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ON UNIVERSALITY OF SEMIGROUP VARIETIES

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To Professor Jiří Rosický on his 60th birthday

Abstract. A category $\mathbb{K}$ is called α-determined if every set of non-isomorphic $\mathbb{K}$-objects such that their endomorphism monoids are isomorphic has a cardinality less than α. A quasivariety $\mathbb{Q}$ is called Q-universal if the lattice of all subquasivarieties of any quasivariety of finite type is a homomorphic image of a sublattice of the lattice of all subquasivarieties of $\mathbb{Q}$. We say that a variety $\mathbb{V}$ is var-relatively alg-universal if there exists a proper subvariety $\mathbb{W}$ of $\mathbb{V}$ such that homomorphisms of $\mathbb{V}$ whose image does not belong to $\mathbb{W}$ contains a full subcategory isomorphic to the category of all graphs. A semigroup variety $\mathbb{V}$ is nearly $\mathcal{J}$-trivial if for every semigroup $\mathbf{S} \in \mathbb{V}$ any $\mathcal{J}$-class containing a group is a singleton. We prove that for a nearly $\mathcal{J}$-trivial variety $\mathbb{V}$ the following are equivalent: $\mathbb{V}$ is Q-universal; $\mathbb{V}$ is var-relatively alg-universal; $\mathbb{V}$ is α-determined for no cardinal α; $\mathbb{V}$ contains at least one of the three specific semigroups. Dually, for a nearly $\mathcal{J}$-trivial variety $\mathbb{V}$ the following are equivalent: $\mathbb{V}$ is 3-determined; $\mathbb{V}$ is not var-relatively alg-universal; the lattice of all subquasivarieties of $\mathbb{V}$ is finite; $\mathbb{V}$ is a subvariety of one of two special finitely generated varieties.

1. Introduction

Questions about the representative power of algebraic structures determined by a given category are quite interesting in general. Here we concentrate on algebraic structures determined by morphisms of a semigroup variety. Adams and Dziobiak [4] surprisingly connected this topic with the representative power of the lattice of all subquasivarieties. The aim of this paper is to develop these connections for semigroup varieties. We continue on the results by Adams and Dziobiak [5], Sapir [30, 31, 32] and the authors of this paper [9, 10, 11].

First, let us recall notions describing algebraic structures determined by morphisms. For a category $\mathbb{K}$, let $\mathrm{End}(A)$ denote the endomorphism monoid of a $\mathbb{K}$--object A (i.e. $\mathrm{End}(A)$ consists of all endomorphisms of A with the operation

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composition and the identity morphism). We say that $\mathbb{K}$-objects A and B are *equimorphic* if $\mathrm{End}(A)$ and $\mathrm{End}(B)$ are isomorphic. The category $\mathbb{K}$ is called *α-determined* (where α is a cardinal) if every set of non-isomorphic equimorphic $\mathbb{K}$-objects has cardinality less than α.

The opposite property is an alg-universality. A concrete category $\mathbb{K}$ is called *alg-universal* if there exists a full embedding F of the category $\mathbb{GR}$ into $\mathbb{K}$ ($\mathbb{GR}$ consists of all undirected graphs and their homomorphisms). If, moreover, $F\mathbf{G}$ is *finite* – i.e., underlying set of $F\mathbf{G}$ is finite – for every finite graph $\mathbf{G}$, then $\mathbb{K}$ is *ff-alg-universal.* If for every monoid $\mathbf{M}$ there exists a $\mathbb{K}$-object A such that $\mathbf{M}$ and $\mathrm{End}(A)$ are isomorphic then we say that $\mathbb{K}$ is *monoid-universal.* Hedrlín and Pultr proved

Theorem 1.1 [19, 20, 28]**.** *Every alg-universal category is monoid universal. If, moreover, $\mathbb{K}$ is ff-alg-universal then for every finite monoid $\mathbf{M}$ there exists a finite $\mathbb{K}$-object A such that $\mathbf{M}$ and $\mathrm{End}(A)$ are isomorphic.*

Rosický [29] constructed complete, cocomplete, well powered and co-well powered concrete monoid universal category that it is not alg-universal.

Many examples of alg-universal categories and their basic properties are presented in the monograph [28] by Pultr and Trnková.

Hedrlín-Sichler theorem says that the properties of determinacy and alg-universality are mutually exclusive. In fact

Theorem 1.2 [20, 28]**.** *If $\mathbb{K}$ is an alg-universal category then for every monoid $\mathbf{M}$ there exists a proper class of non-isomorphic $\mathbb{K}$-objects with endomorphism monoids isomorphic to $\mathbf{M}$.*

Thus any alg-universal category is α-determined for no cardinal α. Hence alg-universal categories and α-determined categories are on the opposite ends of a spectrum of representative power of categories.

For many categories the proof that they are not alg-universal is based on the existence of 'trivial' morphisms. For example, the variety of lattices or the variety of semilattices are not alg-universal because any constant mapping between lattices or semilattices is a homomorphism. This motivates the following definitions. We say that a class I of $\mathbb{K}$-morphisms is an *ideal* if $f \circ g \in I$ whenever $f \in I$ or $g \in I$. Let $\mathcal{C}$ be a class of $\mathbb{K}$-objects then a $\mathbb{K}$-morphism $f : A \to B$ *factorizes through* $\mathcal{C}$ if there exist $C \in \mathcal{C}$ and $\mathbb{K}$-morphisms $g : A \to C$ and $h : C \to B$ with $f = h \circ g$. Clearly, the class $I(\mathcal{C})$ of all $\mathbb{K}$-morphisms factorizing through $\mathcal{C}$ is an ideal. We say that a functor $F : \mathbb{L} \to \mathbb{K}$ is an *I-relatively full embedding* for an ideal I in $\mathbb{K}$ if

F is faithful;
$Ff \notin I$ for all $\mathbb{L}$-morphisms f;
if $f : FA \to FB$ is a $\mathbb{K}$-morphism for $\mathbb{L}$-objects A and B then either $f = Fg$ for some $\mathbb{L}$-morphism $g : A \to B$ or $f \in I$.

A concrete category $\mathbb{K}$ is *I-relatively alg-universal* if for a given ideal I of $\mathbb{K}$ there exists an I-relatively full embedding $F : \mathbb{GR} \to \mathbb{K}$. If, moreover, $F\mathbf{G}$ is finite for every finite graph then we say that $\mathbb{K}$ is *I-relatively ff-alg-universal.* Observe that a category $\mathbb{K}$ is alg-universal (or ff-alg-universal) if and only if $\mathbb{K}$ is I-relatively alg-universal (or I-relatively ff-alg-universal) for the empty ideal I.

We recall that a category of algebraic systems of the same type and their homomorphisms is a *variety* (or a *quasivariety*) if it is closed under all products, subsystems and homomorphic images (or all products, subsystems and ultraproducts, respectively). Any variety is a quasivariety, but the converse does not hold. For a class $\mathcal{C}$ of algebraic systems of the same type, let $\mathrm{Var}\,(\mathcal{C})$ denote the least variety containing $\mathcal{C}$ and $\mathrm{Qua}(\mathcal{C})$ denote the least quasivariety containing $\mathcal{C}$. We modify relative alg-universality for varieties and quasivarieties of algebraic systems. We say that a variety (or quasivariety) $\mathbb{V}$ is $\mathbb{W}$-*relatively alg-universal* where $\mathbb{W}$ is a proper subvariety of $\mathbb{V}$ if $\mathbb{V}$ is $I(\mathbb{W})$-relatively alg-universal. A variety or a quasivariety $\mathbb{V}$ is called *var-relatively alg-universal* if it is $\mathbb{W}$-relatively alg-universal for some proper subvariety $\mathbb{W}$ of $\mathbb{V}$. Analogously we can define $\mathbb{W}$-*relatively ff-alg-universality* and *var-relatively ff-alg-universality*. Many var-relatively alg-universal varieties are α-determined for no cardinal α. On the other hand, if we combine results from [26] and [27] we obtain a finitely generated $I(\mathcal{W})$-relatively ff-alg-universal variety $\mathbb{V}$ of dp-algebras that is α-determined for some finite cardinal α where $\mathcal{W}$ is the union of proper subvarieties of $\mathbb{V}$. Relationship of var-relatively alg-universality to determinacy remains an open problem.

Many papers are devoted to the lattice of subvarieties or the lattice of subquasivarieties. Thus for a quasivariety $\mathbb{Q}$, let $\mathrm{QL}(\mathbb{Q})$ be the lattice of all subquasivarieties of $\mathbb{Q}$ and for a variety $\mathbb{V}$, let $\mathrm{VL}(\mathbb{V})$ be the lattice of all subvarieties of $\mathbb{V}$. The properties of $\mathrm{QL}(\mathbb{Q})$ were examined for many quasivarieties $\mathbb{Q}$. One of the most interesting problems are lattice identities satisfied by $\mathrm{QL}(\mathbb{Q})$ for a concrete quasivariety $\mathbb{Q}$. This motivated Sapir [32] to define the notion of Q-universality. A quasivariety $\mathbb{Q}$ of finite type is called *Q-universal* if $\mathrm{QL}(\mathbb{R})$ is a homomorphic image of a sublattice of $\mathrm{QL}(\mathbb{Q})$ for every quasivariety $\mathbb{R}$ of algebraic systems of finite type. It is known that if $\mathbb{Q}$ is Q-universal then the free lattice over a countable set is isomorphic to a sublattice of $\mathrm{QL}(\mathbb{Q})$ (thus $\mathrm{QL}(\mathbb{Q})$ satisfies no non-trivial lattice identity) and the size of $\mathrm{QL}(\mathbb{Q})$ is $2^{\aleph_0}$. Sapir [32] proved that the variety of all three-nilpotent commutative semigroups is Q-universal. Let $P(\omega_0)$ denote the set of all finite subsets of natural numbers. Dziobiak proved

Theorem 1.3 [12, 13]**.** *If a quasivariety $\mathbb{Q}$ of algebraic systems of finite type contains a family $\{\mathbf{A}_X \mid X \in P(\omega_0)\}$ of finite systems such that*

(P1) *$\mathbf{A}_\emptyset$ is a terminal algebraic system of $\mathbb{Q}$;*
(P2) *if $X, Y, Z \in P(\omega_0)$ with $X = Y \cup Z$ then $\mathbf{A}_X \in \mathrm{Qua}\{\mathbf{A}_Y, \mathbf{A}_Z\}$;*
(P3) *if $X, Y \in P(\omega_0)$ with $X \neq \emptyset$ and $\mathbf{A}_X \in \mathrm{Qua}\{\mathbf{A}_Y\}$ then $X = Y$;*
(P4) *if $X \in P(\omega_0)$ is such that $\mathbf{A}_X$ is an algebraic subsystem of $\mathbf{B} \times \mathbf{C}$ for finite algebraic systems $\mathbf{B}$ and $\mathbf{C}$ from $\mathrm{Qua}\{\mathbf{A}_Y \mid Y \in \mathcal{F}\}$ for a finite $\mathcal{F} \subseteq P(\omega_0)$, then there exist $Y, Z \in P(\omega_0)$ such that if $Y \neq \emptyset$ then $\mathbf{A}_Y$ is an algebraic subsystem of $\mathbf{B}$, if $Z \neq \emptyset$ then $\mathbf{A}_Z$ is an algebraic subsystem of $\mathbf{C}$ and $X = Y \cup Z$,*

then there exists a sublattice of $\mathrm{QL}(\mathbb{Q})$ isomorphic to the lattice of all ideals in the free lattice over a countably infinite set.

Adams and Dziobiak strengthened this result by proving

Theorem 1.4 [2]. *If a quasivariety $\mathbb{Q}$ of finite type contains a family $\{\mathbf{A}_X \mid X \in P(\omega_0)\}$ of finite algebraic systems satisfying the conditions* (P1), (P2), (P3) *and* (P4) *then $\mathbb{Q}$ is Q-universal.*

The result below by Adams and Dziobiak connects alg-universality to Q-universality.

Theorem 1.5 [4]. *If a quasivariety $\mathbb{Q}$ of finite type is ff-alg-universal then it is also Q-universal. In fact, the lattice* $\mathrm{QL}(\mathbb{Q})$ *contains a sublattice isomorphic to the lattice of all ideals of the free lattice over a countably infinite set.*

In many instances the proof that a quasivariety $\mathbb{Q}$ of finite type is var-relatively alg-universal can be modified to a proof that $\mathbb{Q}$ is Q-universal (precisely that $\mathbb{Q}$ contains a family $\{\mathbf{A}_X \mid X \in P(\omega_0)\}$ of finite algebraic systems satisfying the conditions (P1), (P2), (P3) and (P4)). But Koubek and Sichler [27] constructed a finitely generated $I(\mathcal{W})$-relatively ff-alg-universal variety of dp-algebras where $\mathcal{W}$ is the union of proper subvarieties that is not Q-universal.

In [3], Adams and Dziobiak gave also a tool for proving that a quasivariety is not Q-universal. They called a finite algebraic system $\mathbf{A}$ *critical* if

$$\mathbf{A} \notin \mathrm{Qua}\{\mathbf{B} \mid \mathbf{B} \text{ is a proper subsystems of } \mathbf{A}\}.$$

Theorem 1.6 [3]. *Let $\mathbb{Q}$ be a locally finite quasivariety of finite type. If $\mathbb{Q}$ contains only finitely many non-isomorphic critical algebras then $\mathbb{Q}$ is not Q-universal.*

A summary of results concerning Q-universality is given in the survey paper by Adams, Adaricheva, Dziobiak and Kravchenko [1].

We are interested in the relationship of the above notions for semigroup varieties. Hedrlín and Lambek [18, 28] proved that a variety of all semigroups is alg-universal (clearly, it is not ff-alg universal). Koubek and Sichler [25] characterized alg-universal varieties of semigroups. Schein [33] proved that the variety of semilattices is 3-determined and then he generalized this result [34] showing that the variety of normal bands is 5-determined. Sapir [32] proved that the variety of commutative three nilpotent semigroups is Q-universal, in [30] Sapir characterized semigroup varieties $\mathbb{V}$ with finite $\mathrm{QL}(\mathbb{V})$ and in [31] Sapir characterized semigroup varieties $\mathbb{V}$ with countable $\mathrm{QL}(\mathbb{V})$. These results motivate the following open problems:

(a) Characterize var-relatively alg-universal semigroup varieties.
(b) Characterize Q-universal semigroup varieties.
(c) Characterize semigroup varieties that are α-determined for some cardinal α.

The aim of this paper is to contribute to a solution of these problems. We believe that their solution would clarify a connection between var-relatively alg--universality, Q-universality and determinacy.

We recall that a semigroup $\mathbf{S}$ is

$\mathcal{J}$-*trivial* if every $\mathcal{J}$-class of $\mathbf{S}$ is a singleton;
a *band* if every element of $\mathbf{S}$ is idempotent (i.e. $s^2 = s$ for all $s \in S$).

For a semigroup $\mathbf{S}$, let us denote $r(\mathbf{S})$ the union of all subgroups of $\mathbf{S}$. A semigroup variety $\mathbb{V}$ is called *nearly $\mathcal{J}$-trivial* if for every semigroup $\mathbf{S} \in \mathbb{V}$ any $\mathcal{J}$-class of $\mathbf{S}$ having the non-empty intersection with $r(\mathbf{S})$ is a singleton. Let $\mathbb{S}$ denote the variety of all semigroups and $\mathbb{B}$ the variety of all bands.

The specific aim of this paper is to solve problems (a), (b) and (c) for nearly $\mathcal{J}$-trivial semigroup varieties. The second section recalls the results concerning of the band varieties and generalizes these results to semigroup varieties consisting of inflations of bands. The third section investigates the bottom of the lattice $\mathrm{VL}(\mathbb{S})$ and describes the location of nearly $\mathcal{J}$-trivial varieties in $\mathrm{VL}(\mathbb{S})$. For this purpose we recall results about varieties generated by special semigroups $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_3$ (these are defined in the third section). We define another special semigroup $\mathbf{M}_4$ and the main result of this section is that a nearly $\mathcal{J}$-trivial semigroup variety $\mathbb{V}$ either contains one of the semigroups $\mathbf{M}_2$, $\mathbf{M}_4$, $\mathbf{M}_4^{op}$ or $\mathbb{V} \subseteq \mathrm{Var}\,(\mathbf{M}_1)$ or $\mathbb{V} \subseteq \mathrm{Var}\,(\mathbf{M}_1^{op})$. The last section solves problems (a), (b) and (c) for the variety $\mathrm{Var}\,(\mathbf{M}_4)$ (and by the duality also for the variety $\mathrm{Var}\,(\mathbf{M}_4^{op})$). This solves our problems for nearly $\mathcal{J}$-trivial semigroup varieties. Main results of this paper are summarized in Theorems 5.1 and 5.2 in Conclusion, where we also suggest a direction of further research.

Finally, we would like to turn the attention to the following three open general problems:

(d) Is there a var-relatively alg-universal variety (or quasivariety) that is α-determined for some cardinal α? Or is any var-relatively alg-universal variety α-determined for no cardinal α?

(e) Does there exist a Q-universal quasivariety $\mathbb{Q}$ such that no sublattice of $\mathrm{QL}(\mathbb{Q})$ is isomorphic to the lattice of all ideals of the free lattice over an infinite countable set? Or does there exist a Q-universal quasivariety $\mathbb{Q}$ such that no family $\{\mathbf{A}_X \mid X \in P(\omega_0)\}$ of finite algebraic systems from $\mathbb{Q}$ satisfies the conditions (P1), (P2), (P3), and (P4)?

(f) Does there exist a monoid universal variety that is not alg-universal?

We hope that resolving problems (a), (b) and (c) will help in understanding and solution of the latter three problems.

2. Varieties consisting of inflations of bands

The lattice $\mathrm{VL}(\mathbb{B})$ of all band varieties was described independently by Birjukov [6], Fennemore [14] and Gerhard [15]. The bottom of $\mathrm{VL}(\mathbb{B})$ is shown in Figure 1, all connecting lines there indicate covers. The varieties are determined by the associative and idempotent identities and the identity in the brackets.

