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RELATIVE COMMUTATOR ASSOCIATED WITH VARIETIES OF n-NILPOTENT AND OF n-SOLVABLE GROUPS

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Dedicated to Jiří Rosický on the occasion of his sixtieth birthday.

Abstract. In this note we determine explicit formulas for the relative commutator of groups with respect to the subvarieties of n-nilpotent groups and of n-solvable groups. In particular these formulas give a characterization of the extensions of groups that are central relatively to these subvarieties.

1. Introduction

A new notion of commutator was defined in any variety of Ω-groups $\mathcal{A}$, relative to any fixed subvariety $\mathcal{B}$ of $\mathcal{A}$ [1]. This notion gives back the usual commutator of normal subgroups when $\mathcal{A}$ is the variety of groups and $\mathcal{B}$ the subvariety of abelian groups (see Section 2), but also the Peiffer commutator of normal precrossed submodules, when $\mathcal{A}$ is the variety of precrossed modules and $\mathcal{B}$ the subvariety of crossed modules.

It is the aim of this paper to investigate this new notion of relative commutator in the variety of groups with respect to the classical subvarieties of n-nilpotent and of n-solvable groups. Although the characterization of the relative commutator is not trivial in these specific examples, it is fortunate that it can be expressed by a simple formula in terms of the usual commutator of normal subgroups. For instance, when $\mathcal{A} = \mathsf{Gp}$, the variety of groups, and $\mathcal{B} = \mathsf{Nil}_2$, the subvariety of nilpotent groups of class at most 2, the relative commutator $[M,N]_{\mathsf{Nil}_2}$ of normal subgroups M and N of a group A is given by

$$[M,N]_{\mathsf{Nil}_2} = \big[[M,N], M \cdot N\big]\,,$$

where the commutator in the right hand side is the classical one and $M \cdot N$ is the product of normal subgroups. On the other hand, when $\mathcal{B} = \mathsf{Sol}_2$, the subvariety of solvable groups of class at most 2, the relative commutator $[M,N]_{\mathsf{Sol}_2}$ equals

$$\big[[M,N],[M,N]\big] \cdot \big[[M,M],[N,N]\big] \cdot \big[[M,N],[N,N]\big] \cdot \big[[M,N],[M,M]\big]\,.$$

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As shown in [1], the relative commutator describes in particular the central extensions in the sense of Fröhlich [2] [6], a notion which was later generalized by Janelidze and Kelly [5]. More precisely, if $\mathcal{B}$ is a subvariety of a variety of Ω-groups $\mathcal{A}$, then an extension

$$\{1\} \longrightarrow N \longrightarrow A \longrightarrow B \longrightarrow \{1\}$$

in $\mathcal{A}$ is central with respect to $\mathcal{B}$ if and only if the relative commutator $[N, A]_{\mathcal{B}}$ is the trivial Ω-group $\{1\}$. Accordingly, the formulas above imply that an extension of groups is central with respect to Nil_2 if and only if $\big[[N, A], A\big] = \{1\}$, while it is central with respect to Sol_2 if and only if $\big[[N, A], [A, A]\big] = \{1\}$. This last result is also a special case of Proposition 20 in [3].

It would be interesting to know for which varieties of Ω-groups these same formulas are true.

2. The relative commutator

Let us first recall the definition of the relative commutator, and a few of its basic properties.

Suppose $\mathcal{A}$ is a variety of Ω-groups [4]: this means that the theory has among its operations and identities those of the variety of groups and for any n-ary operation $\omega \in \Omega$, one has the identity $\omega(1, \dots, 1) = 1$, where 1 denotes the unit of the group operation. The varieties of groups, abelian groups, (non unital) rings, associative algebras, Lie algebras and precrossed modules are all examples of varieties of Ω-groups.

When $\mathcal{B}$ is a subvariety of the variety $\mathcal{A}$, then $\mathcal{B}$ is completely determined by a set of identities of terms and all these identities are of the form $v(x_1, \dots, x_n) = 1$. These terms $v(x_1, \dots, x_n)$ constitute an Ω-group

$$W = W_{\mathcal{B}} = \{v(x_1, \dots, x_n) \mid v(b_1, \dots, b_n) = 1, \forall B \in \mathcal{B}, \forall b_i \in B\}\,,$$

an Ω-subgroup of the Ω-group of terms $F(\mathbb{N})$, the free Ω-group of $\mathcal{A}$ on a countable set. An Ω-group B belongs to $\mathcal{B}$ if and only if $v(b_1, \dots, b_n) = 1$ for all $v \in W$ and $b_i \in B$.

