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A DESCRIPTION OF DERIVATIONS OF THE ALGEBRA OF SYMMETRIC TENSORS

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Abstract. In this paper the symmetric differential and symmetric Lie derivative are introduced. Using these tools derivations of the algebra of symmetric tensors are classified. We also define a Frölicher-Nijenhuis bracket for vector valued symmetric tensors.

1. Introduction

Frölicher and Nijenhuis described in [2] that any derivation on the algebra of differential forms is the sum of an insertion operator $i(\Phi)$ and a Lie derivative $\Theta(\Psi)$ for tangent bundle valued differential forms Φ and Ψ. It was found in [7] that any derivation of the algebra of vector bundle valued differential forms, if a covariant derivative ∇ is fixed, may uniquely be written as $i(\Phi) + \Theta_{\nabla}(\Psi) + \mu(\Xi)$.

Therefore, in parallel to the space of alternating tensors and existing notions for them, similar notions should exist in the space of symmetric tensors. The main purpose of this paper is to define and study the derivations of the algebra of the symmetric tensors similar to those of the algebra of differential forms.

Grozman described all invariant (with respect to the group of all volume preserving diffeomorphisms or with respect to symplectomorphisms) differential operators of the algebra of tensors [4]. Here, the differential operators of the algebra of the symmetric tensors are invariant with respect to the transformations that preserve a certain connection. Hence they are not on Grozman's list.

Manin described the exterior differential in terms of representations of a Grassmann superalgebra [6]. We would like to investigate Manin's description for symmetric differential in a forthcoming paper.

Throughout this paper all connections on the base manifold M will be assumed to be linear and torsion-free.

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2. Symmetric forms

2.1. **Symmetric forms and vector valued symmetric forms.**

Let TM be the tangent space of C^∞-manifold M, and $\bigvee^k(TM)^*$ be the vector bundle of symmetric covariant tensors of degree k over M. The sections of $\bigvee^k(TM)^*$ are called k-symmetric forms and they span a space denoted by $S^k(M)$. The set of all symmetric forms, i.e., $S(M) := \bigoplus_{k\geq 0} S^k(M)$ with the symmetric product $\vee$ given by

$$(\omega \vee \eta)(X_1, \ldots, X_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)})\eta(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)}),$$

where $\omega \in S^k(M)$, $\eta \in S^l(M)$, is a graded algebra.

Let E be a vector bundle on manifold M, the sections of vector bundle $\bigvee^k(TM)^* \bigotimes E$ are called k-symmetric forms with values in E, and denoted by $S^k(M, E)$. The set of all symmetric forms with values in E i.e., $S(M, E) := \bigoplus_{k\geq 0} S^k(M, E)$ with the above product in which $\omega \in S^k(M)$ and $\eta \in S^l(M, E)$ is a (graded) $S(M)$-module.

2.2. **Insertion operator.**

Let $U \in \mathcal{X}(M)$, where $\mathcal{X}(M)$ is the space of vector fields on M. The insertion operator $i_U : S^k(M) \longrightarrow S^{k-1}(M)$ is a linear map given by

$$i_U\omega(X_1, \ldots, X_{k-1}) = \omega(U, X_1, \ldots, X_{k-1}),$$

where $\omega \in S^k(M)$ and $X_1, \ldots, X_{k-1} \in \mathcal{X}(M)$.
This operator can be defined on vector valued symmetric forms as follows:

$$i_U\Phi(X_1, \ldots, X_{k-1}) = \Phi(U, X_1, \ldots, X_{k-1}),$$

where $\Phi \in S^k(M, E)$ and $X_1, \ldots, X_{k-1} \in \mathcal{X}(M)$.
For any simple vector valued symmetric form $\omega \otimes X \in S^k(M, E)$, we have

$$i_U(\omega \otimes X) = (i_U\omega) \otimes X.$$

3. Derivations of the algebra of symmetric forms

A linear mapping $D : S(M) \longrightarrow S(M)$ is said to be of degree k if $D\big(S^l(M)\big) \subset S^{k+l}(M)$, and D is said to be a derivation of degree k if furthermore

$$D(\omega \vee \eta) = D\omega \vee \eta + \omega \vee D\eta \quad \text{for any} \quad \omega, \eta \in S(M).$$