We recall the terminology of band varieties:

semigroups in the variety $\mathbb{SL}$ are called *semilattices*;
semigroups in the variety $\mathbb{LZ}$ (or $\mathbb{RZ}$) are called *left-zero semigroups* (or *right-zero semigroups*);
semigroups in the variety $\mathbb{RCB}$ are called *rectangular bands*;
semigroups in the variety $\mathbb{LNB}$ (or $\mathbb{RNB}$) are called *left normal bands* (or *right normal bands*);

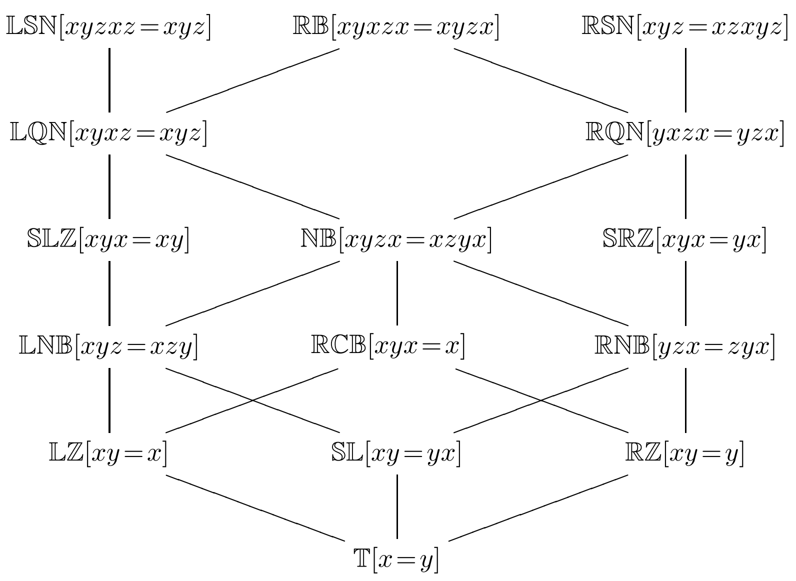

FIG. 1. The bottom of the lattice $\mathrm{VL}(\mathbb{B})$.

semigroups in the variety $\mathbb{NB}$ are called *normal bands*;
semigroups in the variety $\mathbb{SLZ}$ (or $\mathbb{SRZ}$) are called *semilattices of left-zero semigroups* (or *semilattices of right-zero semigroups*);
semigroups in the variety $\mathbb{LQN}$ (or $\mathbb{RQN}$) are called *left quasi-normal bands* (or *right quasi-normal bands*);
semigroups in the variety $\mathbb{RB}$ are called *regular bands*;
semigroups in the variety $\mathbb{LSN}$ (or $\mathbb{RSN}$) are called *left semi-normal bands* (or *right semi-normal bands*).

Next, we recall known results concerning varieties of bands.

Theorem 2.1. *The variety $\mathbb{SLZ}$ of semilattices of left-zero semigroups and the variety $\mathbb{SRZ}$ of semilattices of right-zero senigroups are* 3*-determined. The variety $\mathbb{RNB}$ of right quasi-normal bands and the variety $\mathbb{LNB}$ of left quasi-normal bands are* 5*-determined, see* [8]*.*

The variety $\mathbb{V}$ of bands is var-relatively alg-universal if and only if $\mathbb{LSN} \subseteq \mathbb{V}$ *or* $\mathbb{RSN} \subseteq \mathbb{V}$*, see* [9]*.*

The variety $\mathbb{LSN}$ of all left semi-normal bands and the variety $\mathbb{RSN}$ of all right semi-normal bands are Q-universal, see [5]*.*

For normal bands, the lattice $\mathrm{QL}(\mathbb{NB})$ *is finite and thus $\mathbb{NB}$ is not Q-universal, see* [16]*.*

The cardinality of $\mathrm{QL}(\mathbb{V})$ *is not countable for a band variety $\mathbb{V}$ if and only if* $\mathbb{SLZ} \subseteq \mathbb{V}$ *or* $\mathbb{SRZ} \subseteq \mathbb{V}$*, see* [30, 31]*.*

The announcement of Sapir's result that the variety $\mathbb{LQN}$ of all left quasinormal bands and the variety $\mathbb{RNB}$ of all right quasinormal bands are Q-universal can be found in [5].

We recall that atoms in the lattice $\mathrm{VL}(\mathbb{S})$ are the band varieties $\mathbb{SL}$, $\mathbb{LZ}$ and $\mathbb{RZ}$, and the variety $\mathbb{ZS}$ of all *zero-semigroups* (determined by the identity $xy = uv$) and the varieties $\mathbb{AB}_p$ for a prime p of all *p-elementary Abelian groups* ($\mathbb{AB}_p$ is determined by identities $xy = yx$ and $x^p y = y$).

Next we study quasivarieties of the form $\mathbb{ZS} \vee \mathbb{Q}$ in $\mathrm{QL}(\mathbb{S})$, where $\mathbb{Q}$ is a band quasivariety. Let $\mathbf{Z} = (Z, \cdot)$ be the two-element zero-semigroup with $Z = \{z, 0\}$. Then $\mathbb{ZS} = \mathrm{Qua}(\mathbf{Z}) = \mathrm{Var}\,(\mathbf{Z})$ and hence $\mathbb{ZS} \not\subseteq \mathbb{V}$ for a semigroup variety $\mathbb{V}$ if and only if $\mathbb{V}$ satisfies the identity $x^n = x$ for some $n > 1$.

We recall several notions given by Koubek and Radovanská in [24]. Let $\mathbb{K}$ be a concrete category. We say that $\mathbb{K}$ is *amenable* if for every $\mathbb{K}$-object $\mathbf{A}$ with the underlying set A and for every bijection $f : A \to B$ there exist a $\mathbb{K}$-object $\mathbf{B}$ with the underlying set B and a $\mathbb{K}$-isomorphism $\phi : \mathbf{A} \to \mathbf{B}$ with underlying mapping f. An isomorphism $\phi : \mathrm{End}(\mathbf{A}) \to \mathrm{End}(\mathbf{B})$ for $\mathbb{K}$-objects $\mathbf{A}$ and $\mathbf{B}$ with underlying sets A and B is a *strong isomorphism* if there exists a bijection $\psi : A \to B$ such that $\phi(f) \circ \psi = \psi \circ f$ for all endomorphisms $f \in \mathrm{End}(\mathbf{A})$. We say that $\mathbb{K}$ *has strong isomorphisms* if every isomorphism $\phi : \mathrm{End}(\mathbf{A}) \to \mathrm{End}(\mathbf{B})$ for equimorphic $\mathbb{K}$-objects is strong. We say that $\mathbb{K}$-objects $\mathbf{A}$ and $\mathbf{B}$ with the same underlying set are *strongly equimorphic* if $\mathrm{End}(\mathbf{A}) = \mathrm{End}(\mathbf{B})$. A category $\mathbb{K}$ is *strongly α-determined* for a cardinal α if every set of non-isomorphic strongly equimorphic $\mathbb{K}$-objects has a cardinality less than α. We recall

Proposition 2.2 [24]. *Let $\mathbb{K}$ be an amenable category having strong isomorphisms. Then $\mathbb{K}$ is α-determined if and only if $\mathbb{K}$ is strongly α-determined.*

In fact, [8] contains the proof of the following corollary.

Corollary 2.3. *The variety $\mathbb{SLZ}$ of semilattices of left-zero semigroups and the variety $\mathbb{SRZ}$ of semilattices of right-zero senigroups are strongly 3-determined. The variety $\mathbb{RQN}$ of right quasi-normal bands and the variety $\mathbb{LQN}$ of left quasi-normal bands are strongly 5-determined.*

Next we recall several folklore semigroup notions, see [7]. Let $\mathbf{S} = (S, \cdot)$ be a semigroup, then $s \in S$ is called *irreducible* if $s = uv$ for no $u, v \in S$. We say that $\mathbf{S}$ is an *inflation of* a *semigroup* $\mathbf{T} = (T, \cdot)$ if there exists a subsemigroup of $\mathbf{S}$ on a set $U \subseteq S$ isomorphic to $\mathbf{T}$, every element $s \in S \setminus U$ is irreducible and there exists an idempotent endomorphism f of $\mathbf{S}$ with $\mathrm{Im}(f) = U$. Then we say that f is an *inflation endomorphism.* In this case, $\mathbf{S}$ is isomorphic to a subsemigroup of $\mathbf{T} \times \mathbf{Z}^\alpha$ for some cardinal α. Let $\mathcal{S}$ and $\mathcal{T}$ be classes of semigroups, we say that $\mathcal{S}$ is an *inflation of $\mathcal{T}$-semigroups* if for every semigroup $\mathbf{S} \in \mathcal{S}$ there exists a semigroup $\mathbf{T} \in \mathcal{T}$ such that $\mathbf{S}$ is an inflation of $\mathbf{T}$. The next theorem gives properties of the least quasivariety containing all inflations of $\mathbb{Q}$-semigroups where $\mathbb{Q}$ is a band quasivariety.

Theorem 2.4. *Let $\mathbb{Q}$ be a band quasivariety. Then $\mathbb{Q} \vee \mathbb{ZS}$ is an inflation of*

$\mathbb{Q}$-semigroups, $\mathbb{Q} \vee \mathbb{ZS}$ covers $\mathbb{Q}$ in the lattice QL($\mathbb{S}$) *and*

(1) *$\mathbb{Q} \vee \mathbb{ZS}$ is α-determined for a cardinal α if and only if $\mathbb{Q}$ is α-determined (then $\mathbb{Q} \vee \mathbb{ZS}$ is strongly α-determined);*
(2) *$\mathbb{Q} \vee \mathbb{ZS}$ is Q-universal if and only if $\mathbb{Q}$ is Q-universal;*
(3) *$\mathbb{Q} \vee \mathbb{ZS}$ is $\mathbb{W}$-relatively alg-universal for a proper subvariety of $\mathbb{Q} \vee \mathbb{ZS}$ if and only if $\mathbb{W} \cap \mathbb{Q}$ is a proper subvariety of $\mathbb{Q}$ and $\mathbb{Q}$ is $\mathbb{W} \cap \mathbb{Q}$-relatively alg-universal;*
(4) *$\mathbb{Q} \vee \mathbb{ZS}$ is $\mathbb{W}$-relatively ff-alg-universal for a proper subvariety of $\mathbb{Q} \vee \mathbb{ZS}$ if and only if $\mathbb{W} \cap \mathbb{Q}$ is a proper subvariety of $\mathbb{Q}$ and $\mathbb{Q}$ is $\mathbb{W} \cap \mathbb{Q}$-relatively ff-alg-universal.*

Proof. First we give a proof of the folklore statement characterizing the quasivariety $\mathbb{Q} \vee \mathbb{ZS}$ in QL($\mathbb{S}$). Let $\mathbb{Q}$ be a band quasivariety. Then it is easy to verify that inflations of $\mathbb{Q}$-semigroups are closed under products and subsemigroups. Let $\mathbf{S}$ be an inflation of $\mathbf{T} \in \mathbb{Q}$ and let $\sim$ be a congruence of $\mathbf{S}$ such that $\sim$ is identical on $\mathrm{Im}(f)$ for the inflation endomorphism f of $\mathbf{S}$. Then $\mathbf{S}/\sim$ is an inflation of $\mathbf{T}$. Hence we obtain that inflations of $\mathbb{Q}$-semigroups form a quasivariety and if $\mathbb{Q}$ is a variety then they form also a variety. Thus $\mathbb{Q} \vee \mathbb{ZS}$ consists of all inflations of $\mathbb{Q}$-semigroups. If $\mathbf{S}$ is an inflation of a $\mathbb{Q}$-semigroup and $\mathbf{S} \notin \mathbb{Q}$ then $\mathbf{Z}$ is isomorphic to a subsemigroup of $\mathbf{S}$ and hence $\mathbb{Q} \vee \mathbb{ZS}$ covers $\mathbb{Q}$ in the lattice QL($\mathbb{S}$). Let $\mathbb{Q}_1$ be a subquasivariety of $\mathbb{Q} \vee \mathbb{ZS}$. Let us denote $\mathbb{Q}_2 = \mathbb{Q}_1 \cap \mathbb{Q}$. Then $\mathbb{Q}_2$ is a quasivariety and either $\mathbb{Q}_2 = \mathbb{Q}_1$ or $\mathbf{Z} \in \mathbb{Q}_1$. In the second case, $\mathbb{Q}_1$ consists of inflations of $\mathbb{Q}_2$-semigroups. Hence either $\mathbb{Q}_2 = \mathbb{Q}_1$ or $\mathbb{Q}_1 = \mathbb{Q}_2 \vee \mathbb{ZS}$. Thus QL($\mathbb{Q} \vee \mathbb{ZS}$) is isomorphic to the lattice QL($\mathbb{Q}$) $\times$ **2** where **2** is a two-element lattice. Hence (2) follows.

Since $\mathbb{Q}$ is a subquasivariety of $\mathbb{Q} \vee \mathbb{ZS}$ then from the fact that $\mathbb{Q} \vee \mathbb{ZS}$ is α-determined it follows that $\mathbb{Q}$ is α-determined. Assume that $\mathbb{Q}$ is α-determined. By Proposition 1.7 from [24], $\mathbb{B}$ has strong isomorphisms, thus $\mathbb{Q}$ has strong isomorphisms. Since $\mathbb{Q}$ is amenable we conclude that $\mathbb{Q}$ is strongly α-determined. If $\mathbf{S} = (S, \cdot)$ is a band then every constant mapping from S into itself is an endomorphism of $\mathbf{S}$ that is a left zero of End($\mathbf{S}$). Further, any idempotent endomorphism $f \in \mathrm{End}(\mathbf{S})$ is either a left zero of End($\mathbf{S}$) or there exist at least two distinct left zeros $g \in \mathrm{End}(\mathbf{S})$ with $f \circ g = g$. If $\mathbf{S} = (S, \cdot)$ is an inflation of a band $\mathbf{T}$ such that the inflation endomomorphism f of $\mathbf{S}$ is not an automorphism of $\mathbf{S}$, then

(1) a constant mapping from S into itself with value $s \in S$ is an endomorphism $\mathbf{S}$ if and only if $s \in \mathrm{Im}(f)$;
(2) the set of all left zeros of $\mathbf{S}$ is the set of all constant endomorphisms of $\mathbf{S}$;
(3) for every $s \in S \setminus \mathrm{Im}(f)$ there exists an idempotent endomorphism $g \in \mathrm{End}(\mathbf{S})$ with $\mathrm{Im}(g) = \{s, s^2\}$;
(4) for an idempotent endomorphism g of $\mathbf{S}$ there exists $s \in S \setminus \mathrm{Im}(f)$ with $\mathrm{Im}(g) = \{s, s^2\}$ if and only if there exists exactly one left zero g' of $\mathbf{S}$ with $g \circ g' = g' = g' \circ g \neq g$ and for every idempotent endomorphism h of $\mathbf{S}$ with $g \circ h = h$ we have either $h = g'$ or $h \circ g = g$;
(5) $f = g$ for an idempotent endomorphism g of $\mathbf{S}$ if and only if $g \circ h = h$ for all left zeros $h \in \mathrm{End}(\mathbf{S})$ and if $g' \in \mathrm{End}(\mathbf{S})$ is an idempotent endomorphism

with $g' \circ h = h$ for all left zeros $h \in \mathrm{End}(\mathbf{S})$ then $g' \circ g = g$;

(6) the monoid $\mathrm{End}(\mathbf{T})$ is isomorphic to the subsemigroup $\{f \circ g \circ f \mid g \in \mathrm{End}(\mathbf{S})\}$ of $\mathrm{End}(\mathbf{S})$ (see [24]);

(7) if $\mathbf{S}'$ is an inflation of $\mathbf{T}'$ such that f' is an inflation endomorphism of $\mathbf{S}'$ and if there exists an isomorphism $\phi : \mathrm{Im}(f) \longrightarrow \mathrm{Im}(f')$ between subsemigroups of $\mathbf{S}$ and $\mathbf{S}'$ such that $|f^{-1}(s)| = |(f')^{-1}(\phi(s))|$ for all $s \in \mathrm{Im}(f)$, then there exists an isomorphism $\phi' : \mathbf{S} \longrightarrow \mathbf{S}'$ such that $\phi'(s) = \phi(s)$ for all $s \in \mathrm{Im}(f)$.

First we prove that $\mathbb{Q} \vee \mathbb{ZS}$ has strong isomorphisms. Let $\mathbf{S}_i = (S, \cdot)$ be an inflation of $\mathbf{T}_i$ with an inflation endomorphism f_i for $i = 1, 2$ where $\mathbf{T}_1, \mathbf{T}_2 \in \mathbb{Q}$ such that there exists a monoid isomorphism $\phi : \mathrm{End}(\mathbf{S}_1) \longrightarrow \mathrm{End}(\mathbf{S}_2)$. For $s \in \mathrm{Im}(f_1)$, let g_s be the constant mapping with value s. By (1), g_s is an endomorphism of $\mathbf{S}_1$ and, by (2), $\phi(g_s)$ is a constant endomorphism of $\mathbf{S}_2$. Let us denote $\psi(s)$ the value of $\phi(g_s)$. By (1) and (2), ψ is a bijection from $\mathrm{Im}(f_1)$ onto $\mathrm{Im}(f_2)$ because ϕ is an isomorphism. By (3), for every $s \in S_1 \setminus \mathrm{Im}(f_1)$ there exists an idempotent endomorphism g_s with $\mathrm{Im}(g_s) = \{s, s^2\}$. By (4), $\phi(g_s)$ is an idempotent endomorphism of $\mathbf{S}_2$ such that $\mathrm{Im}(\phi(g_s)) = \{s', (s')^2\}$ for some $s' \in S_2 \setminus \mathrm{Im}(f_2)$ with $\psi(s^2) = (s')^2$. Set $\psi(s) = s'$, then ψ is a bijection because ϕ is a bijection. For every $s \in S_1$ and $f \in \mathrm{End}(\mathbf{S}_1)$ we have

$$\psi \circ (f \circ g_s) = \psi \circ g_{f(s)} = \phi(g_{f(s)}) \circ \psi = \phi(f) \circ \phi(g_s) \circ \psi = \phi(f) \circ \psi \circ g_s$$

and hence ϕ is a strong isomorphism. Thus, by Proposition 1.6 from [24], for every family $\{\mathbf{A}_i \mid i \in I\}$ of equimorphic semigroups from $\mathbb{Q} \vee \mathbb{ZS}$ there exists a family $\{\mathbf{B}_i \mid i \in I\}$ of strongly equimorphic semigroups from $\mathbb{Q} \vee \mathbb{ZS}$ such that $\mathbf{A}_i$ is isomorphic to $\mathbf{B}_i$ for every $i \in I$.