Let M and N be normal Ω-subgroups (=kernels) of an Ω-group $A \in \mathcal{A}$. The relative commutator $[M, N]_{\mathcal{B}}$ is defined as the normal Ω-subgroup of $M \cdot N$ (the smallest normal Ω-subgroup of A containing M and N) generated by all elements

$$v(m_1 n_1, \dots, m_n n_n) v(n_1, \dots, n_n)^{-1} v(m_1, \dots, m_n)^{-1}$$

and

$$v(p_1, \dots, p_n)\,,$$

where the m_i are in M, the n_i in N, the p_i in $M \cap N$, and $v(x_1, \dots, x_n)$ is in $W_{\mathcal{B}}$.

Let us mention the following properties of the relative commutator:

Proposition 2.1 ([1]). *For any Ω-group $A \in \mathcal{A}$ and normal Ω-subgroups M, N, N' of A, one has:*

(1) $[A, A]_{\mathcal{B}} = \{1\} \Leftrightarrow A \in \mathcal{B}$;
(2) $[M, N]_{\mathcal{B}} = [N, M]_{\mathcal{B}}$;

(3) $[M,N]_{\mathcal{B}} \subset M \cap N$;
(4) *universal property:*

$$\Big[\frac{M}{[M,N]_{\mathcal{B}}}, \frac{N}{[M,N]_{\mathcal{B}}}\Big]_{\mathcal{B}} = \{1\};$$

moreover, $[M,N]_{\mathcal{B}}$ *is the smallest normal* Ω*-subgroup* I *of* $M\cdot N$ *such that* $[\frac{M}{I}, \frac{N}{I}]_{\mathcal{B}} = \{1\}$.

From now on, $\mathcal{A}$ will always be Gp, the variety of groups. Thanks to the universal property above, it is now easy to show that the relative commutator with respect to the subvariety Ab of abelian groups coincides with the classical commutator of groups: it suffices to prove that

$$[M,N] = \{1\} \Leftrightarrow [M,N]_{\mathsf{Ab}} = \{1\}\,,$$

for any group A and normal subgroups M and N of A. Indeed, when the equivalence between these two conditions holds, we have, for all A, M and N, that

$$\Big[\frac{M}{[M,N]}, \frac{N}{[M,N]}\Big]_{\mathsf{Ab}} = \{1\}$$

hence, by the universal property of the relative commutator,

$$[M,N]_{\mathsf{Ab}} \subset [M,N]\,.$$

Similarly, thanks to the universal property of the classical commutator of groups, we get that

$$[M,N] \subset [M,N]_{\mathsf{Ab}}\,,$$

hence $[M,N]_{\mathsf{Ab}} = [M,N]$.

So let us first assume that $[M,N] = \{1\}$. Since, obviously,

$$[M\cap N, M\cap N] = \{1\}\,,$$

we only have to prove that

$$v(m_1n_1,\ldots,m_nn_n) = v(m_1,\ldots,m_n)v(n_1,\ldots,n_n)$$

for all $v \in W_{\mathsf{Ab}}$, $m_i \in M$ and $n_i \in N$. It suffices to show this for $v = v_0(x_1,x_2) = [x_1,x_2] = x_1x_2x_1^{-1}x_2^{-1}$. Indeed, every $v \in W_{\mathsf{Ab}}$ is a product $w_1(x_1^1,x_2^1)\ldots w_n(x_1^n,x_2^n)$, with the w_i either v_0 or v_0^{-1}. Furthermore, by the assumption,

$$v_0(m_1,m_2)v_0(n_1,n_2) = v_0(n_1,n_2)v_0(m_1,m_2)$$

for all $m_i \in M$ and $n_i \in N$.

Clearly, under the assumption $[M,N] = \{1\}$,

$$\begin{aligned}[m_1n_1, m_2n_2] &= m_1n_1m_2n_2n_1^{-1}m_1^{-1}n_2^{-1}m_2^{-1}\\ &= [m_1,m_2][n_1,n_2]\,.\end{aligned}$$

Conversely, whenever $[M,N]_{\mathsf{Ab}} = \{1\}$, the identity

$$[m_1n_1, m_2n_2] = [m_1,m_2][n_1,n_2]$$

holds for all $m_i \in M$ and $n_i \in N$. In particular, by choosing $n_1 = m_2 = 1$, we see that the word $[m_1,n_2] = 1$ for all m_1 in M and n_2 in N, thus $[M,N] = \{1\}$.

