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HILLE-WINTNER TYPE COMPARISON CRITERIA FOR HALF-LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. We establish Hille-Wintner type comparison criteria for the half-linear second order differential equation

$$\big(r(t)\Phi(x')\big)' + c(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-2}x\,,\ p > 1\,,$$

where this equation is viewed as a perturbation of another equation of the same form.

1. Introduction

In this paper we deal with the half-linear second order differential equation

$$\big(r(t)\Phi(x')\big)' + c(t)\Phi(x) = 0\,, \tag{1}$$

where $\Phi(x) := |x|^{p-1}\operatorname{sgn} x$, $p > 1$, and r, c are continuous functions, $r(t) > 0$.

It is well known that the oscillation theory of (1) is very similar to that of the second order Sturm-Liouville linear equation (which is the special case $p = 2$ in (1))

$$(r(t)x')' + c(t)x = 0\,.$$

In particular, the Sturm comparison and separation theorems extend verbatim to (1), see, e.g [1, Chap. 3] and [3]. This means that (1) can be classified as oscillatory or nonoscillatory according to whether any nontrivial solution of (1) has or does not have infinitely many zeros on any interval of the form $[T, \infty)$.

In the classical oscillation criteria for half-linear equations, equation (1) is viewed as a perturbation of the one-term differential equation

$$(r(t)\Phi(x'))' = 0 \tag{2}$$

and (non)oscillation criteria are formulated in terms of the asymptotic properties of the function c for large t with respect to the function r. A typical example is

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the Leighton-Wintner type oscillation criterion which states that (1) is oscillatory provided

$$\int^{\infty} r^{1-q}(t)\,dt = \infty \quad \text{and} \quad \int^{\infty} c(t)\,dt = \infty\,,$$

where q is a conjugate number of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$.

The classical Sturm comparison theorem compares the pair of equations with coefficients c, r and C, R *pointwise*, while Hille-Wintner type criteria compare *integrals*. More precisely, together with (1) consider the equation

$$\big(r(t)\Phi(x')\big)' + C(t)\Phi(x) = 0\,. \tag{3}$$

In the case when $\int^{\infty} r^{1-q}(t)\,dt = \infty$ and the integral $\int^{\infty} c(t)\,dt$ converges, the half-linear version of the Hille-Wintner type comparison theorem says that if

$$0 \le \int_t^{\infty} c(s)\,ds \le \int_t^{\infty} C(s)\,ds \quad \text{for large } t \tag{4}$$

and (3) is nonoscillatory, then (1) is nonoscillatory as well, see [10] and also [3, p. 206]. Concerning the complementary case $\int^{\infty} r^{1-q}(t)\,dt < \infty$ (which is treated in [11]), denote $\rho(t) := \int_t^{\infty} r^{1-q}(s)\,ds$ and suppose that $c(t) \ge 0$, $C(t) \ge 0$ for large t. If

$$\int_t^{\infty} c(s)\rho^p(s)\,ds \le \int_t^{\infty} C(s)\rho^p(s)\,ds < \infty \tag{5}$$

for large t, then nonoscillation of (3) implies that of (1).

In this paper we follow the idea introduced in [2, 4, 5] and applied e.g. in [13, 14]. We investigate (1) not as a perturbation of one-term equation (2), but as a perturbation of the general (nonoscillatory) equation of the same form as (1)

$$\big(r(t)\Phi(x')\big)' + \tilde{c}(t)\Phi(x) = 0\,. \tag{6}$$

We compare oscillatory properties of (1) and (3) under the assumption

$$0 \le \int_t^{\infty} \big(c(s) - \tilde{c}(s)\big)h^p(s)\,ds \le \int_t^{\infty} \big(C(s) - \tilde{c}(s)\big)h^p(s)\,ds < \infty\,, \tag{7}$$

where h is the so-called principal solution of (6). If $\tilde{c}(t) \equiv 0$, then this principal solution is either $h(t) \equiv 1$ or $h(t) = \rho(t)$, depending on the divergence/convergence of the integral $\int^{\infty} r^{1-q}(t)\,dt$. Consequently, (7) reduces to (4) or (5) if $\tilde{c}(t) \equiv 0$.