Let $\mathrm{Der}_k\big(S(M)\big)$ be the linear space of all derivations of degree k and let $\mathrm{Der}\big(S(M)\big) = \bigoplus_{k\geq 0} \mathrm{Der}_k\big(S(M)\big)$. A derivation D is called *algebraic*, if $D|_{S^0(M)} = 0$.
If D_1 and D_2 are derivations of degrees k and l, respectively, then $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ is a derivation of degree $k + l$. A derivation is completely determined by its effect on $S^0(M) = C^\infty(M)$ and $S^1(M)$.

3.1. **Insertion of vector valued symmetric forms in symmetric forms.**
For any nonzero $\Phi \in S^{k+1}(M,TM)$, the symmetric insertion operator is the linear map $i(\Phi) : S^l(M) \to S^{k+l}(M)$ homogeneous of degree k, defined by

$$\big(i(\Phi)\omega\big)(X_1,\dots,X_{k+l}) = \frac{1}{(l-1)!(k+1)!}\sum_{\sigma\in S_{k+l}} \omega\big(\Phi(X_{\sigma(1)},\dots,X_{\sigma(k+1)}),X_{\sigma(k+2)},\dots,X_{\sigma(k+l)}\big)\,,$$

where $l \geq 1$, and

$$i(\Phi)f = 0 \quad \text{for any} \quad f \in C^\infty(M)\,.$$

Remark 1. Note that $i(\Phi)\omega = \omega \circ \Phi$ for any $\omega \in S^1(M)$. So, if $i(\Phi) = 0$, then $\Phi = 0$.

Proposition 1. *For any $\eta \otimes U \in S^{k+1}(M,TM)$ and $\omega \in S^l(M)$, we have*

$$i(\eta \otimes U)\omega = \eta \vee i_U\omega\,.$$

Hence $i(\Phi)$ $\big($for $\Phi \in S^k(M,TM)\big)$ is a derivation of degree $k-1$ on $S(M)$.

Remark 2. The insertion of vector valued symmetric forms in a vector valued symmetric form can be defined in a similar way.

Example. Let 1_{TM} be the identity map of TM, then $i(1_{TM})(\omega) = k\omega$, where $\omega \in S^k(M)$.

Lemma 2. *Every algebraic derivation of degree k on $S(M)$ is an insertion of a unique TM-valued $(k+1)$-symmetric form.*

Proof. The proof can be done in a way similar to that of [7, 1.2]. □

3.2. **Nijenhuis-Richardson bracket on symmetric forms.**
Let $\Phi \in S^{k+1}(M,TM)$ and $\Psi \in S^{l+1}(M,TM)$. The bracket $[i(\Phi),i(\Psi)]$ is an algebraic derivation of degree $k+l$. Thus by the Lemma 2 there is a unique $[\Phi,\Psi]^\vee \in S^{k+l+1}(M,TM)$ such that $[i(\Phi),i(\Psi)] = i([\Phi,\Psi]^\vee)$. In view of the bracket $[K,L]^\wedge$ for TM-valued differential forms $K, L \in \Omega(M,TM)$, see [2] and [3], we call $[\Phi,\Psi]^\vee$ Nijenhuis-Richardson bracket of Φ and Ψ. With this bracket $S(M,TM)$ is a Lie superalgebra and it can be deduced that

$$[\Phi,\Psi]^\vee = i(\Phi)\Psi - i(\Psi)\Phi\,.$$

3.3. **The symmetric bracket and symmetric Lie derivative.**
The symmetric bracket was introduced and named *symmetric product* by Crouch [1]. It also arises in the work of Lewis and Murray [5] on a class of mechanical control systems.