Consider a family $\{\mathbf{S}_i \mid i \in I\}$ of equimorphic semigroups from $\mathbb{Q} \vee \mathbb{ZS}$. By the above statement, we can assume that they are strongly equimorphic. By (5), if $\mathbf{S}_i$ is an inflation of $\mathbf{T}_i$ with an inflation endomorphism f_i for all $i \in I$ where $\mathbf{T}_i \in \mathbb{Q}$ for all $i \in I$ then, by (5) $\mathrm{Im}(f_i) = \mathrm{Im}(f_j)$ for all $i, j \in I$. Let $\{\mathbf{U}_i \mid i \in I\}$ be a family of subsemigroups of $\mathbf{S}_i$ on $\mathrm{Im}(f_i)$ for all $i \in I$. Then, by (6), $\{\mathbf{U}_i \mid i \in I\}$ are strongly equimorphic semigroups in $\mathbb{Q}$ and hence $I/\sim$ has cardinality smaller than α where $i \sim j$ if and only if $\mathbf{U}_i$ is isomorphic to $\mathbf{U}_j$. If $\psi_{i,j}$ is an isomorphism between $\mathbf{U}_i$ and $\mathbf{U}_j$ such that $g \circ \psi_{i,j} = \psi_{i,j} \circ g$ for all $g \in \{f_i \circ h \circ f_i \mid h \in \mathrm{End}(\mathbf{S}_i)\}$, then by (3), (4) and (7), it can be extended to an isomorphism between $\mathbf{S}_i$ and $\mathbf{S}_j$. Thus if $\{\mathbf{S}_i \mid i \in I\}$ is a family of non-isomorphic equimorphic semigroups from $\mathbb{Q} \vee \mathbb{ZS}$ then the cardinality of I is less than α and (1) is proved.

If $\mathbb{Q}$ is $\mathbb{W}$-relatively alg-universal (or $\mathbb{W}$-relatively ff-alg-universal) then $\mathbb{W}$ is a proper subvariety of $\mathbb{Q}$ and thus $\mathbb{W}$ is a proper subvariety of $\mathbb{Q} \vee \mathbb{ZS}$ and thus $\mathbb{Q} \vee \mathbb{ZS}$ is $\mathbb{W}$-relatively alg-universal (or $\mathbb{W}$-relatively ff-alg-universal). Conversely, assume that $\mathbb{W}$ is a proper subvariety of $\mathbb{Q} \cap \mathbb{ZS}$ and that $F : \mathbb{GR} \to \mathbb{Q} \vee \mathbb{ZS}$ is $I(\mathbb{W})$-relatively full embedding. First we prove that $\mathbb{Q} \cap \mathbb{W}$ is a subvariety of $\mathbb{Q}$. If $\mathbb{W} \subseteq \mathbb{Q}$ then $\mathbb{W} = \mathbb{W} \cap \mathbb{Q}$ is a subvariety of $\mathbb{Q}$. If $\mathbb{W} \not\subseteq \mathbb{Q}$ then $\mathbb{ZS} \subseteq \mathbb{W}$ and, by (1), there exists a proper subquasivariety $\mathbb{W}_1$ of $\mathbb{Q}$ with $\mathbb{W} = \mathbb{W}_1 \vee \mathbb{ZS}$. Consider a semigroup $\mathbf{S} \in \mathbb{W}_1 \subseteq \mathbb{W}$. If $\mathbf{T}$ is a homomorphic image of $\mathbf{S}$ then $\mathbf{T} \in \mathbb{W}$. Since

$\mathbf{S}$ is a band we conclude that $\mathbf{T}$ is also a band and hence $\operatorname{Var}(\mathbf{T}) \wedge \mathbb{ZS} = \mathbb{T}$. From this it follows that $\mathbf{T} \in \mathbb{W}_1$ and whence $\mathbb{W}_1$ is a variety.

Consider a semigroup $\mathbf{S} = (S, \cdot) \in \mathbb{Q} \vee \mathbb{ZS}$. Clearly, either $\mathbf{S}$ contains an irreducible element or $\mathbf{S} \in \mathbb{Q}$. Thus if $\mathbf{S}$ contains an irreducible element x then $\mathbf{S}$ is an inflation of $\mathbf{T}$ where $\mathbf{T} \in \mathbb{Q}$ is not isomorphic to $\mathbf{S}$. Consider mappings f_x and g_x from S into itself such that

$$f_x(y) = \begin{cases} y & \text{if } y \neq x\,, \\ x^2 & \text{if } y = x\,, \end{cases} \quad g_x(y) = \begin{cases} x^2 & \text{if } y \neq x\,, \\ x & \text{if } y = x\,. \end{cases}$$

It is easy to verify that both f_x and g_x are endomorphisms of $\mathbf{S}$. Clearly, the subsemigroup of $\mathbf{S}$ on $\operatorname{Im}(g_x)$ is isomorphic to $\mathbf{Z}$ and $\mathbf{T}$ is isomorphic to a subsemigroup of the subsemigroup of $\mathbf{S}$ on $\operatorname{Im}(f_x)$.

Consider a rigid graph $\mathbf{G}$ such that $F\mathbf{G}$ has at least three elements. If $F\mathbf{G}$ contains an irreducible element x then subsemigroups of $F\mathbf{G}$ on subsets $\operatorname{Im}(f_x)$ and $\operatorname{Im}(g_x)$ belong to $\mathbb{W}$ because for any non-identical endomorphism f of $F\mathbf{G}$, the subsemigroup of $F\mathbf{G}$ on $\operatorname{Im}(f)$ necessarily belongs to $\mathbb{W}$. Hence $\mathbf{Z}, \mathbf{T} \in \mathbb{W}$ and therefore $F\mathbf{G} \in \mathbb{W}$ – this is a contradiction. Thus we conclude

(A) if $\mathbf{G}$ is a rigid graph such that $F\mathbf{G}$ has at least three elements then $F\mathbf{G} \in \mathbb{Q} \setminus \mathbb{W}$.

It is easy to see that there exists a natural number n_0 depending on F such that $F\mathbf{G}$ has at least three elements whenever $\mathbf{G}$ has at least n_0 vertices. As a consequence of Hedrlín-Sichler theorem we deduce that $\mathbb{Q} \neq \mathbb{Q} \cap \mathbb{W}$.

It is well known that there exists a full embedding $G_1 : \mathbb{GR} \to \mathbb{GR}$ such that $G_1\mathbf{G}$ is connected and has no loops for any graph $\mathbf{G}$, $G_1\mathbf{G}$ is finite for all finite graphs $\mathbf{G}$ and $G_1\mathbf{G}$ has at least n_0 vertices for all graphs $\mathbf{G}$, see [28]. By [22, 23], for every connected graph $\mathbf{H}$ without loops there exists a full embedding $G_{\mathbf{H}} : \mathbb{GR} \to \mathbb{GR}$ such that for every graph $\mathbf{G}$ and every edge e of $G_{\mathbf{H}}\mathbf{G}$ there exists an induced subgraph of $G_{\mathbf{H}}\mathbf{G}$ isomorphic to $\mathbf{H}$ containing the edge e. Moreover, if $\mathbf{H}$ is finite then $G_{\mathbf{H}}\mathbf{G}$ is a finite graph for all finite graphs $\mathbf{G}$. By an easy combination, there exists a full embedding $G : \mathbb{GR} \to \mathbb{GR}$ such that $G\mathbf{G}$ is a finite graph for all finite graphs, $G\mathbf{G}$ has at least n_0 vertices for all graphs $\mathbf{G}$ and for every graph $\mathbf{G}$ there exist a rigid graph $\mathbf{G}_1$ and an injective graph homomorphism $\iota_{\mathbf{G}} : G\mathbf{G} \to \mathbf{G}_1$. If we prove that $F \circ G\mathbf{G} \in \mathbb{Q}$ for all graphs $\mathbf{G}$, then $F \circ G : \mathbb{GR} \to \mathbb{Q}$ is $I(\mathbb{W} \cap \mathbb{Q})$-relatively full embedding and if $F\mathbf{G}$ is finite for every finite graph $\mathbf{G}$ then $F \circ G\mathbf{G}$ is finite for every finite graph $\mathbf{G}$. Thus the proof (3) and (4) will be complete.

Let $\mathbf{G}$ be a graph. To prove that $F \circ G\mathbf{G} \in \mathbb{Q}$ consider that there exists an irreducible element x of $F \circ G\mathbf{G}$. Consider an endomorphism f of $F \circ G\mathbf{G}$ such that

$$f(y) = \begin{cases} y & \text{if } y \neq x, \\ x^2 & \text{if } y = x. \end{cases}$$

¿From the properties of G, there exist a rigid graph $\mathbf{G}_1$ and an injective graph homomorphism $\iota_{\mathbf{G}} : G\mathbf{G} \to \mathbf{G}_1$. Since $G\mathbf{G}$ has at least n_0 vertices we obtain that $\mathbf{G}_1$ has at least n_0 vertices and thus $F\mathbf{G}_1$ has at least three elements and, by (A),

$F\mathbf{G}_1 \in \mathbb{Q}$. Thus $F\iota_{\mathbf{G}}(x) = F\iota_{\mathbf{G}}(x^2)$ because $F\mathbf{G}_1$ is a band. If the subsemigroup of $F \circ G\mathbf{G}$ on $\mathrm{Im}(f)$ does not belong to $\mathbb{W}$ then there exists an endomorphism h of $\mathbf{G}$ such that $F \circ Gh = f$ and $\iota_{\mathbf{G}} \neq \iota_{\mathbf{G}} \circ Gh$. On the other hand

$$F(\iota_{\mathbf{G}} \circ Gh) = F\iota_{\mathbf{G}} \circ (F \circ Gh) = F\iota_{\mathbf{G}} \circ f = F\iota_{\mathbf{G}}$$

and this is a contradiction with the injectivity of $\iota_{\mathbf{G}}$. Thus we can assume that the subsemigroup of $F \circ G\mathbf{G}$ on $\mathrm{Im}(f)$ belongs to $\mathbb{W}$. Since $F\iota_{\mathbf{G}} = F\iota_{\mathbf{G}} \circ f$ we conclude that the subsemigroup of $F\mathbf{G}_1$ on $\mathrm{Im}(F\iota_{\mathbf{G}})$ belongs to $\mathbb{W}$ because $\mathbb{W}$ is a variety, and this contradicts the fact that F is a $\mathbb{W}$-relatively full embedding. Thus $F \circ G\mathbf{G} \in \mathbb{Q}$. □

Combining Theorems 2.1 and 2.4, we obtain that a variety $\mathbb{V} \vee \mathbb{ZS}$ is Q-universal for a band variety $\mathbb{V}$ whenever $\mathbb{LQN} \subseteq \mathbb{V}$ or $\mathbb{RQN} \subseteq \mathbb{V}$ and $\mathbb{V} \vee \mathbb{ZS}$ is not Q-universal for a band variety $\mathbb{V}$ whenever $\mathbb{V} \subseteq \mathbb{NB}$. For varieties $\mathbb{SLZ} \vee \mathbb{ZS}$ and $\mathbb{SRZ} \vee \mathbb{ZS}$ it is an open problem whether they are Q-universal. The varieties $\mathbb{SLZ} \vee \mathbb{ZS}$ and $\mathbb{SRZ} \vee \mathbb{ZS}$ are 3-determined and the varieties $\mathbb{LQN} \vee \mathbb{ZS}$ and $\mathbb{RQN} \vee \mathbb{ZS}$ are 5-determined. For a band variety $\mathbb{V}$ such that $\mathbb{RB} \subseteq \mathbb{V}$ or $\mathbb{LSN} \subseteq \mathbb{V}$ or $\mathbb{RSN} \subseteq \mathbb{V}$ it is an open problem whether $\mathbb{V}$ is α-determined for some cardinal α. We conjecture that the variety $\mathbb{RB}$ (and also $\mathbb{RB} \vee \mathbb{ZS}$) is α-determined for some cardinal α. The variety $\mathbb{V} \vee \mathbb{ZS}$ is var-relatively alg-universal for a band variety $\mathbb{V}$ if and only if $\mathbb{LSN} \subseteq \mathbb{V}$ or $\mathbb{RSN} \subseteq \mathbb{V}$. In this case $\mathbb{V} \vee \mathbb{ZS}$ is var-relatively ff-alg-universal.

3. The bottom of the lattice VL($\mathbb{S}$)

First we recall results from [10]. For this we need the semigroups $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_3$ defined by Table 1. For a semigroup $\mathbf{S} = (S, \cdot)$, let $\mathbf{S}^{\mathrm{op}} = (S, \odot)$ denote the semigroup opposite to $\mathbf{S}$; thus $s \odot t = t \cdot s$ for all $s, t \in S$.

$\mathbf{M}_1$	1	a	0
1	1	a	0
a	0	0	0
0	0	0	0

$\mathbf{M}_2$	a	b	c	0
a	0	c	0	0
b	c	0	0	0
c	0	0	0	0
0	0	0	0	0

$\mathbf{M}_3$	d	a	b	c
d	a	a	a	b
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c

Table 1. Multiplicative tables of semigroups $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_3$.

Semigroup varieties generated by semigroups $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_1^{\mathrm{op}}$ play an important role in investigations of $\mathcal{J}$-trivial varieties. We recall results concerning varieties generated by these semigroups.

Theorem 3.1 [10, 11]**.** *The varieties* $\mathrm{Var}\,(\mathbf{M}_1)$ *and* $\mathrm{Var}\,(\mathbf{M}_1^{\mathrm{op}})$ *are* 3*-determined and are neither var-relatively alg-universal nor* Q*-universal.*

The variety $\mathrm{Var}\,(\mathbf{M}_2)$ *is* $\mathbb{ZS}$*-relatively* ff*-alg-universal,* Q*-universal and* α*-determined for no cardinal* α*. There exists a finite semigroup* $\mathbf{S} \in \mathrm{Var}\,(\mathbf{M}_2)$ *such that* $\mathrm{Qua}(\mathbf{S})$ *is* $\mathbb{ZS}$*-relatively* ff*-alg-universal and* Q*-universal. The quasivariety* $\mathrm{Qua}(\mathbf{M}_2)$ *is neither var-relatively alg-universal nor* Q*-universal.*

The variety $\mathrm{Var}\,(\mathbf{M}_3)$ *is* $(\mathbb{ZS} \vee \mathbb{LZ})$*-relatively* ff*-alg-universal and* Q*-universal and is* α*-determined for no cardinal* α. *The variety* $\mathrm{Var}\,(\mathbf{M}_3^{\mathrm{op}})$ *is* $(\mathbb{ZS} \vee \mathbb{RZ})$*-relatively* ff*-alg-universal and* Q*-universal and is* α*-determined for no cardinal* α. *There exists a finite semigroup* $\mathbf{S} \in \mathrm{Var}\,(\mathbf{M}_3)$ *such that* $\mathrm{Qua}(\mathbf{S})$ *is* $(\mathbb{ZS} \vee \mathbb{LZ})$*-relatively* ff*-alg-universal and* Q*-universal and* $\mathrm{Qua}(\mathbf{S}^{\mathrm{op}})$ *is* $(\mathbb{ZS} \vee \mathbb{RZ})$*-relatively* ff*-alg-universal and* Q*-universal. The quasivarieties* $\mathrm{Qua}(\mathbf{M}_3)$ *and* $\mathrm{Qua}(\mathbf{M}_3^{\mathrm{op}})$ *are neither var-relatively alg-universal nor* Q*-universal.*

Sapir [32] presented a finite semigroup $\mathbf{S} \in \mathrm{Var}\,(\mathbf{M}_2)$ such that $\mathrm{Qua}(\mathbf{S})$ is Q-universal. The semigroup $\mathbf{S}$ from [11] is substantially smaller than the Sapir's semigroup.

Next we recall several properties of varieties $\mathrm{Var}\,(\mathbf{M}_1)$, $\mathrm{Var}\,(\mathbf{M}_1^{\mathrm{op}})$ and $\mathrm{Var}\,(\mathbf{M}_2)$. We recall that $r(\mathbf{S})$ the union of all subgroups of $\mathbf{S}$ for every semigroup $\mathbf{S}$.