3. The case of n-nilpotent groups

Recall that a group A is *n-nilpotent* (=nilpotent of class at most n) if and only if $Z_n(A) = \{1\}$, where $Z_n(A)$ is the n-th term in the descending central series of A defined by

$$\begin{aligned} Z_1(A) &= [A, A]\,, \\ Z_2(A) &= \big[[A, A], A\big]\,, \\ &\vdots \\ Z_n(A) &= [Z_{n-1}(A), A]\,. \end{aligned}$$

We write Nil_n for the variety of n-nilpotent groups (for some positive integer n), which we consider as a subvariety of the variety Gp of groups. We are first going to characterize the commutator relative to the subvariety of 2-nilpotent groups. For this, the following remark will be useful:

Lemma 3.1. *For normal subgroups M and N of a group A, we have the following inclusion:*

$$\big[[M, M], N\big] \subseteq \big[[M, N], M\big]$$

Proof. By the universal property of the commutator of normal subgroups, it suffices to show that $\big[[M, N], M\big] = \{1\}$ implies $\big[[M, M], N\big] = \{1\}$, for every group A and normal subgroups M and N of A. Indeed, the right hand side in the identity

$$\Big[\frac{[M, M]}{[[M, N], M]}, \frac{N}{[[M, N], M]}\Big] = \Big[\Big[\frac{M}{[[M, N], M]}, \frac{M}{[[M, N], M]}\Big], \frac{N}{[[M, N], M]}\Big]$$

equals $\{1\}$ under this assumption.

Now, suppose that $\big[[M, N], M\big] = \{1\}$. We have, for any $m_1, m_2 \in M$ and $n \in N$:

$$\begin{aligned} &m_1 m_2 m_1^{-1} m_2^{-1} n m_2 m_1 m_2^{-1} m_1^{-1} n^{-1} \\ &= m_1 m_2 (m_1^{-1})(m_2^{-1} n m_2 n^{-1}) n m_1 m_2^{-1} m_1^{-1} n^{-1} \\ &= m_1 m_2 (m_2^{-1} n m_2 n^{-1})(m_1^{-1}) n m_1 m_2^{-1} m_1^{-1} n^{-1} \\ &= m_1 n m_2 n^{-1} m_1^{-1} n m_1 m_2^{-1} m_1^{-1} n^{-1} \\ &= m_1 (n m_2 n^{-1})(m_1^{-1} n m_1 n^{-1}) n m_2^{-1} m_1^{-1} n^{-1} \\ &= m_1 (m_1^{-1} n m_1 n^{-1})(n m_2 n^{-1}) n m_2^{-1} m_1^{-1} n^{-1} \\ &= 1\,. \end{aligned}$$

□

The following simple Lemma will prove extremely useful. Roughly speaking, it says that the commutator commutes with the group multiplication, up to conjugation of one of the terms and up to the order of the terms (which may have to be reversed).

Lemma 3.2. *For the ordinary group commutator, the following Witt-Hall identities hold (xyx^{-1} is denoted by ${}^xy$):*

$$[a, b_1b_2] = [a, b_1]\ {}^{b_1}[a, b_2] \tag{A}$$

$$= [a, b_2]\ [{}^{b_2}a, b_1] \tag{B}$$

$$[a_1a_2, b] = {}^{a_1}[a_2, b]\ [a_1, b] \tag{C}$$

$$= [a_1, {}^{a_2}b]\ [a_2, b] \tag{D}$$

We are now ready to characterize the relative commutator with respect to the subvariety of 2-nilpotent groups.

Theorem 3.3. *If $\mathcal{A} = \mathsf{Gp}$ is the variety of groups and Nil_2 its subvariety of 2-nilpotent groups, then, for any group A and normal subgroups M and N of A, one has*

$$[M, N]_{\mathsf{Nil}_2} = [[M, N], M \cdot N]$$

Proof. Recall that $\big[[M, N], M \cdot N\big]$ is the smallest normal subgroup I of $M \cdot N$ such that

$$\Big[\Big[\frac{M}{I}, \frac{N}{I}\Big], \frac{M \cdot N}{I}\Big] = \{1\}\,.$$

Thanks to this observation, together with the universal property of the relative commutator $[\cdot, \cdot]_{\mathsf{Nil}_2}$, it suffices to prove, for any group A and any normal subgroups M and N of A, that

$$\big[[M, N], M \cdot N\big] = \{1\} \Leftrightarrow [M, N]_{\mathsf{Nil}_2} = \{1\}\,.$$