Let ∇ be a connection on M. Since $2\nabla_X Y$ is a bilinear map with respect to two vector fields X and Y, it can be written as the sum of its symmetric and antisymmetric parts as follows

$$2\nabla_X Y = (\nabla_X Y + \nabla_Y X) + (\nabla_X Y - \nabla_Y X) = \nabla_X Y + \nabla_Y X + [X,Y]\,.$$

The symmetric bracket of two vector fields X and Y on M is defined and denoted by

$$[X,Y]^s = \nabla_X Y + \nabla_Y X\,.$$

For any $X, Y \in \mathcal{X}(M)$ and $f \in C^\infty(M)$, we have

$$[fX,Y]^s = f[X,Y]^s + Y(f)X\,.$$

The symmetric Lie derivative along a vector field X is the linear map $\theta^s(X) : \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$ and defined by $\theta^s(X)Y = [X,Y]^s$. For $f \in C^\infty(M)$, $\omega \in S^k(M)$, and $X_1,\ldots,X_k \in \mathcal{X}(M)$, we set

$$(\theta^s(X)\omega)(X_1,\ldots,X_k) = X\omega(X_1,\ldots,X_k) - \sum_{i=1}^{k}\omega(X_1,\ldots,\theta^s(X)X_i,\ldots,X_k)\,,$$

and $\theta^s(X)f = X(f)$.
Then, it is obvious that $\theta^s(X) \in \mathrm{Der}_0\big(S(M)\big)$.

Proposition 3. *Let $\theta(X)$ and $\theta^s(X)$ be the Lie derivative and symmetric Lie derivative along the vector field X with respect to connection ∇ on M. Then $2\nabla_X = \theta(X) + \theta^s(X)$.*

Proof. For every $f \in C^\infty(M)$, $\omega \in S^1(M)$, and $Y \in \mathcal{X}(M)$, we have

$$\big(\theta(X) + \theta^s(X)\big)f = 2X(f) = 2\nabla_X f$$

and

$$\begin{aligned}\big(\theta(X) + \theta^s(X)\big)\omega)(Y) &= X\omega(Y) - \omega([X,Y]) + X\omega(Y) - \omega([X,Y]^s)\\ &= 2X\omega(Y) - 2\omega(\nabla_X Y)\\ &= 2\nabla_X\omega(Y)\,.\end{aligned}$$

□

3.4. **The symmetric differential.**
Let ∇ be a connection on M. The symmetric differential is the derivation $d^s : S(M) \longrightarrow S(M)$ of degree 1, defined by

$$\begin{aligned}(d^s\omega)(X_1,\ldots,X_{k+1}) = &\sum_{i=1}^{k+1} X_i\omega(X_1,\ldots,\hat{X}_i,\ldots,X_{k+1})\\ &- \sum_{i<j}\omega([X_i,X_j]^s,X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_{k+1})\end{aligned}$$

where $\omega \in S^k(M)$ and $X_1,\ldots,X_{k+1} \in \mathcal{X}(M)$.
It is easy to verify that unlike the exterior differential, the symmetric differential dose not satisfies $d^s \circ d^s = 0$.

Lemma 4. *Let d^s be the symmetric differential of ∇. Let $\omega \in S^k(M)$ and $X_1,\ldots,X_{k+1} \in \mathcal{X}(M)$. Then*

i) $(d^s\omega)(X_1,\ldots,X_{k+1}) = \sum_{i=1}^{k+1}(\nabla_{X_i}\omega)(X_1,\ldots,\hat{X}_i,\ldots,X_{k+1})$;
ii) $d^s\omega = (k+1)Symm(\nabla\omega) = \frac{1}{k!}\sum_{\sigma\in S_{k+1}}(\nabla\omega)(X_{\sigma(1)},\ldots,X_{\sigma(k+1)})$;

iii) $d^s\omega = \sum_{i=1}^n \omega^i \vee \nabla_{E_i}\omega$;
where $\{E_i\}_{i=1}^n$ *is a local base of vector fields and* $\{\omega_i\}_{i=1}^n$ *is its dual base.*