2. Preliminaries

In this section we first point out the relationship between nonoscillation of equation (1) and solvability of the Riccati type first order differential equation

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0\,. \tag{8}$$

Let x be a solution of (1), then the function $w = r\Phi(x'/x)$ solves the Riccati equation (8) and it is well known (see [3, p. 171]) that equation (1) is nonoscillatory if and only if there exists a solution of (8) on some interval of the form $[T, \infty)$.

Next we recall half-linear version of the so-called Picone's identity (see [9] or [3, p. 172]), which, in a modified form as needed in our paper, reads as follows. Let w be a solution of (8), then for any $x \in C^1$

$$r(t)|x'|^p - c(t)|x|^p = (w(t)|x|^p)' + pr^{1-q}(t)P\big(r^{q-1}(t)x', \Phi(x)w(t)\big)\,, \tag{9}$$

where

$$P(u,v) := \frac{|u|^p}{p} - uv + \frac{|v|^q}{q} \ge 0 \tag{10}$$

with the equality $P(u,v) = 0$ if and only if $v = \Phi(u)$.

Concerning the function P, we will need its quadratic estimates which are given in the next statement whose proof can be found e.g. in [6].

Lemma 1. *The function $P(u,v)$ defined in* (10) *satisfies the following inequalities*

$$P(u,v) \ge \frac{1}{2}|u|^{2-p}\big(v - \Phi(u)\big)^2 \quad \text{for} \quad p \le 2\,,$$
$$P(u,v) \le \frac{1}{2}|u|^{2-p}\big(v - \Phi(u)\big)^2 \quad \text{for} \quad p \ge 2\,,\ u \ne 0\,.$$

Futhermore, let $T > 0$ be arbitrary. There exists a constant $K = K(T) > 0$ such that

$$P(u,v) \ge K|u|^{2-p}\big(v - \Phi(u)\big)^2 \quad \text{for} \quad p \ge 2$$
$$P(u,v) \le K|u|^{2-p}\big(v - \Phi(u)\big)^2 \quad \text{for} \quad p \le 2\,,$$

and every $u, v \in \mathbb{R}$ satisfying $\left|\frac{v}{\Phi(u)}\right| \le T$.

Now we derive the so-called modified Riccati equation which plays the crucial role in the proof of our main result. Let $x \in C^1$ be any function and w be a solution of the Riccati equation (8). Then from Picone's identity (9) we have

$$(w|x|^p)' = r|x'|^p - c|x|^p - pr^{1-q}|x|^pP(\Phi^{-1}(w_x), w), \tag{11}$$

where $w_x = r\Phi(x'/x)$ and Φ^{-1} is the inverse function of Φ. At the same time, let h be a (positive) solution of (6) and $w_h = r\Phi(h'/h)$ be the solution of the Riccati equation associated with (6), then

$$(w_h|x|^p)' = r|x'|^p - \tilde{c}|x|^p - pr^{1-q}|x|^pP\big(\Phi^{-1}(w_x, w_h)\big)\,. \tag{12}$$

Substituting $x = h$ into (11), (12) and subtracting these equalities we get the equation (in view of the identity $P(\Phi^{-1}(w_h), w_h) = 0$)

$$((w - w_h)h^p)' + (c - \tilde{c})h^p + pr^{1-q}h^pP(\Phi^{-1}(w_h), w) = 0\,. \tag{13}$$

Observe that if $\tilde{c}(t) \equiv 0$ and $h(t) \equiv 1$, then (13) reduces to (8) and this is also the reason why we call this equation the *modified Riccati equation.*