Proposition 3.2 [10]**.** *Let* $\mathbb{V}$ *be a semigroup variety satisfying one of the following conditions:*

(a) $\mathbb{V}$ *satisfies an identity* $x^2 = x^{n+2}$ *for no natural number* $n > 0$*;*
(b) $\mathbf{M}_1, \mathbf{M}_1^{\mathrm{op}} \in \mathbb{V}$*;*
(c) *there exist semigroups* $\mathbf{S}_1, \mathbf{S}_2 \in \mathbb{V}$ *such that* $r(\mathbf{S}_1)$ *is not a right ideal in* $\mathbf{S}_1$ *and* $r(\mathbf{S}_2)$ *is not a left ideal in* $\mathbf{S}_2$*;*
(d) *there exists a semigroup* $\mathbf{S} \in \mathbb{V}$ *such that* $r(\mathbf{S})$ *is not a union of all regular* $\mathcal{J}$*-classes of* $\mathbf{S}$*;*
(e) *there exists a semigroup* $\mathbf{S} \in \mathbb{V}$ *and* $x, y, z \in S \setminus r(\mathbf{S})$ *such that* $s^2 = s^3$ *for all elements* s *of* $\mathbf{S}$*,* $xy = z$ *and* $x^2 y \neq z \neq xy^2$*;*
(f) *there exists a semigroup in* $\mathbb{V}$ *on the set* $\{a, b, c, 0\}$ *such that* $ab = c$ *and all other products equal* 0*.*

Then $\mathbf{M}_2 \in \mathbb{V}$*, and thus* $\mathbb{V}$ *is var-relatively* ff*-alg-universal,* Q*-universal and* α*-determined for no cardinal* α*.*

Proposition 3.3 [10]**.** *Let* $\mathbb{V}$ *be a semigroup variety containing the semigroup* $\mathbf{S}$ *such that* $r(\mathbf{S})$ *is not a right ideal in* $\mathbf{S}$*. Then* $\mathbf{M}_1 \in \mathbb{V}$ *or* $\mathbf{M}_2 \in \mathbb{V}$*.*

Let $\mathbb{V}$ *be a semigroup variety containing the semigroup* $\mathbf{S}$ *such that* $r(\mathbf{S})$ *is not a left ideal in* $\mathbf{S}$*. Then* $\mathbf{M}_1^{\mathrm{op}} \in \mathbb{V}$ *or* $\mathbf{M}_2 \in \mathbb{V}$*.*

Our aim is to solve problems (a), (b) and (c) from Introduction for nearly $\mathcal{J}$-trivial semigroup varieties. From Theorems 2.1 and 3.1 it follows that the the bottom of the lattice $\mathrm{VL}(\mathbb{S})$ plays the key role. Accordingly, we first recall several folklore facts about the bottom of the lattice $\mathrm{VL}(\mathbb{S})$. Recall that a semigroup variety $\mathbb{V}$ covers the variety $\mathbb{ZS}$ in the lattice $\mathrm{VL}(\mathbb{S})$ if and only if $\mathbb{V}$ is one of the varieties $\mathbb{ZS} \vee \mathbb{LZ}$, $\mathbb{ZS} \vee \mathbb{RZ}$, $\mathbb{ZS} \vee \mathbb{SL}$, $\mathbb{ZS} \vee \mathbb{AB}_p$ for a prime p, $\mathrm{Var}\,(\mathbf{M}_2)$. Analogously, if $\mathbb{W}$ is one of the varieties $\mathbb{SL}$, $\mathbb{RZ}$ or $\mathbb{LZ}$ then a variety $\mathbb{V}$ failing the identity $x = x^2$ covers $\mathbb{W}$ in the lattice $\mathrm{VL}(\mathbb{S})$ if and only if $\mathbb{V} = \mathbb{W} \vee \mathbb{ZS}$ or $\mathbb{V} = \mathbb{W} \vee \mathbb{AB}_p$ for a prime. For the full picture we recall that the variety $\mathbb{V}$ covers $\mathbb{ZS} \vee \mathbb{LZ}$ (or $\mathbb{ZS} \vee \mathbb{RZ}$) in the lattice $\mathrm{VL}(\mathbb{S})$ if and only if $\mathbb{V}$ is one of the varieties $\mathbb{ZS} \vee \mathbb{RCB}$, $\mathbb{ZS} \vee \mathbb{LNB}$, $\mathbb{ZS} \vee \mathbb{LZ} \vee \mathbb{AB}_p$ for a prime p, and $\mathrm{Var}\,(\mathbf{M}_3)$ (or $\mathbb{ZS} \vee \mathbb{RCB}$, $\mathbb{ZS} \vee \mathbb{RNB}$, $\mathbb{ZS} \vee \mathbb{RZ} \vee \mathbb{AB}_p$ for a prime p, and $\mathrm{Var}\,(\mathbf{M}_3^{\mathrm{op}})$. These folklore facts are presented, for example in [17]. It is routine to verify that $\mathbb{ZS} \vee \mathbb{SL}$ is determined by the identities $xy = yx$, $x^2 = x^3$ and $xy = (xy)^2$. A folklore statement below describes semigroup the varieties covering $\mathbb{ZS} \vee \mathbb{SL}$.

Proposition 3.4. *A semigroup variety* $\mathbb{V}$ *covers the variety* $\mathbb{ZS} \vee \mathbb{SL}$ *in the lattice* $\mathrm{VL}(\mathbb{S})$ *if and only if* $\mathbb{V}$ *is one of the varieties* $\mathbb{ZS} \vee \mathbb{LNB}$, $\mathbb{ZS} \vee \mathbb{RNB}$, $\mathbb{ZS} \vee \mathbb{AB}_p$ *for a prime* p, $\mathrm{Var}\,(\mathbf{M}_1)$, $\mathrm{Var}\,(\mathbf{M}_1^{\mathrm{op}})$, *or* $\mathbb{SL} \vee \mathrm{Var}\,(\mathbb{M}_2)$.

Proof. By [10], $\mathrm{Var}\,(\mathbf{M}_1)$ and $\mathrm{Var}\,(\mathbf{M}_1^{op})$ cover the variety $\mathbb{ZS} \vee \mathbb{SL}$. We prove that $\mathbb{SL} \vee \mathrm{Var}\,(\mathbb{M}_2)$ covers $\mathbb{ZS} \vee \mathbb{SL}$. It is easy to verify that $\mathbb{SL} \vee \mathrm{Var}\,(\mathbb{M}_2)$ satisfies identities $xy = yx$, $x^2 = x^3$ and $xyz = (xyz)^2$. Consider a semigroup $\mathbf{S} = (S, \cdot) \in \mathbb{SL} \vee \mathrm{Var}\,(\mathbb{M}_2)$. Then $x^2y^2 = (xy)^2 = (xy)^3$ and $(x^2y)^2 = x^2y$ imply that $r(\mathbf{S}) = \{s \in S \mid s^2 = s\}$ is an ideal of $\mathbf{S}$. If there exist $x, y \in S \setminus r(\mathbf{S})$ such that $xy \notin r(\mathbf{S})$ then $\mathbf{M}_2$ is isomorphic to a subsemigroup of the Rees quotient of $\mathbf{S}$ by $r(\mathbf{S})$, if $S \setminus r(\mathbf{S})$ consists of irreducible elements then $\mathbf{S}$ satisfies the identity $xy = (xy)^2$. Hence $\mathbf{S} \in \mathbb{ZS} \vee \mathbb{SL}$ and thus $\mathbb{SL} \vee \mathrm{Var}\,(\mathbb{M}_2)$ covers $\mathbb{ZS} \vee \mathbb{SL}$. From Theorem 2.1 it follows that the other varieties in the list cover the variety $\mathbb{ZS} \vee \mathbb{SL}$ in the lattice $\mathrm{VL}(\mathbb{S})$.

Let $\mathbb{V}$ be a semigroup variety covering $\mathbb{ZS} \vee \mathbb{SL}$. If there exists a semigroup $\mathbf{S} \in \mathbb{V}$ with a non-trivial subgroup, then there exists a prime p and an element s of $\mathbf{S}$ with $s^{p+1} = s$ and $s^p \neq s$. Thus $\mathbb{ZS} \vee \mathbb{SL} \vee \mathbb{AB}_p \subseteq \mathbb{V}$ and hence $\mathbb{V} = \mathbb{ZS} \vee \mathbb{SL} \vee \mathbb{AB}_p$. Therefore we can assume that every group in $\mathbb{V}$ is trivial. If there exists a semigroup $\mathbf{S} \in \mathbb{V}$ with a subsemigroup isomorphic to a nontrivial left zero-semigroup then $\mathbb{ZS} \vee \mathbb{LNB} \subseteq \mathbb{V}$ – hence $\mathbb{V} = \mathbb{ZS} \vee \mathbb{LNB}$. If there exists a semigroup $\mathbf{S} \in \mathbb{V}$ with a subsemigroup isomorphic to a nontrivial right zero-semigroup then $\mathbb{ZS} \vee \mathbb{RNB} \subseteq \mathbb{V}$ – hence $\mathbb{V} = \mathbb{ZS} \vee \mathbb{RNB}$. By Proposition 3.2, if $\mathbb{V}$ does not satisfy the identity $x^2 = x^3$ then $\mathbf{M}_2 \in \mathbb{V}$ – hence $\mathbb{V} = \mathbb{SL} \vee \mathrm{Var}\,(\mathbb{M}_2)$. Thus we can assume that $\mathbb{V}$ satisfies the identity $x^2 = x^3$ and then the $\mathcal{J}$-classes and $\mathcal{D}$-classes coincide for every semigroup $\mathbf{S} \in \mathbb{V}$. By Proposition 3.3, if $\mathbf{M}_2 \notin \mathbb{V}$ and there exists a semigroup $\mathbf{S} \in \mathbb{V}$ such that $r(\mathbf{S})$ is not right ideal (or left ideal) then $\mathbf{M}_1 \in \mathbb{V}$ (or $\mathbf{M}_1^{\mathrm{op}} \in \mathbb{V}$) – in this case $\mathbb{V} = \mathrm{Var}\,(\mathbb{M}_1)$ (or $\mathrm{Var}\,(\mathbf{M}_1^{\mathrm{op}}) \in \mathbb{V}$). Thus we can assume that $r(\mathbf{S})$ is a two-sided ideal of $\mathbf{S}$ and for any element $s \in r(\mathbf{S})$ the $\mathcal{J}$-class of $\mathbf{S}$ containing s is a singleton. If the Rees quotient of $\mathbf{S}$ by $r(\mathbf{S})$ is not zero-semigroup then $\mathbf{M}_2 \in \mathbb{V}$ (see, [10]) and hence $\mathbb{V} = \mathbb{SL} \vee \mathrm{Var}\,(\mathbb{M}_2)$. If the Rees quotient of $\mathbf{S}$ by $r(\mathbf{S})$ is a zero-semigroup then $\mathbf{S}$ satisfies identities $x^2 = x^3$, $xy = (xy)^2$ and $x^2y^2 = y^2x^2$. Choose $s, t \in S$. Then $st, ts \in r(\mathbf{S})$ and $st = stst$, $ts = tsts$ imply that st and ts belong to the same $\mathcal{J}$-class of $\mathbf{S}$. Hence $st = ts$ and $\mathbf{S}$ satisfies identities $xy = yx$ – whence $\mathbf{S} \in \mathbb{ZS} \vee \mathbb{SL}$. $\square$

Let $\mathbf{M}_4$ be the semigroup given in Table 2.

To solve problems (a), (b) and (c) for nearly $\mathcal{J}$-trivial semigroup varieties, it is necessary to describe semigroup varieties covering $\mathrm{Var}\,(\mathbf{M}_1)$ and $\mathrm{Var}\,(\mathbf{M}_1^{\mathrm{op}})$ (observe that $\mathrm{Var}\,(\mathbf{M}_1)$ and $\mathrm{Var}\,(\mathbf{M}_1^{\mathrm{op}})$ are nearly $\mathcal{J}$-trivial). This is the subject of the following theorem.

Theorem 3.5. *If* $\mathbb{V}$ *covers* $\mathrm{Var}\,(\mathbf{M}_1)$, *then either* $\mathbb{V} = \mathbb{LZ} \vee \mathrm{Var}\,(\mathbf{M}_1)$ *or* $\mathbb{V} = \mathbb{RZ} \vee \mathrm{Var}\,(\mathbf{M}_1)$ *or* $\mathbb{V} = \mathbb{AB}_p \vee \mathrm{Var}\,(\mathbf{M}_1)$ *or* $\mathrm{Var}\,(\mathbf{M}_4) = \mathbb{V}$ *or* $\mathbb{V} = \mathrm{Var}\,(\{\mathbf{M}_1, \mathbf{M}_1^{\mathrm{op}}\}) = \mathrm{Var}\,(\{\mathbf{M}_1, \mathbf{M}_2\}) = \mathrm{Var}\,(\{\mathbf{M}_1^{\mathrm{op}}, \mathbf{M}_2\})$.

$\mathbf{M}_4$	a	b	c	d	0
a	b	b	c	d	0
b	b	b	c	d	0
c	d	0	0	0	0
d	0	0	0	0	0
0	0	0	0	0	0

TABLE 2. Multiplicative table of the semigroup $\mathbf{M}_4$.

If $\mathbb{V}$ *covers* $\mathrm{Var}\,(\mathbf{M}_1^{\mathrm{op}})$*, then either* $\mathbb{V} = \mathbb{LZ} \vee \mathrm{Var}\,(\mathbf{M}_1^{\mathrm{op}})$ *or* $\mathbb{V} = \mathbb{RZ} \vee \mathrm{Var}\,(\mathbf{M}_1^{\mathrm{op}})$ *or* $\mathbb{V} = \mathbb{AB}_p \vee \mathrm{Var}\,(\mathbf{M}_1^{\mathrm{op}})$ *or* $\mathrm{Var}\,(\mathbf{M}_4^{\mathrm{op}}) = \mathbb{V}$ *or* $\mathbb{V} = \mathrm{Var}\,(\{\mathbf{M}_1, \mathbf{M}_1^{\mathrm{op}}\}) = \mathrm{Var}\,(\{\mathbf{M}_1, \mathbf{M}_2\}) = \mathrm{Var}\,(\{\mathbf{M}_1^{\mathrm{op}}, \mathbf{M}_2\})$*.*

Proof. Let $\mathbb{V}$ be a variety covering $\mathrm{Var}\,(\mathbf{M}_1)$. Clearly, $\mathbb{LZ} \vee \mathrm{Var}\,(\mathbf{M}_1)$, $\mathbb{RZ} \vee \mathrm{Var}\,(\mathbf{M}_1)$ and $\mathbb{AB}_p \vee \mathrm{Var}\,(\mathbf{M}_1)$ for prime p cover $\mathrm{Var}\,(\mathbf{M}_1)$. By Propositions 3.2 and 3.3, if the identities $x^2 = x^3$ and $x^2y^2 = y^2x^2$ fail in $\mathbb{V}$ and there exists a semigroup $\mathbf{S} \in \mathbb{V}$ such that $r(\mathbf{S})$ is not a right ideal of $\mathbf{S}$, then $\mathbf{M}_2 \in \mathbb{V}$. Thus for any semigroup $\mathbf{S} \in \mathbb{V}$, any element of $r(\mathbf{S})$ is idempotent and $\mathcal{J}$-classes and $\mathcal{D}$-classes of $\mathbf{S}$ coincide. Therefore $r(\mathbf{S})$ is a semilattice and any $\mathcal{J}$-class having a non-empty intersection with $r(\mathbf{S})$ is a singleton for all semigroups $\mathbf{S} \in \mathbb{V}$. Consider a semigroup $\mathbf{S} = (S, \cdot) \in \mathbb{V} \setminus \mathrm{Var}\,(\mathbf{M}_1)$. Then there exist $x, y \in S$ with $xy \neq x^2y$ or else $\mathbf{S}$ satisfies the identities $xy = x^2y$ and $x^2y^2 = y^2x^2 = (xy)^2$ (because $xyxy = x^2y^2x^2y = x^4y^3 = x^2y^2$) and $\mathbf{S} \in \mathrm{Var}\,(\mathbf{M}_1)$, see [10]. Then necessarily $x \notin r(\mathbf{S})$ or else $x^2 = x$. First we prove that $xy \notin r(\mathbf{S})$. Assume that $xy \in r(\mathbf{S})$. Since $r(\mathbf{S})$ is a right ideal we conclude that $xy = (xy)^2$. Then xy and xyx belong to the same $\mathcal{J}$-class of $\mathbf{S}$ and thus $xy = xyx$. Then $xy = xyxxyx$ and thus xy and x^2y belong to the same $\mathcal{J}$-class of $\mathbf{S}$, whence $xy = x^2y$. Therefore $xy \notin r(\mathbf{S})$ and we conclude that $y \notin r(\mathbf{S})$. Consider a subsemigroup $\mathbf{S}'$ of $\mathbf{S}$ generated by $\{x, y\}$. Let T denote the underlying set of $\mathbf{S}'$. It is clear that x^2y^2 is a zero of $\mathbf{S}'$. First we prove that if $x = \alpha$ in $\mathbf{S}'$ for some non-empty word α over $\{x, y\}$ then $\alpha = x$ or $\alpha = y^kx$ for some $k \geq 1$. Assume the opposite. Clearly, $\alpha \neq y$; else $xy = x^2 \in r(\mathbf{S})$ - a contradiction. Thus $\alpha = \alpha'z$ for some $z \in \{x, y\}$ and a non-empty word α' over $\{x, y\}$. If $z = y$ then $xy = \alpha'y^2 \in r(\mathbf{S})$ - a contradiction. If $z = x$ then $x = \alpha'x = (\alpha')^2x$. If x occurs in α' then x and $(\alpha')^2$ belong to the same $\mathcal{J}$-class of $\mathbf{S}'$ and $(\alpha')^2 \in r(\mathbf{S}')$. Thus $x = (\alpha')^2 \in r(\mathbf{S}')$ - a contradiction. Hence we conclude that if $x = \alpha$ in $\mathbf{S}'$ for some word α over $\{x, y\}$ then $\alpha = x$ or $\alpha = y^kx$. Assume that $x = yx$ in $\mathbf{S}'$. Then $y^2x^2 = x^2$ and hence $yx^2 = x^2$. From $xy^2 \in r(\mathbf{S}')$ and $(xy^2)^2 = x^2y^2 = x^2$ we conclude that $\mathbf{S}'$ is isomorphic to the semigroup $\mathbf{M}_4$. Consider that $x = y^2$x. Then $x = y^2x = yy^2x = yx$ and again $\mathbf{S}'$ is isomorphic to $\mathbf{M}_4$. Therefore we can assume that if $x = \alpha$ in $\mathbf{S}'$ for a non-empty word over $\{x, y\}$ then $\alpha = x$.