Proposition 5. *Let* (M, g) *be a Riemmanian manifold with Levi-Civita connection* ∇. *The* 1*-form* ω *is Killing if and only if* $d^s\omega = 0$. (*Recall that* ω *is Killing if the vector field* $\omega^\sharp$ *is Killing, where* $g(\omega^\sharp, X) = \omega(X)$*; recall also that a vector field* X *is said to be Killing one if* $L_X g = 0$.)

Proof. It suffices to show that $d^s\omega = L_{\omega^\sharp} g$. For X and Y in $\mathcal{X}(M)$, we have

$$\begin{aligned}(d^s\omega)(X, Y) &= Xg(\omega^\sharp, Y) + Yg(\omega^\sharp, X) - g(\omega^\sharp, \nabla_X Y) - g(\omega^\sharp, \nabla_Y X)\\ &= g(\nabla_X\omega^\sharp, Y) + g(X, \nabla_Y\omega^\sharp)\\ &= L_{\omega^\sharp} g(X, Y)\,,\end{aligned}$$
□

It is well-known that if ∇ and $\bar{\nabla}$ are two torsion-free connections on M, then there exists a $\Phi \in S^2(M, TM)$ such that $\bar{\nabla} = \nabla + \Phi$.

Lemma 6. *Let* ∇ *and* $\bar{\nabla}$ *be two torsion-free connections with symmetric differentials* d^s *and* $\bar{d}^s$ *respectively and* $\bar{\nabla} = \nabla + \Phi$, *for* $\Phi \in S^2(M, TM)$. *Then* $\bar{d}^s = d^s - 2i(\Phi)$.

Proof. Since, $d^s f = \bar{d}^s f = df$ for any $f \in C^\infty(M)$, it suffices to prove the relation for all 1-forms. If X, Y are two vector fields on M, then

$$\begin{aligned}\bar{d}^s\omega(X, Y) \;&=\; X\omega(Y) + Y\omega(X) - \omega(\bar{\nabla}_X Y + \bar{\nabla}_Y X)\\ &=\; X\omega(Y) + Y\omega(X) - \omega(\nabla_X Y + \nabla_Y X) - 2\omega\big(\Phi(X, Y)\big)\\ &=\; (d^s\omega)(X, Y) - 2\big(i(\Phi)\omega\big)(X, Y)\,.\end{aligned}$$
□

The next result was proved in [11]. Here we prove it in a different way.

Proposition 7 (Proposition 2.4 of [11])**.** *Every derivation of degree* 1 *on symmetric forms, whose value for a function is the differential of the function, is of the form of a symmetric differential of a connection which is also unique.*

Proof. Let D be a derivation of degree 1 on symmetric forms, ∇ an arbitrary connection and d^s its symmetric differential on the manifold M. Since, $D - d^s$ is an algebraic derivation, there exists a unique $\Phi \in S^2(M, TM)$ such that $D - d^s = i(\Phi)$. Now set $\bar{\nabla} = \nabla - \frac{1}{2}\Phi$ and let $\bar{d}^s$ be its symmetric differential, then $D = \bar{d}^s$. If $\tilde{\nabla}$ is a connection with symmetric differential $\tilde{d}^s = D$, then there exists $\Psi \in S^2(M, TM)$, such that $\tilde{\nabla} = \bar{\nabla} + \Psi$. According to Lemma 6 we have $\tilde{d}^s = \bar{d}^s - 2i(\Psi)$. So $2i(\Psi) = 0$ and as a result Ψ=0. □

Proposition 8. *Let* ∇ *be a connection on* M *with symmetric differential* d^s *and let* X *be a vector field. We have* $[i_X, d^s] = \theta^s(X)$.