Finally, let us recall the concept of the principal solution of nonoscillatory equation (1) is introduced by Mirzov in [12] and later independently by Elbert and Kusano in [7]. If (1) is nonoscillatory, as mentioned at the beginning of this section, there exists a solution w of Riccati equation (8) which is defined on some interval $[T, \infty)$. It can be shown that among all solutions of (8) there exists the *minimal* one $\tilde{w}$ (sometimes called the *distinguished* solution), minimal in the sense that any other solution of (8) satisfies the inequality $w(t) > \tilde{w}(t)$ for large t. Then the principal solution of (1) is given by the formula

$$\tilde{x} = K \exp\left\{\int^t r^{1-q}(s)\Phi^{-1}\big(\tilde{w}(s)\big)\, ds\right\},$$

i.e., the principal solution $\tilde{x}$ of (1) is a solution which "produces" the minimal solution $\tilde{w} = r\Phi(\tilde{x}'/\tilde{x})$ of (8).

3. Hille-Wintner type comparison theorem

The main result of our paper is the following statement.

Theorem 1. *Let* $\int^\infty r^{1-q}(t)\, dt = \infty$. *Suppose that equation* (6) *is nonoscillatory and possesses a positive principal solution* h *such that there exist a finite limit*

$$\lim_{t\to\infty} r(t)h(t)\Phi\big(h'(t)\big) =: L > 0 \tag{14}$$

and

$$\int^\infty \frac{dt}{r(t)h^2(t)\big(h'(t)\big)^{p-2}} = \infty\,. \tag{15}$$

Further suppose that $0 \le \int_t^\infty C(s)\, ds < \infty$ *and*

$$0 \le \int_t^\infty \big(c(s) - \tilde{c}(s)\big)h^p(s)\, ds \le \int_t^\infty \big(C(s) - \tilde{c}(s)\big)h^p(s)\, ds < \infty\,, \tag{16}$$

all for large t. *If equation* (3) *is nonoscillatory, then* (1) *is also nonoscillatory.*

Proof. As we have already mentioned before, to prove that (1) is nonoscillatory, it is sufficient to find a solution of associated Riccati equation (8) which is defined on some interval $[T, \infty)$. This solution we will find (using the Schauder-Tychonov theorem) as a fixed point of a suitably constructed integral operator.

By our assumption, equation (3) is nonoscillatory, i.e., there exists an eventually positive principal solution x of this equation. Denote by $w := r\Phi(x'/x)$ the solution of the associated Riccati equation

$$w' + C(t) + (p-1)r^{1-q}(t)|w|^q = 0\,.$$

From the previous section, with (1) replaced by (3), i.e., with c replaced by C, we know that the modified Riccati equation

$$((w - w_h)h^p)' + (C - \tilde{c})h^p + pr^{1-q}h^pP(\Phi^{-1}(w_h), w) = 0$$

holds, where h is the principal solution of (6) and $w_h = r\Phi(h'/h)$ is the minimal solution of the Riccati equation corresponding to equation (6). By integrating we get

$$h^p(w_h - w)|_T^t = \int_T^t \big(C(s) - \tilde{c}(s)\big)h^p(s)\, ds + p\int_T^t r^{1-q}(s)P\big(r^{q-1}h', w\Phi(h)\big)\, ds\,. \tag{17}$$

Since $\int^\infty r^{1-q}(t)\,dt = \infty$ and $0 \le \int_t^\infty C(s)\,ds < \infty$, w solves also the integral Riccati equation (see [3, p. 207])

$$w(t) = \int_t^\infty C(s)\,ds + (p-1)\int_t^\infty r^{1-q}(s)|w(s)|^q\,ds,$$

and therefore $w(t) \ge 0$ for large t. Hence

$$h^p(w_h - w)|_T^t \le h^p w_h(t) + h^p\big(w(T) - w_h(T)\big)$$

and letting $t \to \infty$ in (17) we have (with L given by (14))

$$\begin{aligned} L + h^p\big(w(T) - w_h(T)\big) \ge & \int_T^\infty \big(C(s) - \tilde c(s)\big)h^p(s)\,ds \\ & + p\int_T^\infty r^{1-q}(s)P\big(r^{q-1}h', w\Phi(h)\big)\,ds\,. \end{aligned}$$