Next we prove that if $y = \alpha$ in $\mathbf{S}'$ for a non-empty word α over $\{x, y\}$ then $\alpha = y$. Assume the contrary. Observe that $\alpha = x$ implies $x^2 = xy$ – this is a contradiction. Thus $\alpha = z\alpha'$ for some $z \in \{x, y\}$ and a non-empty word α' over $\{x, y\}$. If $z = x$ then $xy = xx\alpha = xxx\alpha = x^2y$ – a contradiction. If $z = y$ then

$y = y\alpha' = y(\alpha')^2 \in r(\mathbf{S})$ – a contradiction. Thus if $\alpha = y$ in $\mathbf{S}'$ for some non-empty word α over $\{x, y\}$ then $y = \alpha$. If $xy \neq \alpha$ in $\mathbf{S}'$ for all non-empty words $\alpha \neq xy$ over $\{x, y\}$ then $T \setminus \{x, y, xy\}$ is a two-sided ideal in $\mathbf{S}'$ and the Rees quotient of $\mathbf{S}'$ by this ideal and Proposition 3.2(f) imply that $\mathbf{M}_2 \in \mathbb{V}$. Thus we can assume that $xy = \alpha$ in $\mathbf{S}'$ for some non-empty word α over $\{x, y\}$ distinct from x, y and xy. First assume that $\alpha = x^2\alpha'$ for some word α' over $\{x, y\}$ (α' can be an empty word). Then

$$xy = \alpha = x^2\alpha' = x^3\alpha' = x\alpha = x^2y$$

and this is a contradiction. Secondly, assume that $\alpha = xy\alpha'$ for some non-empty word over $\{x, y\}$. Then

$$xy = \alpha = xy\alpha' = \alpha\alpha' = xy(\alpha')^2 \in r(\mathbf{S}),$$

again a contradiction. Thirdly, assume that $\alpha = y^2\alpha'$ for some non-empty word over $\{x, y\}$ (if α' is an empty word then $xy = y^2 \in r(\mathbf{S})$ – a contradiction). Since $\mathbb{V}$ satisfies the identity $x^2 = x^3$ we can assume that the first letter of α' is x. Thus $\alpha' = x\alpha''$ for some non-empty word α'' over $\{x, y\}$ (if α'' is an empty word then $xy = y^2x = y^4x = y^2xy$ and hence $y^2xy^2 = (y^2xy)y = y^2xy = xy \in r(\mathbf{S})$ – a contradiction). Then $\alpha'' \neq x\alpha'''$ for some word α''' over $\{x, y\}$ because $xy = y^2x^2\alpha'''$ in $\mathbf{S}'$ implies $x^2y = x^2xy = x^2y^2x^2\alpha''' = y^2x^2\alpha''' = xy$ – a contradiction. If $\alpha'' = y\alpha'''$ for a non-empty word α''' over $\{x, y\}$ then $xy = y^2xy\alpha''' = y^2xy(\alpha''')^2 \in r(\mathbf{S})$ – a contradiction. Hence $\alpha = y^2xy$. From $xy = y^2xy$ in $\mathbf{S}'$ it follows that $yxy = y^2xy = xy$ and $x^2y^2 = xy^2xy = x^2y$. Let $\sim$ be the least equivalence on $\mathbf{S}'$ such that $yx \sim y^2x$, $x^2 \sim x^2y^2$ and $x \sim yx$. By a routine calculation we obtain that $\sim$ is a congruence of $\mathbf{S}'$ and $\mathbf{S}'/\sim$ is isomorphic to $\mathbf{M}_4$. Finally consider that $\alpha = yx\alpha'$ for some word α' over $\{x, y\}$. If α' is an empty word then $T \setminus \{x, y, xy\}$ is a two-sided ideal in $\mathbf{S}'$ and the Rees quotient of $\mathbf{S}'$ by this ideal is isomorphic to $\mathbf{M}_2$. Thus we can assume that α' is non-empty. If $\alpha' = y$ then in $\mathbf{S}'$ $xy = yxy = y^2xy$ and, by the foregoing case, $\mathbf{M}_4$ is isomorphic to a quotient semigroup of $\mathbf{S}'$. If $\alpha' = x$ then $xy = yx^2 \in r(\mathbf{S})$ – this is a contradiction. If $\alpha' = y\alpha''$ for a non-empty word α'' over $\{x, y\}$ then $xy = yxy\alpha'' = y^2xy(\alpha'')^2 \in r(\mathbf{S})$ – a contradiction. Analogously, if $\alpha' = xy\alpha''$ for a non-empty word α''over $\{x, y\}$ then $xy = yx^2y\alpha'' = yx^2y(\alpha'')^2 \in r(\mathbf{S}')$ – a contradiction. It remains that $\alpha = yx^2y$. Clearly, x^2y^2 is a zero of $\mathbf{S}'$. Since $r(\mathbf{S})$ is a right ideal of $\mathbf{S}$ and $x^2 \in r(\mathbf{S})$ we conclude that $yx^2 \in r(\mathbf{S})$ and hence $(yx^2)^2 = yx^2$. Then yx^2 and yx^2y belong to the same $\mathcal{J}$-class of $\mathbf{S}$ and whence $yx^2 = yx^2y = yx^2y^2 = x^2y^2$. Hence $xy = yx^2y = x^2y^2$ – a contradiction. Thus we proved that if a variety $\mathbb{V}$ covers $\mathrm{Var}\,(\mathbf{M}_1)$ then either $\mathbb{V} = \mathrm{Var}\,(\mathbf{M}_1) \vee \mathbb{LZ}$ or $\mathbb{V} = \mathrm{Var}\,(\mathbf{M}_1) \vee \mathbb{RZ}$ or $\mathbb{V} = \mathrm{Var}\,(\mathbf{M}_1) \vee \mathbb{AB}_p$ for a prime p or $\mathbf{M}_2 \in \mathbb{V}$ or $\mathbf{M}_4 \in \mathbb{V}$. Since $\mathrm{Var}\,(\mathbf{M}_1) \vee \mathbb{LZ}$, $\mathrm{Var}\,(\mathbf{M}_1) \vee \mathbb{RZ}$, $\mathrm{Var}\,(\mathbf{M}_1) \vee \mathbb{AB}_p$ for prime p, $\mathrm{Var}\,(\{\mathbf{M}_1, \mathbf{M}_2\})$, and $\mathrm{Var}\,(\mathbf{M}_4)$ are pairwise distinct semigroup varieties and since $\mathrm{Var}\,(\mathbf{M}_1)$ is a proper subvariety of $\mathrm{Var}\,(\mathbf{M}_4)$ the proof of the first statement is complete because, by Proposition 3.3, $\mathrm{Var}\,(\{\mathbf{M}_1, \mathbf{M}_1^{\mathrm{op}}\}) = \mathrm{Var}\,(\{\mathbf{M}_1, \mathbf{M}_2\}) = \mathrm{Var}\,(\{\mathbf{M}_1^{\mathrm{op}}, \mathbf{M}_2\})$. The proof of the second statement is dual. $\square$

Clearly, if $\mathbb{V}$ is a nearly $\mathcal{J}$-trivial semigroup variety then $\mathbb{V} \cap \mathbb{LZ} = \mathbb{V} \cap \mathbb{RZ} = \mathbb{V} \cap \mathbb{AB}_p = \mathbb{T}$ for all primes p. On the other hand, if $\mathbf{S}$ is a semigroup such that $r(\mathbf{S})$ is a band and any $\mathcal{J}$-class of $\mathbf{S}$ having the non-empty intersection with $r(\mathbf{S})$ is a singleton then, by a direct verification, the variety $\mathrm{Var}\,(\mathbf{S})$ is nearly $\mathcal{J}$-trivial. Thus the varieties $\mathrm{Var}\,(\mathbf{M}_1)$, $\mathrm{Var}\,(\mathbf{M}_1^{\mathrm{op}})$, $\mathrm{Var}\,(\mathbf{M}_2)$, $\mathrm{Var}\,(\mathbf{M}_4)$, and $\mathrm{Var}\,(\mathbf{M}_4^{\mathrm{op}})$ are nearly $\mathcal{J}$-trivial. Whence from Theorem 3.5 we immediately obtain

Corollary 3.6. *Let $\mathbb{V}$ be a nearly $\mathcal{J}$-trivial semigroup variety. Then one of the following holds:*

(1) $\mathbb{V} \subseteq \mathrm{Var}\,(\mathbf{M}_1)$ *or* $\mathbb{V} \subseteq \mathrm{Var}\,(\mathbf{M}_1^{\mathrm{op}})$*;*
(2) $\mathrm{Var}\,(\mathbf{M}_2) \subseteq \mathbb{V}$ *or* $\mathrm{Var}\,(\mathbf{M}_4) \subseteq \mathbb{V}$ *or* $\mathrm{Var}\,(\mathbf{M}_4^{\mathrm{op}}) \subseteq \mathbb{V}$*.*

4. The variety $\mathrm{Var}\,(\mathbf{M}_4)$

In this section we show that $\mathrm{Var}\,(\mathbf{M}_4)$ is $\mathrm{Var}\,(\mathbf{M}_1)$-relatively ff-alg universal, Q-universal and α-determined for no cardinal α.

We recall that $\mathbb{GR}$ is a category of all undirected graphs and their homomorphisms. Let $\mathbb{DG}$ be a category of all directed graphs and their homomorphisms. If we identify any undirected graph (V, E) with a directed graph (V, R) where $R = \{(u, v) \mid \{u, v\} \in E\}$ then it is well-known that $\mathbb{GR}$ is a full subcategory of $\mathbb{DG}$. We exploit this fact in this section without a further reference. We give an outline of the proof for the following folklore statement:

Theorem 4.1. *For every finite, irreflexive, asymmetric digraph $\mathcal{Z} = (Z, S)$ there exists an ff-alg-universal full subcategory $\mathbb{DG}_{\mathcal{Z}}$ of $\mathbb{DG}$ such that*

(1) *if (X, R) is a digraph from $\mathbb{DG}_{\mathcal{Z}}$ then $Z \subsetneq X$, (X, R) is irreflexive, asymmetric and strongly connected digraph, $S = R \cap (Z \times Z)$, and for every $z \in Z$ there exist $x, y \in X \setminus Z$ with $(x, z), (z, y) \in R$;*
(2) *if (X, R) and (X', R') are digraphs from $\mathbb{DG}_Z$ then $f^{-1}(z) = \{z\}$ for all $z \in Z$ and for every digraph homomorphism $f : (X, R) \to (X', R')$, there exists no digraph homomorphism $f : (X, R) \to (X', S')$ for $S' = \{(x, y) \mid (y, x) \in R'\}$.*

We recall that a digraph (X, R) is

irreflexive if $(x, x) \in R$ for no $x \in X$;
asymmetric if $(x, y) \notin R$ for all $(y, x) \in R$;
strongly connected if for every ordered pair $(x, y) \in X \times X$ there exists a sequence $x_0, x_1, \dots, x_k$ of nodes from X such that $x_0 = x$, $x_k = y$ and $(x_i, x_{i+1}) \in R$ for all $i = 0, 1, \dots, k-1$.

Outline of the proof of Theorem 4.1. By [10], there exists an ff-alg-universal full subcategory $\mathbb{DG}_s$ of $\mathbb{DG}$ such that

(1) if $(X, R) \in \mathbb{DG}_s$ then (X, R) is irreflexive, asymmetric and strongly connected and for every edge $(x, y) \in R$ there exist edges $(y, z), (z, x) \in R$;
(2) for every $\mathcal{X} = (X, R) \in \mathbb{DG}_s$ there exist distinct nodes $a_{\mathcal{X}}, b_{\mathcal{X}} \in X$ such that $f(a_{\mathcal{X}}) = a_{\mathcal{X}'}$ and $f(b_{\mathcal{X}}) = b_{\mathcal{X}'}$ for every digraph homomorphism $f : \mathcal{X} \to \mathcal{X}' \in \mathbb{DG}_s$;

(3) if $(X,R),(X',R') \in \mathbb{DG}_s$ then there exists no digraph homomorphism $f : (X,R) \longrightarrow (X',S')$ for $S' = \{(u,v) \mid (v,u) \in R'\}$.

Let $\mathcal{Z} = (Z,S)$ be a finite irreflexive and asymmetric digraph. Then there exist finite rigid digraphs $\mathcal{Z}_1 = (Z_1,S_1), \mathcal{Z}_2 = (Z_2,S_2) \in \mathbb{DG}_s$ such that there exists no digraph homomorphism between $\mathcal{Z}_1$ and $\mathcal{Z}_2$ and $|Z_1|,|Z_2| > |Z|+2$. Choose an injective mapping $\phi_i : Z \longrightarrow Z_i$ such that $a_{\mathcal{Z}_i}, b_{\mathcal{Z}_i} \notin \mathrm{Im}(\phi_i)$ for $i = 1,2$. Define a functor $\Omega : \mathbb{DG}_s \longrightarrow \mathbb{DG}$ so that $\Omega(X,R) = (Y,T)$ where $Y = X \cup Z \cup Z_1 \cup Z_2$ (we assume that X, Z, Z_1, and Z_2 are pairwise disjoint) and $T = R \cup S \cup S_1 \cup S_2 \cup \{(\phi_1(z),z),(z,\phi_2(z) \mid z \in Z\} \cup \{(a_{\mathcal{X}},a_{\mathcal{Z}_1}),(a_{\mathcal{Z}_2},a_{\mathcal{X}})\}$. Clearly $\Omega(X,R)$ is irreflexive, asymmetric, strongly connected, $Z \subsetneq Y$, $S = T \cap (Z \times Z)$, and for every $z \in Z$ there exist $x,y \in Y \setminus Z$ with $(x,z),(z,y) \in T$. Thus $\Omega(X,R)$ satisfies the first requirement.

Let (X,R) and (X',R') be digraphs from $\mathbb{DG}_s$ and let $\Omega(X,R) = (Y,T)$ and $\Omega(X',R') = (Y',T')$. Set $S' = \{(u,v) \mid (v,u) \in T'\}$. Let $f : (Y,T) \longrightarrow (Y',T')$ or $f : (Y,T) \longrightarrow (Y',S')$ be a digraph homomorphism. The induced subgraphs of (Y,T) on the sets X, Z_i for $i = 1,2$ are strongly connected and for every arc (x,y) of the subgraph there exist arcs (y,z) and (z,x) of the subgraph. Hence we conclude that $f(X)$, $f(Z_1)$ and $f(Z_2)$ are subsets of one of the sets X', Z_1, Z_2, Z. Since there exists exactly one arc from X to Z_1 and exactly one arc from Z_2 to X and for every $z \in Z$ there exists exactly one arc from Z_1 to z and one arc from z to Z_2 we conclude that $f(X) \subseteq X'$, $f(Z_i) \subseteq Z_i$ for $i = 1,2$ and $f(Z) = Z$. By the properties of $\mathcal{Z}_1$ and $\mathcal{Z}_2$, we conclude that $f(z) = z$ for all $z \in Z_1 \cup Z_2$ and thus also $f(z) = z$ for all $z \in Z$. Hence $f^{-1}\{z\} = \{z\}$ for all $z \in Z$. The domain-range restriction g of f on X and X' is a digraph homomorphism from (X,R) into (X',R') or from (X,R) into (X',U') where $U' = \{(u,v) \mid (v,u) \in R'\}$. In the first case $f = \Omega g$ and the second case is impossible by the properties of $\mathbb{DG}_s$. Thus Ω is a full embedding satisfying the second requirement. To complete the proof it suffices to take $\mathbb{DG}_{\mathcal{Z}}$ as a full subcategory formed by $\Omega\mathcal{X}$ for $\mathcal{X} \in \mathbb{DG}_s$.$\square$

Choose a finite irreflexive, asymmetric, strongly connected graph $\mathcal{Z} = (Z,S)$ and two disjoint sets Z_1 and Z_2 such that $Z = Z_1 \cup Z_2$ and $S \cap (Z_1 \times Z_1) \neq \emptyset \neq S \cap (Z_2 \times Z_2)$. Fix a triple $\mathfrak{Z} = (\mathcal{Z}, Z_1, Z_2)$. For a digraph $\mathcal{X} = (X,R) \in \mathbb{DG}_{\mathcal{Z}}$ we shall construct a groupoid $\Lambda_{\mathfrak{Z}}\mathcal{X} = (Y,\cdot)$ where $Y = X \cup R \cup \{a_0,a_1,a_2,a_3,a_4,0\}$ (we assume that $X \cap R = \emptyset$, $\{a_0,a_1,a_2,a_3,a_4,0\} \cap (X \cup R) = \emptyset$ and $a_0,a_1,a_2,a_3,a_4,0$ are pairwise distinct elements) and

$$x \cdot y = \begin{cases} a_4 & \text{if } \ x,y \in \{a_0,a_1,a_2,a_3,a_4\}\,, \\ 0 & \text{if } \ x,y \in X \cup R \cup \{0\} \text{ or } x \in X \cup \{0\} \text{ or } 0 \in \{x,y\}\,, \\ y & \text{if } \ x \in \{a_0,a_1,a_2,a_3,a_4\} \text{ and } y \in X \cup R\,, \\ u & \text{if } \ x = (u,v) \in R \text{ and } y = a_0\,, \\ v & \text{if } \ x = (u,v) \in R \text{ and } y = a_1\,, \\ u & \text{if } \ x = (u,v) \in R \cap (Z_i \times Z_i) \text{ and } y = a_{1+i} \text{ for } i = 1,2, \\ 0 & \text{if } \ x = (u,v) \in R \setminus (Z_i \times Z_i) \text{ and } y = a_{1+i} \text{ for } i = 1,2, \\ 0 & \text{if } \ x = (u,v) \in R \text{ and } y = a_4\,. \end{cases}$$

It is routine to verify the correctness of the definition.

Proposition 4.2. $\Lambda_3\mathcal{X} \in \mathrm{Var}\,(\mathbf{M}_4)$ *for every digraph* $\mathcal{X} \in \mathbb{DG}_{\mathcal{Z}}$.

Proof. Consider a subsemigroup $\mathbf{S}_1$ of the semigroup $(\mathbf{M}_4)^4$ on the set

$$\begin{aligned}S_1 = \{&a_0 = (a,b,b,b), a_1 = (b,a,b,b), a_2 = (b,b,a,b), a_3 = (b,b,b,a), a_4 = (b,b,b,b),\\ &c = (c,c,c,c), d_0 = (d,0,0,0), d_1 = (0,d,0,0), d_2 = (0,0,d,0), d_3 = (0,0,0,d),\\ &0 = (0,0,0,0)\}\,.\end{aligned}$$

Assume that $\mathcal{X} = (X,R) \in \mathbb{DG}_{\mathcal{Z}}$ is a digraph. Let $\mathbf{S}_{\mathcal{X}}$ be the subsemigroup of $(\mathbf{S}_1)^R$ on the set $S_{\mathcal{X}}$ consisting of mappings $\alpha : R \longrightarrow S_1$ such that

(1) α is a constant mapping with the value a_i for $i = 0,1,2,3$;
(2) mappings α such that there exists $r \in R$ and $\alpha(r) \in \{c, d_0, d_1, d_2, d_3\}$ and $\alpha(q) = 0$ for all $q \in R \setminus \{r\}$;
(3) the constant mapping with value 0.

By a straightforward calculation, we obtain that this set is closed under multiplication, thus it is a semigroup from $\mathrm{Var}\,(\mathbf{M}_4)$. Consider the least equivalence $\sim$ on $S_{\mathcal{X}}$ such that $\alpha \sim \beta$ whenever one of the following cases occurs

there exist $(x,y), (x,z) \in R$ such that $\alpha(x,y) = d_0$ and $\beta(x,z) = d_0$;
there exist $(z,x), (x,y) \in R$ such that $\alpha(z,x) = d_1$ and $\beta(x,y) = d_0$;
there exist $(y,x), (z,x) \in R$ such that $\alpha(y,x) = d_1$ and $\beta(z,x) = d_1$;
there exists $(x,y) \in R \cap (Z_i \times Z_i)$ such that $\alpha(x,y) = d_0$ and $\beta(x,y) = d_{1+i}$ for $i = 1,2$;
there exists $(x,y) \in R \setminus (Z_i \times Z_i)$ such that $\alpha(x,y) = d_{i+1}$ for $i = 1,2$ and β is the constant mapping with value 0.