Proof. For $f \in C^\infty(M)$, we have

$$[i_X, d^s](f) = i_X d^s f - d^s(i_X f) = X(f) = \theta^s(X)f.$$

For any $\omega \in S^1(M), Y \in \mathcal{X}(M)$, we also have

$$\begin{aligned}\big([i_X, d^s](w)\big)(Y) &= \big(i_X(d^s\omega)\big)(Y) - \big(d^s(i_X\omega)\big)(Y)\\ &= d^s\omega(X,Y) - Y\big(\omega(X)\big)\\ &= X\omega(Y) - \omega([X,Y]^s)\\ &= \big(\theta^s(X)\omega\big)(Y).\end{aligned}$$

□

By considering a fixed connection on M, we can define the symmetric Lie derivative along a TM-valued symmetric forms. For any $\Phi \in S(M,TM)$, we put

$$\theta^s(\Phi) = [i(\Phi), d^s]\,.$$

Proposition 9. *Let $\eta \otimes X$ be a simple symmetric form. For any $\omega \in S(M)$, we have*

$$\theta^s(\eta \otimes X)\omega = \eta \vee \theta^s(X)\omega - d^s\eta \vee i_X\omega.$$

Proof. We compute as follows

$$\begin{aligned}\theta^s(\eta\otimes X)\omega &= \big(i(\eta\otimes X)\circ d^s\big)\omega - \big(d^s\circ i(\eta\otimes X)\big)\omega\\ &= \eta\vee i_X d^s\omega - d^s(\eta\vee i_X\omega)\\ &= \eta\vee i_X d^s\omega - d^s\eta\vee i_X\omega - \eta\vee d^s i_X\omega\\ &= \eta\vee(i_X d^s - d^s i_X)\omega - d^s\eta\vee i_X\omega\\ &= \eta\vee\theta^s(X)\omega - d^s\eta\vee i_X\omega\,.\end{aligned}$$

□

Lemma 10. *Let Φ and Ψ be two symmetric forms. If $\theta^s(\Phi)f = \theta^s(\Psi)f$ for every $f \in C^\infty(M)$, then $\Phi = \Psi$.*

Proof. The assertion easily follows from $\theta^s(\Phi)f = df \circ \Phi$.

Lemma 11. *Let ∇ and $\bar{\nabla}$ be two connections having symmetric Lie derivatives $\theta^s(\Phi)$ and $\bar{\theta}^s(\Phi)$ along $\Phi \in S^k(M,TM)$ on M, and $\bar{\nabla} = \nabla + \Psi$ for some $\Psi \in S^2(M,TM)$. Then*

$$\bar{\theta}^s(\Phi) = \theta^s(\Phi) - 2i([\Phi,\Psi]^\vee).$$

Proof. Use Lemma 6. □

Theorem 12. *Let ∇ be a connection on M. Every derivation $D \in \mathrm{Der}_k\big(S(M)\big)$ can be uniquely written in the form $D = i(\Phi)+\theta^s(\Psi)$, for some $\Phi \in S^{k+1}(M,TM)$ and $\Psi \in S^k(M,TM)$. Moreover, Ψ is independent of ∇.*

Proof. Let $X_1,\dots,X_k \in \mathcal{X}(M)$. Then $f \longmapsto (Df)(X_1,\dots,X_k)$ is a derivation of degree 0 on $C^\infty(M)$, so it is given by the action of a vector field $\Psi(X_1,\dots,X_k)$, which is symmetric and $C^\infty(M)$-linear in X_j, thus $\Psi \in S^k(M,TM)$. Then $D - \theta^s(\Psi)$ is algebraic, therefore equals $i(\Phi)$ for some $\Phi \in S^{k+1}(M,TM)$.
We are now in a position to show uniqueness of Φ and Ψ. Let $i(\Phi) + \theta^s(\Psi) = i(\Phi') + \theta^s(\Psi')$ for some Φ' and Ψ'. Applying this relation to functions, Lemma 10 yields $\Psi = \Psi'$. By Remark 1 we get $\Phi = \Phi'$.
Using Lemma 11 and uniqueness of Ψ implies that Ψ is independent of ∇. □