Since $P(u,v) \ge 0$ and (16) holds, this means that

$$\int^\infty r^{1-q}(t)P\big(r^{q-1}(t)h'(t), w(t)\Phi\big(h(t)\big)\big)\,dt < \infty\,. \tag{18}$$

Now, since (14), (16), (18) hold, from (17) it follows that there exists a finite limit

$$\lim_{t\to\infty} h^p(t)\big(w(t) - w_h(t)\big) =: \beta$$

and also the limit

$$\lim_{t\to\infty}\frac{w(t)}{w_h(t)} = \lim_{t\to\infty}\frac{h^p(t)w(t)}{h^p(t)w_h(t)} = \frac{L+\beta}{L}\,. \tag{19}$$

Therefore, letting $t \to \infty$ in (17) and then replacing T by t, we get the equation

$$\begin{aligned} h^p(t)\big(w(t) - w_h(t)\big) - \beta = & \int_t^\infty \big(C(s) - \tilde c(s)\big)h^p(s)\,ds \\ & + p\int_t^\infty r^{1-q}(s)P\big(r^{q-1}h', w\Phi(h)\big)\,ds\,. \end{aligned} \tag{20}$$

Since (19) holds, according to Lemma 1 there exists a positive constant K such that

$$K|\Phi^{-1}(w_h)|^{2-p}(w - w_h)^2 \le P\big(\Phi^{-1}(w_h), w\big)\,,$$

and hence

$$Kr^{1-q}h^p w_h^{q-2}(w - w_h)^2 \le r^{1-q}h^pP\big(\Phi^{-1}(w_h), w\big) = r^{1-q}P\big(r^{q-1}h', w\Phi(h)\big)\,.$$

Now, using the fact that $w_h^{q-2} = r^{q-2}(h')^{2-p}h^{p-2}$, we get the inequality

$$\begin{aligned} & \frac{K}{r(t)h^2(t)\big(h'(t)\big)^{p-2}}\big[\big(w(t) - w_h(t)\big)h^p(t)\big]^2 \\ & \qquad \le r^{1-q}(t)P\big(r^{q-1}(t)h'(t), w(t)\Phi(t)\big)\,. \end{aligned} \tag{21}$$

Denote $G(t) = r^{-1}(t)h^{-2}(t)\big(h'(t)\big)^{2-p}$, then the last inequality after integrating over $[T,\infty)$ reads

$$K\int_T^\infty G(t)\big[\big(w(t) - w_h(t)\big)h^p(t)\big]^2\,dt \le \int_T^\infty r^{1-q}(t)P\big(r^{q-1}(t)h'(t), w(t)\Phi(h(t)\big)\,dt\,.$$

By (15) we have $\int^t G(s)\,ds \to \infty$ as $t \to \infty$. This implies that $\beta = \lim_{t\to\infty} h^p(t)\big(w(t)-w_h(t)\big) = 0$ since if $\beta \neq 0$, we have

$$\int^\infty G(t)\big[\big(w(t)-w_h(t)\big)h^p(t)\big]^2\,dt = \infty\,,$$

which, in view of (21), implies that $\int^\infty r^{1-q}P\big(r^{q-1}h', w\Phi(h)\big)\,dt = \infty$ and this contradicts (18). Consequently from (20), we get the integral equation

$$\begin{aligned} h^p(t)\big(w(t)-w_h(t)\big) = &\int_t^\infty \big(C(s)-\tilde c(s)\big)h^p(s)\,ds \\ &+\ p\int_t^\infty r^{1-q}(s)P\big(r^{q-1}h', w\Phi(h)\big)\,ds\,, \end{aligned} \tag{22}$$

and this equation we use in constructing the integral operator whose fixed point is a solution of (8) which we are looking for.