By a direct verification, we find that if α and β are distinct mappings from $S_{\mathcal{X}}$ with $\alpha \sim \beta$ then $\alpha \cdot \beta$ is the constant mapping with value 0. Hence $\sim$ is a congruence of $\mathbf{S}_{\mathcal{X}}$ and, by a simple calculation, we find that $\mathbf{S}_{\mathcal{X}}/\sim$ is isomorphic to $\Lambda_3\mathcal{X}$. Whence $\Lambda_3\mathcal{X} \in \mathrm{Var}\,(\mathbf{M}_4)$. $\square$

Observe that if $\mathcal{X} = (X,R) \in \mathbb{DG}_{\mathcal{Z}}$ is a digraph then $R \neq \emptyset$, and for every $(x,y) \in R$ the subsemigroup of $\Lambda_3\mathcal{X}$ generated by $\{a_0, (x,y)\}$ is isomorphic to $\mathbf{M}_4$.

Let $\mathcal{X}_1 = (X_1, R_1), \mathcal{X}_2 = (X_2, R_2) \in \mathbb{DG}_{\mathcal{Z}}$ be digraphs and let $f : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ be a digraph homomorphism. Let us define $\Lambda_3 f : \Lambda_3\mathcal{X}_1 \longrightarrow \Lambda_3\mathcal{X}_2$ by

$$\Lambda_3 f(x) = \begin{cases} x & \text{if } x \in \{a_0, a_1, a_2, a_3, a_4, 0\}\,, \\ f(x) & \text{if } x \in X_1\,, \\ (f(u), f(v)) & \text{if } x = (u,v) \in R_1\,. \end{cases}$$

It is easy to see that $\Lambda_3 f$ is a semigroup homomorphism from $\Lambda_3\mathcal{X}_1$ into $\Lambda_3\mathcal{X}_2$. Then we can summarize

Proposition 4.3. Λ_3 *is a faithful functor from* $\mathbb{DG}_{\mathcal{Z}}$ *into* $\mathrm{Var}\,(\mathbf{M}_4)$ *such that* $\Lambda_3\mathcal{X}$ *is a finite semigroup for every finite digraph* $\mathcal{X}$. *For every digraph homomorphism* $f : \mathcal{X}_1 \to \mathcal{X}_2 \in \mathbb{DG}_{\mathcal{Z}}$, *the variety* $\mathrm{Var}\,(\mathbf{M}_4)$ *is generated by the subsemigroup of* $\Lambda_3\mathcal{X}_2$ *on the set* $\mathrm{Im}(\Lambda_3 f)$.

Next we give a simple technical lemma.

Lemma 4.4. *For every digraph* $(X, R) \in \mathbb{DG}_{\mathcal{Z}}$ *and for every congruence* $\sim$ *on* $\Lambda_3(X, R)$ *such that* $x \sim 0$ *for all* $x \in X$ *we have* $\Lambda_3(X, R)/\sim \in \mathrm{Var}\,(\mathbf{M}_1)$.

Proof. By a direct verification, $\Lambda_3(X, R)/\sim$ satisfies the identities $x^2 = x^3$, $xy = x^2y$ and $x^2y^2 = y^2x^2 = (xy)^2$. Thus, by [10], $\Lambda_3(X, R)/\sim \in \mathrm{Var}\,(\mathbf{M}_1)$ and the proof is complete. □

Assume that digraphs $\mathcal{X}_1 = (X_1, R_1)$ and $\mathcal{X}_2 = (X_2, R_2)$ from $\mathbb{DG}_{\mathcal{Z}}$ are given. First we investigate semigroup homomorphisms from $\Lambda_3\mathcal{X}_1 = (Y_1, \cdot)$ to $\Lambda_3\mathcal{X}_2 = (Y_2, \cdot)$.

Lemma 4.5. *Let* $f : \Lambda_3\mathcal{X}_1 \to \Lambda_3\mathcal{X}_2$ *be a semigroup homomorphism. If* $f(a_i) = a_i$ *for* $i = 0, 1$ *and* $f(0) = 0$ *then one of the following cases occurs*

1. *there exists a digraph homomorphism* $g : \mathcal{X}_1 \to \mathcal{X}_2$ *with* $\Lambda_3 g = f$;
2. $f(a_2), f(a_3) \in \{a_0, a_1, a_2, a_3, a_4\}$, $f(a_4) = a_4$, $f(X_1 \cup \{0\}) = \{0\}$, $f(R_1) \subseteq X_2 \cup \{0\}$.

Conversely, any mapping satisfying these conditions is a semigroup homomorphism from $\Lambda_3\mathcal{X}_1$ *into* $\Lambda_3\mathcal{X}_2$.

Proof. Assume that $f : \Lambda_3\mathcal{X}_1 \to \Lambda_3\mathcal{X}_2$ is a semigroup homomorphism with $f(a_0) = a_0$, $f(a_1) = a_1$ and $f(0) = 0$. Then $f(a_4) = a_4$ and $f(a_2), f(a_3) \in \{a_0, a_1, a_2, a_3, a_4\}$. From $f(0) = 0$ it follows that $f(R_1 \cup X_1) \subseteq R_2 \cup X_2 \cup \{0\}$. Since $(x, y) \in R_2$ is irreducible we conclude that $f(X_1) \cap R_2 = \emptyset$. Assume that there exists $(x, y) \in R_1$ with $f(x, y) = (u, v) \in R_2$. From $(x, y)a_0 = x$ and $(x, y)a_1 = y$ it follows $f(x) = u$ and $f(y) = v$. If $f(x) = u \in X_2$ then for $(x, y), (z, x) \in R_1$ we have $f(x, y)a_0 = u = f(z, x)a_1$ and thus there exist $(u, v), (w, u) \in R_2$ with $f(x, y) = (u, v)$ and $f(z, x) = (w, u)$. Since $\mathcal{X}_1$ is strongly connected, by an easy induction we obtain that $f(X_1) \subseteq X_2$, $f(R_1) \subseteq R_2$ and $f(x, y) = (f(x), f(y))$ for all $(x, y) \in R_1$. Let g be the domain-range restriction of f to X_1 and X_2. Then $g : \mathcal{X}_1 \to \mathcal{X}_2$ is a digraph homomorphism such that if $f(a_2) = a_2$ and $f(a_3) = a_3$ then $\Lambda_3 g = f$. If $(x, y) \in R_1 \cap (Z_i \times Z_i)$ for $i = 1, 2$ then $f(x, y)f(a_{1+i}) = f(x)$ implies that $f(a_2), f(a_3) \in \{a_0, a_2, a_3\}$. If $(x, y) \in R_1 \setminus (Z \times Z)$ then $(x, y)a_{1+i} = 0$. Since $(x, y)a_0 = x$ and $f(0) = 0$, we conclude that $f(a_2), f(a_3) \neq a_0$. Since $R_1 \cap (Z_1 \times Z_1)$ and $R \cap (Z_2 \times Z_2)$ are non-empty and disjoint we analogously obtain that $f(a_{1+i}) = a_{i+i}$ for $i = 1, 2$, thus $f = \Lambda_3 g$.

If $f(x, y) \in X_2 \cup \{0\}$ for some $(x, y) \in R_1$ then $f(x) = f(y) = 0$. If $f(x) = 0$ for some $x \in X_1$ then for every $(x, y), (z, x) \in R_1$ we have $f(x, y)a_0 = f(z, x)a_1 = 0$ and hence $f(x, y), f(z, x) \in X_2 \cup \{0\}$. Since $\mathcal{X}_1$ is strongly connected, by an easy induction we obtain that $f(X_1 \cup \{0\}) = \{0\}$ and $f(R_1) \subseteq X_2 \cup \{0\}$. The first statement is proved.

The proof of the converse statement is obtained by a direct computation. □

Lemma 4.6. *Let $f : \Lambda_3\mathcal{X}_1 \to \Lambda_3\mathcal{X}_2$ be a semigroup homomorphism. If $f(a_i) = a_{1-i}$ for $i = 0, 1$ and $f(0) = 0$ then $f(a_2), f(a_3) \in \{a_0, a_1, a_2, a_3, a_4\}$, $f(a_4) = a_4$, $f(X_1 \cup \{0\}) = \{0\}$, and $f(R_1) \subseteq X_2 \cup \{0\}$.*

Conversely, any mapping satisfying these conditions is a semigroup homomorphism from $\Lambda_3\mathcal{X}_1$ into $\Lambda_3\mathcal{X}_2$.

Proof. Assume that $f : \Lambda_3\mathcal{X}_1 \to \Lambda_3\mathcal{X}_2$ is a semigroup homomorphism with $f(a_0) = a_1$, $f(a_1) = a_0$ and $f(0) = 0$. Then $f(a_4) = a_4$ and $f(a_2), f(a_3) \in \{a_0, a_1, a_2, a_3, a_4\}$. From $f(0) = 0$ it follows that $f(R_1 \cup X_1) \subseteq R_2 \cup X_2 \cup \{0\}$. Since $(x, y) \in R_2$ is irreducible, we conclude that $f(X_1) \cap R_2 = \emptyset$. Assume that there exists $(x, y) \in R_1$ with $f(x, y) = (u, v) \in R_2$. From $(x, y)a_0 = x$ and $(x, y)a_1 = y$ it follows that $f(x) = v$ and $f(y) = u$. If $f(x) = u \in X_2$ then for $(x, y), (z, x) \in R_1$ we have $f(x, y)a_1 = u$, $f(z, x)a_0 = v$ and thus there exist $(u, v), (w, u) \in R_2$ with $f(x, y) = (w, u)$ and $f(z, x) = (u, v)$. Since $\mathcal{X}_1$ is strongly connected, by an easy induction we obtain that $f(X_1) \subseteq X_2$, $f(R_1) \subseteq R_2$ and $f(x, y) = (f(y), f(x))$ for all $(x, y) \in R_1$. Let g be the domain-range restriction of f to X_1 and X_2. Then $g : \mathcal{X}_1 \to (X_2, S_2)$ is a digraph homomorphism for $S_2 = \{(u, v) \mid (v, u) \in R_2\}$. By the properties of $\mathbb{DG}_{\mathcal{Z}}$, such digraph homomorphism does not exist. Thus $f(R_1) \cap R_2 = \emptyset$.

If $f(x, y) \in X_2 \cup \{0\}$ for some $(x, y) \in R_1$ then $f(x) = f(y) = 0$ and hence we obtain that $f(X_1 \cup \{0\}) = \{0\}$, $f(R_1) \subseteq X_2 \cup \{0\}$. The first statement is proved.

The proof of the converse statement is obtained by a direct computation. □

Lemma 4.7. *Let $f : \Lambda_3\mathcal{X}_1 \to \Lambda_3\mathcal{X}_2$ be a semigroup homomorphism with $f(0) = 0$. If $f(a_0) = f(a_1) \in \{a_0, a_1, a_2, a_3, a_4\}$ or $a_4 \in \{f(a_0), f(a_1)\}$ or $f(a_0), f(a_1) \in \{a_0, a_2, a_3\}$ then $f(X_1) \subseteq \{0\}$ and one of the following cases occurs*

(1) *$f(a_0), f(a_1) f(a_2), f(a_3) \in \{a_0, a_1, a_2, a_3, a_4\}$, $f(a_4) = a_4$, $f(R_1) \subseteq X_2 \cup \{0\}$;*
(2) *for some $i = 1, 2$ $f(a_0) = a_{1+i}$, $f(a_1) \in \{a_{i+1}, a_4\}$ $f(a_2), f(a_3) \in \{a_0, a_1, a_2, a_3, a_4\}$, $f(a_4) = a_4$, $f(R_1) \subseteq \big(R_2 \setminus (Z_i \times Z_i)\big) \cup X_2 \cup \{0\}$ and the following conditions hold:*
 – if $f(R_1 \cap (Z_j \times Z_j)) \cap R_2 \neq \emptyset$ for some $j = 1, 2$ then $f(a_{1+j}) \neq a_0, a_1$,
 – if $f(R_1 \cap (Z_j \times Z_j)) \cap \big(R_2 \setminus (Z_i \times Z_i)\big) \neq \emptyset$ for some $j = 1, 2$ then $f(a_{1+j}) \neq a_0, a_1, a_{4-i}$;
(3) *$f(a_0) = f(a_1) = f(a_4) = a_4$, $f(a_2), f(a_3) \in \{a_0, a_1, a_2, a_3, a_4\}$, $f(X_1 \cup \{0\}) = \{0\}$, $f(R_1) \subseteq R_2 \cup X_2 \cup \{0\}$ and the following conditions hold:*
 – if $f(R_1 \cap (Z_i \times Z_i)) \cap R_2 \neq \emptyset$ for some $i = 1, 2$ then $f(a_{1+i}) \neq a_0, a_1$,
 – if $f(R_1 \cap (Z_i \times Z_i)) \cap \big(R_2 \setminus (Z_j \times Z_j)\big) \neq \emptyset$ for some $i = 1, 2$ and $j = 1, 2$ then $f(a_{1+i}) \neq a_0, a_1, a_{1+j}$.

Conversely, any mapping satisfying these conditions is a semigroup homomorphism from $\Lambda_3\mathcal{X}_1$ into $\Lambda_3\mathcal{X}_2$.

Proof. Let $f : (Y_1, \cdot) \to (Y_2, \cdot)$ be a semigroup homomorphism with $f(0) = 0$ and one of the following conditions holds: $f(a_0) = f(a_1) \in \{a_0, a_1, a_2, a_3, a_4\}$ or $a_4 \in \{f(a_0), f(a_1)\}$ or $f(a_0), f(a_1) \in \{a_0, a_2, a_3\}$. Then $f(a_4) = a_4$ and $f(a_2), f(a_3) \in \{a_0, a_1, a_2, a_3, a_4\}$. From $f(0) = 0$ it follows that $f(R_1 \cup X_1) \subseteq R_2 \cup X_2 \cup \{0\}$.

Since $(x,y) \in R_2$ is irreducible we conclude that $f(X_1) \cap R_2 = \emptyset$. First we prove that $f(X_1) = \{0\}$. This is clear if $f(a_0) = a_4$ or $f(a_1) = a_4$ because $(x,y)a_4 = 0$ and (X_1, R_1) is strongly connected (it suffices that for every $x \in X_1$ there exist $y, z \in X_1$ with $(x,y)(z,x) \in R_1$). Secondly assume that $f(a_0) = f(a_1)$. Consider that $f(x) = u$ for some $x \in X_1$. Then

$$u = f(x) = f((x,y)a_0) = f(x,y)f(a_0) = f((z,x)a_1) = f(z,x)f(a_1)$$

for all $(x,y),(z,x) \in R_1$. From $f(a_0) = f(a_1)$ it follows that

$$u = f(x,y)f(a_1) = f((x,y)a_1) = f(y) = f(z,x)f(a_0) = f((z,x)a_0) = f(z).$$

¿From the fact that $\mathcal{X}_1$ is strongly connected we conclude that $f(X) = \{u\}$ and one of the following possibilities occurs:

(a) $f(a_0) = a_0$ and $f(R_1) \subseteq \{(u,v) \mid (u,v) \in R_2\}$;
(b) $f(a_0) = a_1$ and $f(R_1) \subseteq \{(v,u) \mid (v,u) \in R_2\}$;
(c) $f(a_0) = a_2$ and $f(R_1) \subseteq \{(u,v) \mid (u,v) \in R_2 \cap (Z_1 \times Z_1)\}$;
(d) $f(a_0) = a_3$ and $f(R_1) \subseteq \{(u,v) \mid (u,v) \in R_2 \cap (Z_2 \times Z_2)\}$.

Choose $j = 2,3$. Then there exists $(x,y) \in R_1$ with $(x,y)a_0 = x \neq 0 = (x,y)a_j$. Hence $f(x,y)f(a_0) = u \neq 0 = f(x,y)f(a_j)$ and thus $f(a_j) \neq a_0, a_1, f(a_0)$. Since there exists $(x',y') \in R_1$ with $(x',y')a_0 = (x',y')a_j$ we conclude that the case $f(a_0), f(a_j) \in \{a_2, a_3\}$ does not occur. Thus the cases (b), (c) and (d) are not possible. Consider the case (a). If $u \notin Z_i$ and $f(a_j) = a_{i+1}$ for some $i = 1,2$ then $(x',y')a_0 = (x',y')a_j$ imply that $f(a_0) = f(a_j)$ – a contradiction. Thus we can assume that $u \in Z_i$ and $f(a_2) = f(a_3) = a_{i+1}$ for some $i = 1,2$. Then the existence of an arc $(x'',y'') \in R_1$ with $(x'',y'')a_2 \neq (x'',y'')a_3$ yields a contradiction because $\{(x'',y'')a_2, (x'',y'')a_3\} = \{x'', 0\}$. This contradicts the assumption that $f(x) = u \in X_2$ for some $x \in X_1$. Hence $f(a_0) = f(a_1)$ implies $f(X_1) = \{0\}$. In the remaining case $f(a_0) \neq f(a_1)$ and $a_4 \notin \{f(a_0), f(a_1)\}$. Then $f(a_0), f(a_1) \in \{a_0, a_2, a_3\}$. If $\{f(a_0), f(a_1)\} = \{a_2, a_3\}$ and $f(X_1) \neq \{0\}$ then there exist $x \in X_1$ and $i = 1,2$ such that $f(x) = u \in X_2 \setminus Z_i$. Since there exist $(x,y),(z,x) \in R_1$ with $x = (x,y)a_0 = (z,x)a_1$ and there exists $j = 0,1$ with $f(a_j) = a_{i+1}$ we find a contradiction because $ta_{i+1} \neq u$ for all $t \in Y_2$. Whence $\{f(a_0), f(a_1)\} = \{a_2, a_3\}$ implies $f(X_1) = \{0\}$. If $\{f(a_0), f(a_1)\} = \{a_0, a_{i+1}\}$ for some $i = 1,2$ and $f(X_1) \neq \{0\}$ then there exists $x \in X_1$ with $f(x) = u \in X_2$. Since there exist $(x,y),(z,x) \in R_1$ we conclude that $u = f(x,y)f(a_0) = f(z,x)f(a_1)$ and since $ta_{i+1} \in X_2$ for $t \in Y_2$ just when $t \in R_2 \cap (Z_i \times Z_i)$ and $ta_{i+1} \in Z_i$ we state that $u \in Z_i$ and the following is true:
if $f(a_0) = a_{i+1}$ then $f(x,y) \in R_2 \cap (Z_i \times Z_i)$ and $f(y) = u$,
if $f(a_1) = a_{i+1}$ then $f(z,x) \in R_2 \cap (Z_i \times Z_i)$ and $f(z) = u$.
Since (X_1, R_1) is strongly connected we obtain, by an easy induction that $f(X_1) = \{u\}$ and $f(R_1) \subseteq R_2 \cap (Z_i \times Z_i)$. If $f(a_2) \in \{a_{4-i}, a_4\}$ then for $(x',y') \in R_1$ with $(x',y')a_2 = x'$ we have $u = f(x') = f(x',y')f(a_2) = 0$ – a contradiction, if $f(a_2) = a_1$ then for $(x',y') \in R_1$ with $(x',y')a_2 = x'$ we have $u = f(x') = f(x',y')a_1 \neq u$ – a contradiction because $f(x',y') \in R_2$ and $ta_1 \notin \{ta_0, ta_2, ta_3\}$

for all $t \in R_2$, if $f(a_2) \in \{a_0, a_{i+1}\}$ then for $(x', y') \in R_1$ with $(x', y')a_2 = 0$ we have $0 = f(0) = f(x', y')f(a_2) = u$ – a contradiction. Thus we prove that $f(X_1) = \{0\}$.