Corollary 13. *D is algebraic if and only if $\Psi = 0$.*

Corollary 14. *$D = d^s$ if and only if $\Psi = 1_{TM}$.*

3.5. **Frölicher-Nijenhuis bracket on symmetric forms.**
In the remainder of this paper let us fix a connection on M. Let $\Phi \in S^k(M, TM)$ and $\Psi \in S^l(M, TM)$. Then $[\theta^s(\Phi), \theta^s(\Psi)]$ is a derivation of degree $k+l$ on $S(M)$. By Theorem 12 there exist unique $\Theta \in S^{k+l}(M, TM)$ and $\Omega \in S^{k+l+1}(M, TM)$ such that $[\theta^s(\Phi), \theta^s(\Psi)] = i(\Omega) + \theta^s(\Theta)$. We define the Frölicher-Nijenhuis bracket of symmetric forms Φ and Ψ to be

$$[\Phi, \Psi] = \Theta\,.$$

Proposition 15. *Let $\Phi = \phi \otimes X$ and $\Psi = \psi \otimes Y$ be two simple symmetric forms. Then*

$$\begin{aligned}[\phi \otimes X, \psi \otimes Y] = \phi \vee \psi \otimes [X, Y] + \phi \vee (\theta^s(X)\psi) \otimes Y - (\theta^s(Y)\phi) \vee \psi \otimes X \\ - d^s\phi \vee i_X\psi \otimes Y + d^s\psi \vee i_Y\phi \otimes X.\end{aligned}$$

Proof. For simplicity we denote the right hand side of the above relation by Θ. It suffices to show that $[\theta^s(\phi \otimes X), \theta^s(\psi \otimes Y)]f = \theta^s(\Theta)f$ for any $f \in C^\infty(M)$:

$$\begin{aligned}[\theta^s(\phi \otimes X), \theta^s(\psi \otimes Y)]f\theta^s(\phi \otimes X) \circ \theta^s(\psi \otimes Y)(f) - \theta^s(\psi \otimes Y) \circ \theta^s(\phi \otimes X)(f) \\ = \theta^s(\phi \otimes X)(Y(f)\psi) - \theta^s(\psi \otimes Y)(X(f)\phi) \\ = \phi \vee \theta(X)(Y(f)\psi) - d^s\phi \vee i_X(Y(f)\psi) \\ - \psi \vee \theta^s(Y)(X(f)\phi) + d^s\psi \vee i_Y(X(f)\phi) \\ = [X, Y](f)\phi \vee \psi + Y(f)\phi \vee \theta(X)\psi - Y(f)d^s\phi \vee i_X\psi \\ - X(f)\psi \vee \theta^s(Y)\phi + X(f)d^s\psi \vee i_Y\phi \\ = \theta^s(\Theta)f\,.\end{aligned}$$

□

Remark 3. Note that the operators: symmetric bracket, symmetric differential, Frölicher-Nijenhuis bracket on symmetric forms are not invariant with respect to the group of all volume preserving diffeomorphisms or with respect to symplectomorphisms. Hence they are not be covered in Grozman's list [4].

4. Derivations of the algebra of vector valued symmetric forms

Let E be a vector bundle on M. A derivation of degree k of $S(M, E)$ is a linear mapping $D : S(M, E) \longrightarrow S(M, E)$ with $D\big(S^l(M, E)\big) \subset S^{k+l}(M, E)$ such that for any $\omega \in S(M)$ and $\Phi \in S(M, E)$, we have

$$D(\omega \vee \Phi) = \overline{D}(\omega) \vee \Phi + \omega \vee D(\Phi),$$

where $\overline{D} : S(M) \longrightarrow S(M)$ is a linear mapping.