Define the function set U and the mapping F by

$$U = \{u \in C[T,\infty) : w_h(t) \le u(t) \le w(t) \text{ for } t \in [T,\infty)\}\,,$$

where T is sufficiently large,

$$\begin{aligned} F(u)(t) = w_h(t) + h^{-p}(t)\bigg\{&\int_t^\infty \big(c(s)-\tilde c(s)\big)h^p(s)\,ds \\ &+ p\int_t^\infty r^{1-q}(s)h^p(s)P\big(\Phi^{-1}(w_h), u\big)\,ds\bigg\} \end{aligned}$$

Observe that the set U is well defined since $w(t) \ge w_h(t)$ for large t by (16) and (22). Obviously, U is a convex and closed subset of the Frechet space $C[T,\infty)$ with the topology of the uniform convergence on compact subintervals of $[T,\infty)$. Denote $H(s) := \frac{|s|^q}{q} - \Phi^{-1}(w_h)s$. Then $H'(s) = \Phi^{-1}(s) - \Phi^{-1}(w_h) \ge 0$ for $s \ge w_h$. This means that $P(\Phi^{-1}(w_h), u)$ is nondecreasing in the second variable and hence if $w_h(t) \le u_1(t) \le u_2(t) \le w(t)$, $t \in [T,\infty)$, we have $F(u_1)(t) \le F(u_2)(t)$ for $t \in [T,\infty)$.

Next we show that F maps U into itself. To this end, it is sufficient to show that $w_h(t) \le F(w_h)(t) \le F(u)(t) \le F(w)(t) \le w(t)$ for large t. We have

$$F(w_h)(t) = w_h(t) + h^{-p}(t)\left\{\int_t^\infty \big(c(s)-\tilde c(s)\big)h^p(s)\,ds\right\} \ge w_h(t)$$

and, at the same time, using (16) and (22) (suppressing the argument t)

$$\begin{aligned} F(w) &= w_h + h^{-p}\left\{\int_t^\infty (c-\tilde c)h^p + p\int_t^\infty r^{1-q}h^pP(\Phi^{-1}(w_h), w)\right\} \\ &\le w_h + h^{-p}\left\{\int_t^\infty (C-\tilde c)h^p + p\int_t^\infty r^{1-q}h^pP\big(\Phi^{-1}(w_h), w\big)\right\} \\ &= w\,. \end{aligned}$$

Let $T_1 > T$ be arbitrary. As $w_h(t) \le F(u)(t) \le w(t)$ for $u \in U$ and w_h, w exist on the whole interval $[T,\infty)$, the set $F(U)|_{[T,T_1]}$ is bounded. Next we show that this set is also uniformly continuous. Let $u \in U$ be arbitrary, $\varepsilon > 0$, and $t_1, t_2 \in [T, T_1]$, without a loss of generality we may suppose that $t_1 < t_2$. Denote

$$f(t) := \big(c(t) - \tilde{c}(t)\big)h^p(t) + pr^{1-q}(t)h^p(t)P\big(\Phi^{-1}\big(w_h(t)\big), u(t)\big)\,,$$

then by the monotonicity of P in the second argument

$$\int_T^\infty f(s)\,ds \le \int_T^\infty \big[c(t) - \tilde{c}(t))h^p(t) + pr^{1-q}(t)P\big(\Phi^{-1}(w_h(t)), w(t)\big)\big]\;dt =: R$$

and hence

$$\begin{aligned}
|F(u)(t_2) - F(u)(t_1)| &\le |w_h(t_2) - w_h(t_1)| \\
&\quad + \left|h^{-p}(t_2)\int_{t_2}^\infty f(s)\,ds - h^{-p}(t_1)\int_{t_1}^\infty f(s)\,ds\right| \\
&= |w_h(t_2) - w_h(t_1)| + \left|h^{-p}(t_2)\int_{t_2}^\infty f(s)\,ds - h^{-p}(t_1)\int_{t_2}^\infty f(s)\,ds\right. \\
&\quad \left. + h^{-p}(t_1)\int_{t_2}^\infty f(s)\,ds - h^{-p}(t_1)\int_{t_1}^\infty f(s)\,ds\right| \\
&\le |w_h(t_2) - w_h(t_1)| + |h^{-p}(t_2) - h^{-p}(t_1)|\int_{t_2}^\infty f(s)\,ds + h^{-p}(t_1)\int_{t_1}^{t_2} f(s)\,ds \\
&\le |w_h(t_2) - w_h(t_1)| + |h^{-p}(t_2) - h^{-p}(t_1)|\int_{T}^\infty f(s)\,ds + h^{-p}(t_1)\int_{t_1}^{t_2} f(s)\,ds
\end{aligned}$$