If $f(a_0) \in \{a_0, a_1\}$ or $f(a_1) \in \{a_0, a_1\}$ then $f(R_1) \subseteq X_2 \cup \{0\}$ because $f(x, y)f(a_0) = 0$ or $f(x, y)f(a_1) = 0$ for all $(x, y) \in R_1$. If $f(a_0) = a_{1+1}$ for $i = 1, 2$ then, by the same argument, $f(R_1) \subseteq \big(R_2 \setminus (Z_i \times Z_i)\big) \cup X_2 \cup \{0\}$, if $\{f(a_0), f(a_1)\} = \{a_2, a_3\}$ then $f(R_1) \subseteq R_2 \setminus \big((Z_1 \times Z_1) \cup (Z_2 \times Z_2)\big) \cup X_2 \cup \{0\}$ and if $f(a_0) = f(a_1) = a_4$ then $f(R_1) \subseteq R_2 \cup X_2 \cup \{0\}$. If $(x, y) \in R_1 \cap (Z_j \times Z_j)$ for $j = 1, 2$ then $(x, y)a_{1+j} = x$ therefore if $f(x, y) \in R_2$ then $f(a_{1+j}) \neq a_0, a_1$, if $f(x, y) \in R_2 \cap (Z_i \times Z_i)$ for $i = 1, 2$ then $f(a_{1+j}) \neq a_0, a_1, a_{1+i}$. Thus the first statement is proved and the second statement is proved by verification of all possibilities. □

Lemma 4.8. *Let $f : \Lambda_3\mathcal{X}_1 \to \Lambda_3\mathcal{X}_2$ be a semigroup homomorphism. If $f(a_0) = a_i$ for $i = 2, 3$, $f(a_1) = a_1$ and $f(0) = 0$ then $f(a_2), f(a_3) \in \{a_0, a_1, a_2, a_3, a_4\}$, $f(a_4) = a_4$, $f(X_1 \cup \{0\}) = \{0\}$, $f(R_1) \subseteq X_2 \cup \{0\}$.*

Conversely, any mapping satisfying these conditions is a semigroup homomorphism from $\Lambda_3\mathcal{X}_1$ into $\Lambda_3\mathcal{X}_2$.

Proof. Assume that $f : \Lambda_3\mathcal{X}_1 \to \Lambda_3\mathcal{X}_2$ is a semigroup homomorphism with $f(a_0) = a_i$ for $i = 2, 3$, $f(a_1) = a_1$ and $f(0) = 0$. Then $f(a_4) = a_4$ and $f(a_2), f(a_3) \in \{a_0, a_1, a_2, a_3, a_4\}$. ¿From $f(0) = 0$ it follows that $f(R_1 \cup X_1) \subseteq R_2 \cup X_2 \cup \{0\}$. Since $(x, y) \in R_2$ is irreducible, we conclude that $f(X_1) \cap R_2 = \emptyset$. Observe that if there exists $(x, y) \in R_1$ with $f(x, y) \in R_2 \cap (Z_{i-1} \times Z_{i-1})$ then $f(x), f(y) \in Z_{i-1}$ and $f(x, y) = (f(x), f(y))$ and if $f(x) \in Z_{i-1}$ then $f(x, y) \in R_2 \cap (Z_{i-1} \times Z_{i-1})$ for all $(x, y) \in R_1$. Since (X_1, R_1) is strongly connected, by an easy induction we obtain that $f(X_1) \subseteq Z_{i-1} \subseteq Z$, $f(R_1) \subseteq R_2 \cap (Z_{i-1} \times Z_{i-1})$ with $f(x, y) = (f(x), f(y))$. Thus the domain-range restriction g of f to X_1 and X_2 is a digraph homomorphism from $\mathcal{X}_1$ into $\mathcal{X}_2$ with $\mathrm{Im}(g) \subseteq Z$ – this contradicts to the property of $\mathbb{DG}_{\mathcal{Z}}$. Hence $f(R_1) \cap R_2 \cap (Z_{i-1} \times Z_{i-1}) = \emptyset$. Thus for every $(x, y) \in R$, $f(x) = f(x, y)f(a_0) = 0$ and we conclude that $f(X_1) = \{0\}$ and $f(R_1) \subseteq X_2 \cup \{0\}$ because $f(x, y)a_1 = 0$. The first statement is proved and the second statement follows by a direct verification. □

Lemma 4.9. *Let $f : \Lambda_3\mathcal{X}_1 \to \Lambda_3\mathcal{X}_2$ be a semigroup homomorphism. If $f(a_1) = a_i$ for $i = 2, 3$, $f(a_0) = a_1$ and $f(0) = 0$ then $f(a_2), f(a_3) \in \{a_0, a_1, a_2, a_3, a_4\}$, $f(a_4) = a_4$, $f(X_1 \cup \{0\}) = \{0\}$, and $f(R_1) \subseteq X_2 \cup \{0\}$.*

Conversely, any mapping satisfying these conditions is a semigroup homomorphism from $\Lambda_3\mathcal{X}_1$ to $\Lambda_3\mathcal{X}_2$.

Proof. Assume that $f : \Lambda_3\mathcal{X}_1 \to \Lambda_3\mathcal{X}_2$ is a semigroup homomorphism with $f(a_1) = a_i$ for $i = 2, 3$, $f(a_0) = a_1$ and $f(0) = 0$. Then $f(a_4) = a_4$ and $f(a_2), f(a_3) \in \{a_0, a_1, a_2, a_3, a_4\}$. ¿From $f(0) = 0$ it follows that $f(R_1 \cup X_1) \subseteq R_2 \cup X_2 \cup \{0\}$. Since $(x, y) \in R_2$ is irreducible, we conclude that $f(X_1) \cap R_2 = \emptyset$. Observe that if there exists $(x, y) \in R_1$ with $f(x, y) \in R_2 \cap (Z_{i-1} \times Z_{i-1})$ then $f(x), f(y) \in Z_{i-1}$ and $f(x, y) = (f(y), f(x))$ and if $f(y) \in Z_{i-1}$ then $f(x, y) \in R_2 \cap (Z_{i-1} \times Z_{i-1})$ for all $(x, y) \in R_1$. Since (X_1, R_1) is strongly connected, by an easy induction we obtain that $f(X_1) \subseteq Z_{i-1} \subseteq Z$, $f(R_1) \subseteq R_2 \cap (Z_{i-1} \times Z_{i-1})$ with $f(x,y) = (f(y), f(x))$. Thus the domain-range restriction g of f to X_1 and X_2 is a digraph homomorphism from $\mathcal{X}_1$ into (X_2, U_2) for $U_2 = \{(u,v) \mid (v,u) \in R_2\}$ with $\operatorname{Im}(g) \subseteq Z$ – this contradicts to the property of $\mathbb{DG}_{\mathcal{Z}}$. Hence $f(R_1) \cap R_2 \cap (Z_{i-1} \times Z_{i-1}) = \emptyset$. Thus for every $(x,y) \in R$, $f(y) = f(x,y)f(a_1) = 0$ and we conclude that $f(X_1) = \{0\}$ and $f(R_1) \subseteq X_2 \cup \{0\}$ because $f(x,y)a_0 = 0$. The first statement is proved, and the second statement follows by a direct verification. □

Lemma 4.10. *Let $f : \Lambda_3\mathcal{X}_1 \to \Lambda_3\mathcal{X}_2$ be a semigroup homomorphism. If $f(a_4) = f(0) = a_4$ then $f(\{a_0, a_1, a_2, a_3\}) \subseteq \{a_0, a_1, a_2, a_3, a_4\}$ and $f(R_1 \cup X_1 \cup \{a_4\}) = \{a_4\}$.*

Conversely, any mapping satisfying these conditions is a semigroup homomorphism from $\Lambda_3\mathcal{X}_1$ into $\Lambda_3\mathcal{X}_2$.

Proof. Let $f : (Y_1, \cdot) \to (Y_2, \cdot)$ be a semigroup homomorphism with $f(a_4) = f(0) = a_4$. Clearly, $f(R_1 \cup X_1 \cup \{a_0, a_1, a_2, a_3\}) \subseteq \{a_0, a_1, a_2, a_3, a_4\}$. Since $a_4 x = x$ for all $x \in X_1$ and $a_4(x,y) = (x,y)$ for all $(x,y) \in R_1$ and because $a_i a_j = a_4$ for all $i, j \in \{0,1,2,3,4\}$, we conclude that $f(R_1 \cup X_1) = \{a_4\}$ and the first statement is proved. A straightforward calculation proves the second statement. □

Lemma 4.11. *Let $f : \Lambda_3\mathcal{X}_1 \to \Lambda_3\mathcal{X}_2$ be a semigroup homomorphism. If $f(a_4) = 0$ then $f(R_1 \cup X_1 \cup \{0\}) = \{0\}$ and $f(\{a_0, a_1, a_2, a_3\}) \subseteq R_2 \cup X_2 \cup \{0\}$.*

Conversely, any mapping satisfying these conditions is a semigroup homomorphism from $\Lambda_3\mathcal{X}_1$ into $\Lambda_3\mathcal{X}_2$.

Proof. Let $f : (Y_1, \cdot) \to (Y_2, \cdot)$ be a semigroup homomorphism with $f(a_4) = 0$. From $a_4 0 = 0$ it follows $f(0) = 0$ and $f(Y_1) \subseteq R_2 \cup X_2 \cup \{0\}$. From $a_4 x = x$ for all $x \in X_1$ and $a_4(x,y) = (x,y)$ for all $(x,y) \in R_1$ it follows that $f(R_1 \cup X_1 \cup \{0\}) = \{0\}$ and the first statement is proved. A direct verification proves the second statement. □

Theorem 4.12. *The variety* $\operatorname{Var}(\mathbf{M}_4)$ *is* $\operatorname{Var}(\mathbf{M}_1)$*-relatively ff-alg-universal.*

Proof. For digraphs $\mathcal{X}_1, \mathcal{X}_2 \in \mathbb{DG}_{\mathcal{Z}}$ let $f : \Lambda_3\mathcal{X}_1 \to \Lambda_3\mathcal{X}_2$ be a semigroup homomorphism. Then idempotents of $\Lambda_3\mathcal{X}_i$ are only 0 and a_4 and hence $f(\{0, a_4\}) \subseteq \{0, a_4\}$. Thus Proposition 4.3 and Lemmas 4.4 – 4.11 imply that Λ_3 is $\operatorname{Var}(\mathbf{M}_1)$-relatively full embedding, and Theorem 4.1 completes the proof. □

Theorem 4.13. *The variety* $\operatorname{Var}(\mathbf{M}_4)$ *is α-determined for no cardinal α.*

Proof. Let (X_1, R_1) and (X_2, R_2) be digraphs from $\mathbb{DG}_{\mathcal{Z}}$ such that X_1 and X_2 have the same cardinality and R_1 and R_2 have the same cardinality. Choose bijections $\phi : X_1 \to X_2$ and $\psi : R_1 \to R_2$. Then, by Lemmas 4.5 – 4.11, these bijections induce a semigroup isomorphism between transformation semigroups

$$\begin{aligned}\{f \in \Lambda_3(X_1, R_1) \mid &\text{the subsemigroup of } \Lambda_3(X_1, R_1) \\ &\text{on } \operatorname{Im}(f) \text{ belongs to } \operatorname{Var}(\mathbf{M}_1)\}\end{aligned}$$

and

$$\{f \in \Lambda_3(X_2, R_2) \mid \text{the subsemigroup of } \Lambda_3(X_2, R_2) \\ \text{on } \operatorname{Im}(f) \text{ belongs to } \operatorname{Var}(\mathbf{M}_1)\}\,.$$

The proof is completed by the fact that for every infinite cardinal α there exist 2^α non-isomorphic rigid digraphs (X, R) in $\mathbb{DG}_{\mathcal{Z}}$ such that the cardinality of X and the cardinality of R are α. □

Let $P(\omega)$ be the set of non-empty finite subsets of natural numbers (we recall that $P(\omega_0) = P(\omega) \cup \{\emptyset\}$). We recall a construction from [11]. There exists a family $\{\mathbf{G}_A \mid A \in P(\omega)\}$ of unordered finite connected graphs such that

(1) if $A \subseteq B$ then there exists an extremal epimorphism $g_{B,A} : \mathbf{G}_B \longrightarrow \mathbf{G}_A$;
(2) if $f : \mathbf{G}_A \longrightarrow \mathbf{G}_B$ is a graph homomorphism then $B \subseteq A$ and $f = g_{A,B}$ (thus $g_{A,A}$ is the identity);
(3) a finite family $\{g_{A,B_i} : \mathbf{G_A} \longrightarrow \mathbf{G}_{B_i} \mid i \in I\}$ of graph homomorphisms is separating if and only if $A = \bigcup_{i\in I} B_i$ ($A, B_i \in P(\omega)$ for all $i \in I$).

Let $\mathbb{N}$ be the category whose objects are sets from $P(\omega)$ and there exists a $\mathbb{N}$-morphism from B to A just when $A \subseteq B$ – and there is only one such morphism, denoted by $g_{B,A}$. Thus the construction is a functor from $\mathbb{N}$ into $\mathbb{GR}$. For a graph $\mathbf{G}_A = (V_A, E_A)$ consider $R_A = \{(x,y) \mid \{x,y\} \in E_A\}$ and a groupoid $\Theta(A) = (U_A, \cdot)$ where $U_A = \{a_0, a_1, a_2, 0\} \cup V_A \cup R_A$ (assume that $V_A \cap R_A = \emptyset = \{a_0, a_1, a_2, 0\} \cap (V_A \cup R_A)$ and $a_0, a_1, a_2, 0$ are pairwise distinct elements) and

$$xy = \begin{cases} a_2 & \text{if } x, y \in \{a_0, a_1, a_2\}\,, \\ y & \text{if } y \in V_A \cup R_A \cup \{0\} \text{ and } x \in \{a_1, a_2, a_3\}\,, \\ 0 & \text{if } 0 \in \{x,y\} \text{ or } x \in V_A \cup \{0\} \text{ or } x, y \in V_A \cup R_A\,, \\ u & \text{if } x = (u,v) \in R_A \text{ and } y = a_0\,, \\ v & \text{if } x = (u,v) \in R_A \text{ and } y = a_1\,, \\ 0 & \text{if } x = (u,v) \in R_A \text{ and } y = a_2\,. \end{cases}$$

By a direct verification, we obtain the correctness of the definition. For $A, B \in P(\omega)$ with $B \subseteq A$ define $\Theta g_{A,B}$ such that

$$\Theta g_{A,B}(x) = \begin{cases} x & \text{if } x \in \{a_0, a_1, a_2, 0\}\,, \\ g_{B,A}(x) & \text{if } x \in V_A\,, \\ (g_{B,A}(u), g_{B,A}(v)) & \text{if } x = (u,v) \in R_B\,. \end{cases}$$

Proposition 4.14. $\Theta : \mathbb{N} \longrightarrow \operatorname{Var}(\mathbf{M}_4)$ *is a functor such that* $\Theta g_{B,A}$ *is surjective for all* $B, A \in P(\omega)$ *with* $A \subseteq B$*, and a finite family* $\{\Theta g_{B,A_i} \mid i \in I\}$ *is separating for* $B, A_i \in P(\omega)$ *for* $i \in I$ *if and only if* $B = \bigcup_{i\in I} A_i$.

Proof. Consider the subsemigroup $\mathbf{S}_2$ of the semigroup $\mathbf{S}_1$ from the proof of Proposition 4.2 on the set $S_2 = \{a_0, a_1, a_4, c, d_0, d_1, 0\}$. By a direct calculation we obtain that $\mathbf{S}_2$ is a subsemigroup of $\mathbf{S}_1$, thus $\mathbf{S}_2 \in \operatorname{Var}(\mathbf{M}_4)$. Consider a subsemigroup $\mathbf{S}_A$ of $(\mathbf{S}_2)^{R_A}$ consisting of mappings $\alpha : R_A \longrightarrow S_2$ such that

α is the constant mapping with the value a_i for $i = 0, 1, 2$;
there exists $r \in R_A$ such that $\alpha(r) \in \{c, d_0, d_1\}$ and $\alpha(q) = 0$ for all $q \in R_A \setminus \{r\}$;
α is the constant mapping with the value 0.

It is easy to prove that $\mathbf{S}_A$ is a subsemigroup of $(\mathbf{S}_2)^{R_A}$. Let $\sim$ be the least equivalence such that

there exist $(x, y), (x, z) \in R_A$ such that $\alpha(x, y) = d_0$ and $\beta(x, z) = d_0$;
there exist $(z, x), (x, y) \in R_A$ such that $\alpha(z, x) = d_1$ and $\beta(x, y) = d_0$;
there exist $(y, x), (z, x) \in R_A$ such that $\alpha(y, x) = d_1$ and $\beta(z, x) = d_1$.