Lemma 16. *$\overline{D}$ is uniquely determined by D and is a derivation of degree k on $S(M)$.*

Proof. The action of $S(M)$ on $S(M,E)$ is effective: $\omega \vee \Phi = \eta \vee \Phi$ for all $\Phi \in S(M,E)$ implies $\omega = \eta$. The assertion follows easily from the derivation property of D. $\square$

A derivation D of $S(M,E)$ is called *algebraic* if the associated derivation $\overline{D}$ of $S(M)$ is algebraic. Thus $D(f\Phi) = fD(\Phi)$ for any $f \in C^\infty(M)$ and $\Phi \in S(M,E)$. Therefore D is of tensorial character.

Suppose E_p is the fiber of E at $p \in M$. The bundle of linear endomorphisms of fibers of E (i.e., $L(E,E)$) naturally acts on E

$$\begin{aligned} L(E,E)_p \times E_p &\longrightarrow \ E_p \\ (\Xi_p, \Phi_p) &\longmapsto \Xi_p(\Phi_p) \end{aligned}$$

induces a symmetric product between $S(M,L(E,E))$ and $S(M,E)$. Namely, for $\Xi \in S\big(M, L(E,E)\big)$, we define

$$\begin{aligned} \mu(\Xi) : S(M,E) &\longrightarrow S(M,E) \\ \Phi \ &\longmapsto \Xi \vee \Phi \,. \end{aligned}$$

More precisely,

$$\big(\mu(\Xi)\Phi\big)(X_1,\ldots,X_{k+l}) = \frac{1}{k!l!}\sum_\sigma \Xi(X_{\sigma(1)},\ldots,X_{\sigma(k)})\big(\Phi(X_{\sigma(k+1)},\ldots,X_{\sigma(k+l)})\big)\,.$$

Thus $\mu(\Xi)$ for $\Xi \in S^k\big(M, L(E,E)\big)$ is a derivation of degree k of $S(M,E)$ and $\overline{\mu(\Xi)} = 0$.

The following results are analogs of Theorems 3.4, 3.8, 4.7, Lemmas 3.2, 4.3, 4.4, and Corollary 3.3 of [7]. Their proofs can be done with proper modifications in the arguments of [7].

Lemma 17. *Let D be a derivation of degree k on $S(M,E)$ with $\overline{D} = 0$. Then there exists a unique Ξ in $S^k\big(M, L(E,E)\big)$ such that $D = \mu(\Xi)$.*

So the graded $S(M)$-module endomorphisms of $S(M,E)$ are exactly of the form $\mu(\Xi)$.

Corollary 18. *If D is an algebraic derivation of $S(M,E)$ of degree k, then there exist unique $\Psi \in S^{k+1}(M,TM)$ and $\Xi \in S^k\big(M, L(E,E)\big)$ such that $D = i(\Psi) + \mu(\Xi)$.*

Proposition 19. *The space $\operatorname{Der} S(M,E)$ of derivations of the $S(M)$-module $S(M,E)$ is a Lie algebra with the commutator as bracket. We have $\overline{[D_1,D_2]} = [\overline{D}_1, \overline{D}_2]$.*

Let ∇ be a connection on E, and denote by d^{∇^s} the exterior symmetric covariant derivative $S^k(M,E) \longrightarrow S^{k+1}(M,E)$, given by

$$\begin{aligned} (d^{\nabla^s}\Phi)(X_1,\ldots,X_{k+1}) = &\sum_i \nabla_{X_i}\Phi(X_1,\ldots,\hat{X}_i,\ldots,X_{k+1}) \\ &- \sum_{i<j} \Phi([X_i,X_j]^s, X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_{k+1})\,. \end{aligned}$$

It is a graded derivation of degree 1 and

$$d^{\nabla^s}(\omega \vee \Phi) = d^s\omega \vee \Phi + \omega \vee d^{\nabla^s}\Phi$$

for any $\omega \in S(M)$ and $\Phi \in S(M, E)$.