Since w_h is continuous, there exists δ_1 such that $|w_h(t_2) - w_h(t_1)| < \frac{\varepsilon}{3}$ provided $|t_2 - t_1| < \delta_1$. Similarly, as h^{-p} is continuous, there exists δ_2 such that $|h^{-p}(t_2) - h^{-p}(t_1)| < \frac{\varepsilon}{3R}$ if $|t_2 - t_1| < \delta_2$. Finally, for $\tilde{R} := \sup_{t \in [T,T_1]} h^{-p}(t)$ there exists δ_3 such that $\int_{t_1}^{t_2} f(s)\,ds < \frac{\varepsilon}{3\tilde{R}}$ provided $|t_2 - t_1| < \delta_3$.

Altogether,

$$|F(u)(t_2) - F(u)(t_1)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3R}R + \tilde{R}\frac{\varepsilon}{3\tilde{R}} = \varepsilon$$

if $|t_2 - t_1| < \min\{\delta_1, \delta_2, \delta_3\}$. Hence $F(U)|_{[T,T_1]}$ is uniformly continuous.

It is obvious that F is a continuous mapping and using the Arzela-Ascoli theorem, $F(U)$ is relatively compact subset of $C[T, \infty)$. Now, from the Schauder-Tychonov fixed point theorem follows that there exists $v \in U$ such that $v = F(v)$. Hence v satisfies the modified Riccati integral equation

$$h^p(t)\big(v(t) - w_h(t)\big) = \int_t^\infty (c(s) - \tilde{c}(s)h^p(s)\,ds + p\int_t^\infty r^{1-q}(s)P\big(r^{q-1}h', v\Phi(h)\big)\,ds.$$

By differentiating one can see that v satisfies the modified Riccati equation (13) and hence v solves also (8). This implies that equation (1) is nonoscillatory and the proof is complete. □

As an immediate consequence of the previous theorem we have the following statement.

Corollary 1. *Let the assumptions of Theorem 1 be satisfied. Then the oscillation of equation* (1) *implies that of* (6).

Corollary 2. *Let $r(t) \equiv 1$, $\tilde c = \frac{\tilde\gamma}{t^p}$, where $\tilde\gamma = \left(\frac{p-1}{p}\right)^p$, i.e., (6) is the generalized Euler equation with the critical coefficient*

$$\big(\Phi(y')\big)' + \frac{\tilde\gamma}{t^p}\Phi(y) = 0\,. \tag{23}$$

If equation (3) is nonoscillatory, $\int_t^\infty C(s)\,ds \ge 0$ for large t, and

$$0 \le \int_t^\infty \Big(c(s) - \frac{\tilde\gamma}{s^p}\Big)s^{p-1}\,ds \ \le \int_t^\infty \Big(C(s) - \frac{\tilde\gamma}{s^p}\Big)s^{p-1}\,ds < \infty \tag{24}$$

for large t, then (1) is also nonoscillatory.