It is routine to verify that $\sim$ is a congruence and $\mathbf{S}_A / \sim$ is isomorphic to ΘA for every $A \in P(\omega)$. The correctness of $\Theta g_{A,B}$ is clear and by a straightforward computation, $\Theta g_{A,B} : \Theta A \longrightarrow \Theta B$ is a semigroup homomorphism whenever $B \subseteq A$. These homomorphisms are closed under composition, and hence $\Theta : \mathbb{N} \longrightarrow \operatorname{Var}(\mathbf{M}_4)$ is a functor. The other statements immediately follow from the definition of Θ and from the properties of $\{\mathbf{G}_A \mid A \in P(\omega)\}$. $\square$

Lemma 4.15. *Let $f : \Theta A \longrightarrow \Theta B$ be a semigroup homomorphism for $A, B \in P(\omega)$. Then one of the following cases occurs:*

(1) $\{f(a_0), f(a_1)\} = \{a_0, a_1\}$, $f(a_2) = a_2$, $f(0) = 0$ *and there exists a graph homomorphism* $g : \mathbf{G}_A \longrightarrow \mathbf{G}_B$ *such that* $f(v) = g(v)$ *for all* $v \in V_A$ *and either* $f((u, v)) = (g(u), g(v))$ *or* $f((u, v)) = (g(v), g(u))$ *for all* $(u, v) \in R_A$*;*
(2) $f(a_0) = f(a_1) \in \{a_0, a_1\}$, $f(a_2) = a_2$, $f(0) = 0$ *and there exists* $v \in V_B$ *such that* $f(V_A) = \{v\}$ *and either* $f(R_A) \subseteq \{(u, v) \mid (u, v) \in R_B\}$ *or* $f(R_A) \subseteq \{(v, u) \mid (v, u) \in R_B\}$*;*
(3) $f(a_0) = f(a_1) = f(a_2) = a_2$, $f(V_A \cup \{0\}) = \{0\}$ *and* $f(R_A) \subseteq R_B \cup V_B \cup \{0\}$*;*
(4) $f(a_0), f(a_1) \in \{a_0, a_1, a_2\}$, $f(a_2) = a_2$, $f(V_A \cup \{0\}) = \{0\}$ *and* $f(R_A) \subseteq V_B \cup \{0\}$*;*
(5) $f(a_0), f(a_1) \in \{a_0, a_1, a_2\}$ *and* $f(R_A \cup V_A \cup \{0, a_2\}) = \{a_2\}$*;*
(6) $f(\{a_0, a_1\} \cup R_A) \subseteq R_B \cup V_B \cup \{0\}$ *and* $f(V_A \cup \{0, a_2\}) = \{0\}$*.*

Proof. Observe that $f(\{a_2, 0\}) \subseteq \{0, a_2\}$ and

if $\{f(a_0), f(a_1)\} \cap \{a_0, a_1, a_2\} \neq \emptyset$ then $f(a_0), f(a_1) \in \{a_0, a_1, a_2\}$ and $f(a_2) = a_2$,
if $\{f(a_0), f(a_1)\} \cap (R_B \cup V_B \cup \{0\}) \neq \emptyset$ then $f(a_0), f(a_1) \in R_B \cup V_B \cup \{0\}$ and $f(a_2) = 0$,
if $f(R_A \cup V_A \cup \{0\}) \cap \{a_0, a_1, a_2\} \neq \emptyset$ then $f(R_A \cup V_A) \subseteq \{a_0, a_1, a_2\}$ and $f(0) = a_2$,
if $f(R_A \cup V_A \cup \{0\}) \cap (R_B \cup V_B \cup \{0\}) \neq \emptyset$ then $f(R_A \cup V_A) \subseteq R_B \cup V_B \cup \{0\}$ and $f(0) = 0$.

Since $(u, v)a_0 = u$ and $(u, v)a_1 = v$ for all $(u, v) \in R_A$ and since a_0, a_1 and elements of V_B are irreducible and $\mathbf{G}_A$ is connected, we conclude that $f(V_A) \cap (R_B \cup \{a_0, a_1\}) = \emptyset$.

Steps below are simple modifications of Lemmas 4.5 – 4.11. First let $f(a_0)$, $f(a_1) \in \{a_0, a_1, a_2\}$ and $f(R_A \cup V_A \cup \{0\}) \subseteq R_B \cup V_B \cup \{0\}$. Then $f(a_2) = a_2$, $f(0) = 0$ and $f(V_A) \subseteq V_B \cup \{0\}$. From $(u,v)a_0 = u$, $(u,v)a_1 = v$ for all $(u,v) \in R_A$ and from the fact that (V_A, R_A) is strongly connected (because $\mathbf{G}_A$ is connected) we conclude that if $f(a_0), f(a_1) \in \{a_0, a_1\}$ then of the following cases occurs:

1. (1) $f(a_0) = a_0$, $f(a_1) = a_1$ and there exists a graph homomorphism $g : \mathbf{G}_A \to \mathbf{G}_B$ such that $f(v) = g(v)$ for all $v \in V_A$ and $f(u,v) = (f(u), f(v)) = (g(u), g(v))$ for all $(u,v) \in R_A$;
2. (2) $f(a_0) = a_1$, $f(a_1) = a_0$ and there exists a graph homomorphism $g : \mathbf{G}_A \to \mathbf{G}_B$ such that $f(v) = g(v)$ for all $v \in V_A$ and $f(u,v) = (f(v), f(u)) = (g(v), g(u))$ for all $(u,v) \in R_A$ (because (V_A, R_A) is a symmetric digraph we conclude that g is a graph homomorphism);
3. (3) $f(a_0) = f(a_1) = a_0$ and there exists $v \in V_B$ such that $f(V_A) = \{v\}$ and $f(R_A) \subseteq \{(v,u) \mid (v,u) \in R_B\}$;
4. (4) $f(a_0) = f(a_1) = a_1$ and there exists $v \in V_B$ such that $f(V_A) = \{v\}$ and $f(R_A) \subseteq \{(u,v) \mid (u,v) \in R_B\}$;
5. (5) $f(V_A) = \{0\}$ and $f(R_A) \subseteq V_B \cup \{0\}$.

If $a_2 \in \{f(a_0), f(a_1)\}$ then $(u,v)a_0 = u$, $(u,v)a_1 = v$ for all $(u,v) \in R_A$, and hence the facts that (V_A, R_A) is strongly connected and $ta_2 = 0$ for all $t \in R_B \cup V_B \cup \{0\}$ imply that $f(V_A) = \{0\}$. If $\{f(a_0, f(a_1)\} \setminus \{a_2\} \neq \emptyset$ then, moreover, we deduce that $f(R_A) \subseteq V_B \cup \{0\}$ (because $f(u,v)a_2 = 0$ for all $(u,v) \in R_A$) and if $f(a_0) = f(a_1) = a_2$ then $f(R_A) \subseteq R_B \cup V_B \cup \{0\}$.

Secondly if $f(a_2) = f(0)$ then $a_2 t = t$ for all $t \in R_A \cup V_A$ implies that $f(R_A \cup V_A) = \{f(0)\}$ and the rest follows from the foregoing observations. Since $f(a_2) = 0$ and $f(0) = a_2$ is impossible the proof is complete. □

Lemma 4.15 is exploited for the proof that the family $\{\Theta A \mid A \in P(\omega)\}$ extended by the terminal object satisfies Dziobiak's conditions. Formally define $\mathbf{S}_\emptyset$ is a singleton semigroup and $\mathbf{S}_A = \Theta A$ for all $A \in P(\omega)$. Then

Theorem 4.16. *The family* $\{\mathbf{S}_A \mid A \in P(\omega_0)\}$ *of finite semigroups from the variety* $\mathrm{Var}\,(\mathbf{M}_4)$ *satisfies Dziobiak's conditions* (P1)–(P4) *from Theorem 1.3. The variety* $\mathrm{Var}\,(\mathbf{M}_4)$ *is Q-universal.*

Proof. Clearly, (P1) holds. From Proposition 4.14 follows the condition (P2). To prove (P3), suppose that $\mathbf{S}_A \in \mathrm{Qua}(\mathbf{S}_B)$ for $A \in P(\omega)$. Then the family of all semigroup homomorphism from $\mathbf{S}_A$ into $\mathbf{S}_B$ is separating. By Lemma 4.15, then there exists an injective graph homomorphism from $\mathbf{G}_A$ into $\mathbf{G}_B$ and this is true just when $A = B$. Thus (P3) holds.

We prove (P4). Let $\mathbf{T}$ and $\mathbf{U}$ be finite semigroups from $\mathrm{Qua}(\{\mathbf{S}_A \mid A \in \mathcal{F}\}$ for a finite set $\mathcal{F} \subseteq P(\omega)\}$ such that there exists an injective homomorphism $f : \mathbf{S}_A \to \mathbf{T} \times \mathbf{U}$ for some $A \in P(\omega)$. Let $\pi_0 : \mathbf{T} \times \mathbf{U} \to \mathbf{T}$ and $\pi_1 : \mathbf{T} \times \mathbf{U} \to \mathbf{U}$ be the two projections. We can assume that $\pi_0 \circ f$ and $\pi_1 \circ f$ are surjective – it is easy to see that if the statement of (P4) holds for the restricted case then (P4) is true.

Let $\{g_i : \mathbf{T} \to \mathbf{S}_{B_i} \mid i \in I\}$ and $\{h_j : \mathbf{U} \to \mathbf{S}_{C_j} \mid j \in J\}$ be separating families of semigroup homomorphisms (these exist because $\mathbf{T}, \mathbf{U} \in \mathrm{Qua}(\{\mathbf{S}_A \mid A \in \mathcal{F}\}$, for finite $\mathcal{F} \subseteq P(\omega)\})$ where $B_i, C_j \in P(\omega)$ for all $i \in I$ and $j \in J$. We can assume that I and J are finite because $\mathbf{T}$ and $\mathbf{U}$ and $\mathcal{F}$ are finite. Set

$$\begin{aligned} I' = \{i \in I \mid & \text{ the domain-range restriction of } g_i \circ \pi_0 \circ f \\ & \text{ is a graph homomorphism from } \mathbf{G}_A \text{ into } \mathbf{G}_{B_i}\}\,, \\ J' = \{j \in J \mid & \text{ the domain-range restriction of } h_j \circ \pi_1 \circ f \\ & \text{ is a graph homomorphism from } \mathbf{G}_A \text{ into } \mathbf{G}_{C_j}\}\,. \end{aligned}$$

Set $B = \bigcup_{i \in I'} B_i$ and $C = \bigcup_{j \in J'} C_j$. By Lemma 4.15, $\{g_i \circ \pi_0 \circ f \mid i \in I'\} \cup \{h_j \circ \pi_1 \circ f \mid j \in J'\}$ is a separating family, and from the properties of $\{\mathbf{G}_A \mid A \in P(\omega)\}$ we conclude that $A = B \cup C$. According to Lemma 4.15, for every $i \in I'$ (or $j \in J'$) there exists a semigroup homomorphism $\bar{g}_i : \mathbf{S}_B \to \mathbf{S}_{B_i}$ (or $\bar{h}_j : \mathbf{S}_C \to \mathbf{S}_{C_j}$) such that the domain-range restriction of $\bar{g}_i$ (or $\bar{h}_j$) to V_B and V_{B_i} (or V_C and V_{C_j}) is a graph homomorphism from $\mathbf{G}_B$ into $\mathbf{G}_{B_i}$ (or from $\mathbf{G}_C$ into $\mathbf{G}_{C_j}$), $g_i \circ \pi_0 \circ f = \bar{g}_i \circ \Theta g_{A,B}$ (or $h_j \circ \pi_1 \circ f = \bar{h}_j \circ \Theta g_{A,C}$) and the family $\{\bar{g}_i \mid i \in I\}$ (or $\{\bar{h}_j \mid j \in J\}$) is separating. Thus, by the diagonalization property, there exist semigroup homomorphisms $g_0 : \mathbf{T} \to \mathbf{S}_B$ and $h_0 : \mathbf{U} \to \mathbf{S}_C$ such that $g_0 \circ \pi_0 \circ f = \Theta g_{A,B}$, $h_0 \circ \pi_1 \circ f = \Theta g_{A,C}$, $\bar{g}_i \circ g_0 = g_i$ for all $i \in I'$, $\bar{h}_j \circ h_0 = h_j$ for all $j \in J'$ because $\pi_0 \circ f$ and $\pi_1 \circ f$ are surjective. Since $\Theta g_{A,B}$ and $\Theta g_{A,C}$ are surjective, we conclude that g_0 and h_0 are also surjective. Since $\Theta g_{A,B}^{-1}(a_i)$ and $\Theta g_{A,C}^{-1}(a_i)$ are singletons for $i = 0, 1, 2$ we find that $(g_0)^{-1}(a_i)$ and $(h_0)^{-1}(a_i)$ are singletons for $i = 0, 1, 2$. By Lemma 4.15, $(g_0)^{-1}(u)$ is a singleton for all $u \in V_B$ and $(h_0)^{-1}(v)$ is a singleton for all $v \in V_C$ (because $(\Theta g_{A,B})^{-1}(u), (\Theta g_{A,C})^{-1}(v) \subseteq V_A$ and $g_i \circ \pi_0 \circ f(V_A)$ is a singleton for all $i \in I \setminus I'$, $h_j \circ \pi_1 \circ f(V_A)$ is a singleton for all $j \in J \setminus J'$). Then $T' = (g_0)^{-1}(\{a_0, a_1, a_2\} \cup V_B) \cup \{\pi_0 \circ f(0)\}$ is a subsemigroup of $\mathbf{T}$ such that elements from the complement of T' are irreducible and analogously $U' = (h_0)^{-1}(\{a_0, a_1, a_2\} \cup V_C) \cup \{\pi_1 \circ f(0)\}$ is a subsemigroup of $\mathbf{U}$ such that elements from the complement of U' are irreducible. Therefore mappings $g_1 : \mathbf{S}_B \to \mathbf{T}$ and $h_1 : \mathbf{S}_C \to \mathbf{U}$ such that $g_0 \circ g_1$ and $h_0 \circ h_1$ are the identity mappings are semigroup homomorphisms, and (P4) is proved. The second statement follows from Theorem 1.4. □

By the duality, we obtain

Corollary 4.17. *The semigroup variety* $\mathrm{Var}\,(\mathbf{M}_4^{\mathrm{op}})$ *is* $\mathrm{Var}\,(\mathbf{M}_1^{\mathrm{op}})$*-relatively ff-alg-universal, Q-universal and α-determined for no cardinal α.*

5. Conclusion

Main results of the paper concern nearly $\mathcal{J}$-trivial varieties. By Corollary 3.6, the varieties $\mathrm{Var}\,(\mathbf{M}_1)$, $\mathrm{Var}\,(\mathbf{M}_1^{\mathrm{op}})$, $\mathrm{Var}\,(\mathbf{M}_2)$, $\mathrm{Var}\,(\mathbf{M}_4)$, and $\mathrm{Var}\,(\mathbf{M}_4^{\mathrm{op}})$ play the key role. The solutions of problems (a), (b) and (c) for varieties $\mathrm{Var}\,(\mathbf{M}_1)$, $\mathrm{Var}\,(\mathbf{M}_1^{\mathrm{op}})$ and $\mathrm{Var}\,(\mathbf{M}_2)$ are known, see Theorem 3.1. The solution for the varieties $\mathrm{Var}\,(\mathbf{M}_4)$ and $\mathrm{Var}\,(\mathbf{M}_4^{\mathrm{op}})$ is presented in the fourth section, see Theorems 4.12, 4.13, 4.16 and Corollary 4.17. As a combination of these facts we obtain

Theorem 5.1. *For a nearly $\mathcal{J}$-trivial variety $\mathbb{V}$ the following are equivalent*

(1) $\mathbb{V}$ *contains one of the semigroups* $\mathbf{M}_2$, $\mathbf{M}_4$, $\mathbf{M}_4^{\mathrm{op}}$*;*
(2) $\mathbb{V}$ *is var-relatively ff-alg-universal (more precisely,* $\mathbb{V}$ *is* $\mathrm{Var}\,(\mathbf{M}_1)$*-relatively ff-alg-universal or* $\mathrm{Var}\,(\mathbf{M}_1^{\mathrm{op}})$*-relatively ff-alg-universal or* $\mathbb{ZS}$*-relatively ff-alg-universal);*
(3) $\mathbb{V}$ *is Q-universal;*
(4) *there exists a sublattice of* $\mathrm{QL}(\mathbb{V})$ *isomorphic to the lattice of all ideals of the free lattice over an infinite countable set;*
(5) $\mathbb{V}$ *is α-determined for no cardinal α.*

Theorem 5.2. *For a nearly $\mathcal{J}$-trivial variety $\mathbb{V}$ the following are equivalent*

(1) $\mathbb{V}$ *is a subvariety of* $\mathrm{Var}\,(\mathbf{M}_1)$ *or of* $\mathrm{Var}\,(\mathbf{M}_1^{\mathrm{op}})$*;*
(2) $\mathbb{V}$ *is not var-relatively alg-universal;*
(3) $\mathbb{V}$ *is not Q-universal;*
(4) $\mathrm{QL}(\mathbb{V})$ *is finite;*
(5) $\mathbb{V}$ *is 3-determined.*

The second section contains solutions of problems (a), (b) and (c) for inflations of band quasivarieties. First we recall a partial solution of these problems for band varieties. The characterization of var-relatively alg-universal band varieties is complete, to characterize Q-universal band varieties it is necessary to decide Q-universality for the varieties $\mathbb{SLZ}$ and $\mathbb{SRZ}$. It is known that the varieties $\mathbb{LQN}$ and $\mathbb{RQN}$ are 5-determined, but for larger band varieties this problem is open. We hope that a solution of these problems for band varieties would help us fully understand properties of homomorphisms. Sapir's results [S1,S2] say that the lattice $\mathrm{QL}(\mathbb{RNB} \vee \mathrm{Var}\,(\mathbf{M}_1))$ is finite and the lattices $\mathrm{QL}(\mathbb{LNB} \vee \mathrm{Var}\,(\mathbf{M}_1))$ and $\mathrm{QL}(\mathbb{SLZ} \vee \mathrm{Var}\,(\mathbf{M}_1))$ are uncountably infinite. Therefore it would be interesting to solve problems (a), (b) and (c) for these varieties.

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