Suppose first that $[M, N]_{\mathsf{Nil}_2} = \{1\}$. Then, in particular,

$$\big[[m_1n_1, m_2n_2], m_3n_3\big] = \big[[m_1, m_2], m_3\big]\ \big[[n_1, n_2], n_3\big]$$

for all $m_i \in M$ and $n_i \in N$. In particular (choose $n_1 = m_2 = n_3 = 1$),

$$\big[[m_1, n_2], m_3\big] = 1$$

for all $m_1, m_3 \in M$ and $n_2 \in N$, hence $\big[[M, N], M\big] = \{1\}$. Similarly, one has $\big[[M, N], N\big] = \{1\}$.

Conversely, suppose $\big[[M, N], M{\cdot}N\big] = \{1\}$. Note that in this case also $\big[[M, M], N\big] = \{1\} = \big[[N, N], M\big]$, by Lemma 3.1.

Clearly, we have

$$\big[[M \cap N, M \cap N], M \cap N\big] = \{1\}\,.$$

It remains to be shown that

$$v(m_1n_1, \dots, m_nn_n) = v(m_1, \dots, m_n)v(n_1, \dots, n_n)$$

for all $v \in W_{\mathsf{Nil}_2}$, $m_i \in M$ and $n_i \in N$.

It suffices to show this for $v = v_0(x_1, x_2, x_3) = \big[[x_1, x_2], x_3\big]$. Indeed, every $v \in W_{\mathsf{Nil}_2}$ is a product $w_1(x_1^1, x_2^1, x_3^1) \dots w_n(x_1^n, x_2^n, x_3^n)$, with the w_i either v_0 or v_0^{-1}. Furthermore, by the assumptions and Lemma 3.1,

$$v_0(m_1, m_2, m_3)v_0(n_1, n_2, n_3) = v_0(n_1, n_2, n_3)v_0(m_1, m_2, m_3)$$

for all $m_i \in M$ and $n_i \in N$. Let us therefore prove that

$$\big[[m_1n_1, m_2n_2], m_3n_3\big] = \big[[m_1, m_2], m_3\big]\,\big[[n_1, n_2], n_3\big]$$

for all $m_i \in M$ and $n_i \in N$. From the commutator identity (**A**), it follows that

$$\big[[m_1n_1, m_2n_2], m_3n_3\big] = \big[[m_1n_1, m_2n_2], m_3\big]\ {}^{m_3}\big[[m_1n_1, m_2n_2], n_3\big]$$

Now, consider the following identities (that follow from (**A**) and (**C**)):

$$\begin{aligned}\big[[m_1n_1, m_2n_2], m_3\big] &= \big[[m_1n_1, m_2]\ {}^{m_2}[m_1n_1, n_2], m_3\big]\\ &= {}^{[m_1n_1, m_2]}\big[\ {}^{m_2}[m_1n_1, n_2], m_3\big]\ \big[[m_1n_1, m_2], m_3\big].\end{aligned}$$

Then, by using the conditions $[[M, N], M] = \{1\}$ and $[[N, N], M] = \{1\}$, it suffices to apply (**C**) or (**D**) to see that $[\ {}^{m_2}[m_1n_1, n_2], m_3] = \{1\}$. On the other hand, the term $[[m_1n_1, m_2], m_3]$ can be developed as follows (by (**C**) and (**D**)):

$$\begin{aligned}\big[[m_1n_1, m_2], m_3\big] &= \big[{}^{m_1}[n_1, m_2]\ [m_1, m_2], m_3\big]\\ &= \big[{}^{m_1}[n_1, m_2],\ {}^{[m_1, m_2]}m_3\big]\ \big[[m_1, m_2], m_3\big].\end{aligned}$$

By the condition $\big[[N, M], M]\big] = \{1\}$, one has that

$$[{}^{m_1}[n_1, m_2],\ {}^{[m_1, m_2]}m_3] = \{1\},$$

and we can conclude that

$$\big[[m_1n_1, m_2n_2], m_3\big] = \big[[m_1, m_2], m_3\big].$$

Similarly one shows that

$$\big[[m_1n_1, m_2n_2], n_3\big] = \big[[n_1, n_2], n_3\big].$$

Finally, since $\big[[N, N], M\big] = \{1\}$, it follows that

$${}^{m_3}\big[[n_1, n_2], n_3\big] = \big[[n_1, n_2], n_3\big],$$

as desired. □

Let $\mathcal{A} = \mathsf{Gp}$ be the variety of groups and let Nil_n be the subvariety of n-nilpotent groups (for some positive integer n). For any group A and normal subgroups M and N of A we write

$$\begin{aligned}Z_1(M, N) &= [M, N],\\ Z_2(M, N) &= \big[[M, N], M \cdot N\big],\\ &\ \ \vdots\\ Z_n(M, N) &= [Z_{n-1}(M, N), M \cdot N].\end{aligned}$$