Proof. The function $h(t) = t^{\frac{p-1}{p}}$ is the principal solution of (23) (see [8]),

$$\lim_{t\to\infty} h(t)\Phi(h'(t)) = \lim_{t\to\infty} t^{\frac{p-1}{p}}\Big(\frac{p-1}{p}t^{-\frac{1}{p}}\Big)^{p-1} = \Big(\frac{p-1}{p}\Big)^{p-1},$$

and

$$\int^\infty \frac{dt}{h^2(t)(h'(t))^{p-2}} = \Big(\frac{p}{p-1}\Big)^{p-2}\int^\infty \frac{dt}{t} = \infty\,.$$

Since all remaining assumptions of Theorem 1 are obviously satisfied, the statement follows from this theorem. ☐

Remark 1. (i) The assumptions $\int^\infty r^{1-q}(t)\,dt = \infty$ and (14), (15) are used in the proof of Theorem 1 to prove that $\frac{w(t)}{w_h(t)} \to 1$ as $t\to\infty$ and this fact is then used in the quadratization of the function P and the proof that $w(t) > w_h(t)$ for large t. It is an open question whether Theorem 1 can be modified in such a way that it remains to hold without these assumptions. Also, the assumption of convergence of the integrals $\int^\infty C(t)\,ds$, $\int^\infty \tilde c(t)\,dt$ is natural in view of the Leighton-Wintner oscillation criterion mentioned at the beginning of the paper since equation (3) and (6) are supposed to be nonoscillatory.

(ii) If $\tilde c(t) \equiv 0$, no function of the form P appears in the proof of Hille-Wintner type theorem (this proof follows essentially the linear case, see [3, p. 171]) and hence this statement can be proved without assumptions (14), (15). If we suppose that $\int^\infty r^{1-q}(t)\,dt = \infty$, then $h(t) \equiv 1$ is the principal solution of one-term equation (2), i.e., $w_h \equiv 0$ and the assumption $\int_t^\infty c(s)\,ds \ge 0$ (see (4)) ensures that the minimal solution of (8) satisfies $w(t) \ge 0$. This means that the crucial requirement $w(t) > w_h(t)$ (to construct the set U) is satisfied without assuming (14). A similar situation we have if $\int^\infty r^{1-q}(t)\,dt < \infty$. Then $h(t) = \int_t^\infty r^{1-q}(s)\,ds$ is the principal solution of (2) and $w_h(t) < 0$. The inequality $w(t) > w_h(t)$ is then ensured by the assumption $c(t) \ge 0$ since this assumption implies $w(t) > w_h(t)$ by the comparison theorem for minimal solutions of Riccati-type equations, see [3, p. 234].

(iii) In Corollary 2 we have used Euler equation (23) as "unperturbed" equation (6). Another example of the nonoscillatory equation which can be used at this place is the half-linear Euler-Weber differential equation (an alternative terminology is Riemann-Weber equation, see [15])

$$(\Phi(x'))' + \Big[\frac{\tilde\gamma}{t^p} + \frac{\hat\gamma}{t^p\log^2 t}\Big]\Phi(x) = 0, \quad \hat\gamma := \frac{1}{2}\Big(\frac{p-1}{p}\Big)^{p-1}. \tag{25}$$

However, the principal solution of this equation is not known explicitly and only its asymptotic estimate is known, see [8, 15]. This fact suggests the idea to replace the assumption that h is a principal solution of (6) by the assumption that h is a function close to this solution, in a certain sense. This idea is a subject of the present investigation.

(iv) The fact that equation (25) is nonoscillatory suggests a specification of Corollary 2, namely, we will take $C(t) = t^{-p}\left[\tilde{\gamma} t + \hat{\gamma}\log^{-2} t\right]$ in this statement. Then we get the following statement which is a modification of [4, Theorem 2].

Corollary 3. *Suppose that*

$$0 \le \int_t^\infty \Big(c(s) - \frac{\tilde{\gamma}}{s^p}\Big) s^{p-1}\, ds < \infty$$

for large t*. If*

$$\log t \int_t^\infty \Big(c(s) - \frac{\tilde{\gamma}}{s^p}\Big) s^{p-1}\, ds \le \hat{\gamma} = \frac{1}{2}\Big(\frac{p-1}{p}\Big)^{p-1}$$

for large t*, then equation* (1) *is nonoscillatory.*

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