varphi(x_1, y) \\ &\geq d(x_1 - x_2) + \varphi(x_1, x_2) \,, \end{aligned}$$

which means that:

$$\psi_2(y) \geq \psi_2(x_2) \,, \quad \forall y \in S_2 \,.$$

Since $x_2 \in S_2, x_1 \in S_1$ and $S_1 \cap S_2 = \emptyset, \psi_2$ is Fréchet differentiable at x_2.

Consider now the function $\psi_1(x) = d(x - x_2) + \varphi(x, x_2)$. The same techniques show that ψ_1 is Fréchet differentiable at x_1 and $\psi_1(x) \geq \psi_1(x_1)$ for all $x \in S_1$.

According to one proposition of Ioffe [9]: *Suppose that f is Fréchet differentiable at $x \in S$ and attains a local minimum on S at this point. Then $-f'(x) \in N(S, x)$.*

We deduce that:

$$\left(-\psi_1'(x_1), -\psi_2'(x_2)\right) \in N(S_1, x_1) \times N(S_2, x_2) \,.$$

On the other hand, we have:

$$\psi_1'(x_1) = d'(x_1 - x_2) + \varphi_x'(x_1, x_2)$$

and

$$\psi_2'(x_2) = -d'(x_1 - x_2) + \varphi_y'(x_1, x_2) \,.$$

Therefore $\| - \psi_1'(x_1) - \psi_2'(x_2)\| = \| - \varphi_x'(x_1, x_2) - \varphi_y'(x_1, x_2)\| < 2\varepsilon' < \varepsilon$.

Since $\|\varphi'\|_\infty < \varepsilon'$ and $2\varepsilon' < \varepsilon$, then $1 - (\varepsilon/2) < \|\psi_i'\|_\infty < 1 + (\varepsilon/2)$, $i = 1, 2$ and the proof is complete. $\square$

Remark. Following [13], we say that two closed subsets S_1 and S_2 of a Banach space X generate a extremal system $\{S_1, S_2\}$ if for some $x \in S_1 \cap S_2$ and for any $\delta > 0$ there is $u \in X$ such that $\|u\| < \delta$ and $(S_1 + u) \cap S_2 = \emptyset$, (for more about this property and its characterizations see [7], [14]).

1) One can prove Theorem 2.7 replacing the hypothesis $S_1 \cap S_2 = \emptyset$ by the fact that S_1 and S_2 generate a extremal system $\{S_1, S_2\}$.

2) It is trivial that the space $X = \mathbb{R}^n$ admits a C^1-Fréchet smooth norm. Then X admits a C^1-Fréchet smooth Lipschitz function. So that, Theorem 2.7 holds true if $X = \mathbb{R}^n$. This particular case was proved by Mordukhovich [12] for two closed subsets S_1 and S_2 generate an extremal system $\{S_1, S_2\}$.

3) The existence of smooth norm implies the presence of smooth Lipschitz bump function, but the converse is not true in general. Haydon [8] gives an example of Banach space with smooth Lipschitz bump function but not have an equivalent smooth norm.

We deduce that our theorem holds true if the space X have an equivalent Fréchet smooth norm. This particular case was proved by Ioffe [9] for two closed subsets S_1 and S_2 generate an extremal system $\{S_1, S_2\}$.

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Unité de Recherche Mathématiques et Applications
Faculté des Sciences et Techniques
B.P. 523, Beni - Mellal, Maroc
E-mail: hmaaden2002@yahoo.fr