As recalled in Section 2, $[M, N] = Z_1(M, N) = [M, N]_{\mathsf{Nil}_1 = \mathsf{Ab}}$, while the previous Theorem says that $Z_2(M, N) = [M, N]_{\mathsf{Nil}_2}$. By using similar arguments to the ones above, one proves that the following more general result holds:

Proposition 3.4. *If $\mathcal{A} = \mathsf{Gp}$, the variety of groups and Nil_n is the subvariety of n-nilpotent groups, then,*

$$[M,N]_{\mathsf{Nil}_n} = Z_n(M,N) = \big[\dots[[[M,N], M\cdot N], M\cdot N],\dots, M\cdot N\big]$$

where the right hand side expression involves n group commutators.

From this, an explicit description of central extensions of groups relative to the subvariety Nil_n can be deduced (see Section 2.2 in [1]):

Corollary 3.5. *Let Nil_n be the subvariety of Gp consisting of all n-nilpotent groups (for some positive integer n). An extension of groups*

$$\{1\} \longrightarrow N \longrightarrow A \longrightarrow B \longrightarrow \{1\}$$

is central *with respect to the subvariety Nil_n if and only if*

$$[N,A]_{\mathsf{Nil}_n} = \big[\dots[[[N,A],A],A],\dots,A\big] = \{1\}\,,$$

where the right hand side expression involves n group commutators.

4. The case of n-solvable groups

Recall that a group A is *n-solvable* (= solvable of class at most n) if and only if $D_n(A) = \{1\}$, where $D_n(A)$ is the n-th term in the derived series of A defined by

$$\begin{aligned} D_1(A) &= [A,A]\,,\\ D_2(A) &= \big[[A,A],[A,A]\big]\,,\\ &\vdots\\ D_n(A) &= [D_{n-1}(A), D_{n-1}(A)]\,. \end{aligned}$$

Theorem 4.1. *Let Gp be the variety of groups and Sol_2 its subvariety of 2-solvable groups. Then, for any group A and normal subgroups M and N of A one has that $[M,N]_{\mathsf{Sol}_2}$ equals*

$$\big[[M,N],[M,N]\big]\cdot\big[[M,M],[N,N]\big]\cdot\big[[M,N],[N,N]\big]\cdot\big[[M,N],[M,M]\big]\,.$$

Proof. The structure of the proof is similar to the one of Theorem 3.3. Again, it suffices to prove that $[M,N]_{\mathsf{Sol}_2} = \{1\}$ if and only if the following conditions are satisfied:

(1) $\big[[M,N],[M,N]\big] = \{1\}$,
(2) $\big[[M,M],[N,N]\big] = \{1\}$,
(3) $\big[[M,N],[N,N]\big] = \{1\}$,
(4) $\big[[M,N],[M,M]\big] = \{1\}$.

Suppose $[M,N]_{\mathsf{Sol}_2} = \{1\}$. Then, in particular,

$$\big[[m_1n_1, m_2n_2],[m_3n_3, m_4n_4]\big] = \big[[m_1,m_2],[m_3,m_4]\big]\cdot\big[[n_1,n_2],[n_3,n_4]\big]$$

for all $m_i \in M$ and $n_i \in N$. By choosing $n_1 = m_2 = n_3 = m_4 = 1$ one obtains that

$$\big[[m_1,n_2],[m_3,n_4]\big] = 1$$

for all $m_1, m_3 \in M$ and $n_2, n_4 \in N$, hence $\big[[M,N],[M,N]\big] = \{1\}$. The other three conditions are proved similarly.