Let us fix a connection ∇ on E. Then, for any $\Phi \in S^k(M, TM)$, we consider the derivation $\theta^s_\nabla(\Phi) = [i(\Phi), d^{\nabla^s}]$ of degree k on $S(M, E)$, which we call the covariant symmetric Lie derivative along Φ. Clearly $\overline{\theta^s_\nabla(\Phi)} = \theta^s(\Phi)$ on $S(M)$.

Theorem 20. *If d^{∇^s} is an exterior symmetric covariant derivative on E, then any derivation D in $\mathrm{Der}_k\, S(M, E)$ can be written in the form $D = i(\Phi) + \theta^s_\nabla(\Psi) + \mu(\Xi)$ for unique $\Phi \in S^{k+1}(M, TM)$, $\Psi \in S^k(M, TM)$ and $\Xi \in S^k\big(M, L(E, E)\big)$. Moreover, D is algebraic if and only if $\Psi = 0$.*

5. Derivations on the algebra of symmetric forms into the module of vector valued symmetric forms

Let E be a vector bundle on M. A linear mapping D from $S(M)$ into $S(M)$-module $S(M, E)$ is called a derivation of degree k if $D\big(S^l(M)\big) \subset S^{k+l}(M, E)$ and $D(\omega \vee \eta) = D\omega \vee \eta + \omega \vee D\eta$ for any $\omega \in S(M)$ and $\eta \in S(M)$. The space of all derivations of degree k from $S(M)$ into $S(M, E)$ is denoted by $\mathrm{Der}_k\,\big(S(M), S(M, E)\big)$. Let D' be a derivation of degree l on $S(M, E)$. Then $[D', D] = D' \circ D - D \circ \overline{D'}$ is a derivation of degree $k + l$ from $S(M)$ into $S(M, E)$.

For any $\Phi \otimes T \in S^{k+1}(M, TM \otimes E)$, where $\Phi \in S^{k+1}(M, TM)$ and $T \in \Gamma E$, we define $\rho(\Phi \otimes T) \in \mathrm{Der}_k\,\big(S(M), S(M, E)\big)$ by setting

$$\rho(\Phi \otimes T)\omega = i(\Phi)\omega \otimes T \quad \text{for any} \quad \omega \in S(M)\,.$$

We extend ρ linearly to $S^{k+1}(M, TM \otimes E)$.

Lemma 21. *Every algebraic derivation from $S(M)$ into $S(M, E)$ is in the form $\rho(\Xi)$, for a unique $\Xi \in S^{k+1}(M, TM \otimes E)$.*

Let ∇ be a connection on E. For $\Xi \in S^{k+1}(M, TM \otimes E)$, we call the derivation $\theta^{\nabla^s}(\Xi) := [\rho(\Xi), d^{\nabla^s}]$ the E-valued covariant symmetric Lie derivative along Ξ.

For any $\Phi \in S^{k+1}(M, TM)$ and $\Xi \in S^{l+1}(M, TM \otimes E)$, the derivation $[i(\Phi), \rho(\Xi)]$ is algebraic, so it is of the form $\rho([\Phi, \Xi]^\vee)$ for unique $[\Phi, \Xi]^\vee \in S^{k+l+1}(M, TM \otimes E)$.

Proposition 22. *We have $[\Phi, \Xi]^\vee = i(\Phi)\Xi - \rho(\Xi)\Phi$, where $\rho(\Xi)(\phi \otimes X) = \rho(\Xi)\phi \otimes X$.*

Theorem 23. *Let D be a derivation from $S(M)$ into $S(M, E)$. Then there exist unique $\Xi \in S^{k+1}(M, TM \otimes E)$ and $\Pi \in S^k(M, TM \otimes E)$ such that $D = \rho(\Xi) + \theta^{\nabla^s}(\Pi)$.*

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