Conversely, let us assume that the four above mentioned identities hold. This immediately implies that

$$\big[[M\cap N, M\cap N],[M\cap N, M\cap N]\big] = \{1\}\,.$$

We have now to prove that

$$v(m_1n_1,\ldots,m_nn_n) = v(m_1,\ldots,m_n)v(n_1,\ldots,n_n)$$

for all $v \in W_{\mathsf{Sol}_2}$, $m_i \in M$ and $n_i \in N$. Similar to the case of Nil_2, it suffices to show this for $v = v_0(x_1,x_2,x_3,x_4) = [[x_1,x_2],[x_3,x_4]]$. Hence, we are to prove that

$$\big[[m_1n_1,m_2n_2],[m_3n_3,m_4n_4]\big] = \big[[m_1,m_2],[m_3,m_4]\big]\cdot\big[[n_1,n_2],[n_3,n_4]\big]$$

for all $m_i \in M$ and $n_i \in N$. By applying identity $(\mathbf{A})$ twice to the left hand side of this expression, one obtains

$$\big[[m_1n_1,m_2n_2],[m_3n_3,m_4]\big]\ {}^{[m_3n_3,m_4]}\big[[m_1n_1,m_2n_2],\ {}^{m_4}[m_3n_3,n_4]\big].$$

By using the identities $(\mathbf{A}),(\mathbf{B}),(\mathbf{C}),(\mathbf{D})$ as in the proof of Theorem 3.3, from the conditions $1,2,3,4$ it follows that

$$\big[[m_1n_1,m_2n_2],[m_3n_3,m_4]\big] = \big[[m_1,m_2],[m_3,m_4]\big]\,.$$

On the other hand,

$$\big[[m_1n_1,m_2n_2],\ {}^{m_4}[m_3n_3,n_4]\big] = \big[[m_1n_1,m_2n_2],\ [{}^{m_4}m_3^{m_4}n_3,{}^{m_4}n_4]\big]\,,$$

hence, with similar calculations to the ones above, we obtain the identity

$$\big[[m_1n_1,m_2n_2],\ {}^{m_4}[m_3n_3,n_4]\big] = \big[[n_1,n_2],\ {}^{m_4}[n_3,n_4]\big]\,.$$

Since $[m_3n_3,m_4] = {}^{m_3}[n_3,m_4][m_3,m_4]$, this element commutes with the element $\big[[n_1,n_2],\ {}^{m_4}[n_3,n_4]\big]$ (by conditions $[[M,M],[N,N]] = \{1\}$ and $\big[[N,M],[N,N]\big] = \{1\}$), so that

$${}^{[m_3n_3,m_4]}\big[[m_1n_1,m_2n_2],\ {}^{m_4}[m_3n_3,n_4]\big] = \big[[n_1,n_2],{}^{m_4}[n_3,n_4]\big]\,.$$

Finally,

$$\begin{aligned}\big[[n_1,n_2],{}^{m_4}[n_3,n_4]\big] &= \big[[n_1,n_2],\big[m_4,[n_3,n_4]\big]\ [n_3,n_4]\big]\\ &= \big[[n_1,n_2],[n_3,n_4]\big]\,.\end{aligned}$$

□

In order to state the next Proposition it is useful to introduce the following notation.

Given normal subgroups $X_1, X_2, X_3, \dots$ of a group A, one defines inductively

$$\begin{aligned} D_1(X_1, X_2) &= [X_1, X_2]\,, \\ D_2(X_1, X_2, X_3, X_4) &= \Big[[X_1, X_2], [X_3, X_4]\Big]\,, \\ &\vdots \\ D_n(X_1, \dots, X_{2^n}) &= \big[D_{n-1}(X_1, \dots, X_{2^{n-1}}), D_{n-1}(X_{2^{n-1}+1}, \dots, X_{2^n})\big]\,. \end{aligned}$$

Similarly to the above Theorem and Proposition 3.4, one can prove the following:

Proposition 4.2. *Let* Gp *be the variety of groups and* Sol_n *its subvariety of* n*-solvable groups* (*for some positive integer* n). *Then, for any group* A *and normal subgroups* M *and* N *of* A*:*

$$[M, N]_{\mathsf{Sol}_n} = \prod_{(X_i)_{i \in [2^n]} \in \{M,N\}^{2^n} \setminus (\{M\}^{2^n} \cup \{N\}^{2^n})} D_n(X_1, \dots, X_{2^n})$$

where $\prod$ *denotes the product of normal subgroups.*

In particular, the previous Proposition gives a description of central extensions:

Corollary 4.3. *Let* Sol_n *be the subvariety of* Gp *consisting of* n*-solvable groups* (*for some positive integer* n). *An extension of groups*

$$\{1\} \longrightarrow N \longrightarrow A \longrightarrow B \longrightarrow \{1\}$$

is central *with respect to the subvariety* Sol_n *if and only if*

$$[N, A]_{\mathsf{Sol}_n} = D_n(N, A, \dots, A) = \{1\}\,.$$

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