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INITIAL NORMAL COVERS IN BI-HEYTING TOPOSES

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To Jiří Rosický, on his sixtieth birthday

Abstract. The dual of the category of pointed objects of a topos is semi-abelian, thus is provided with a notion of semi-direct product and a corresponding notion of action. In this paper, we study various conditions for representability of these actions. First, we show this to be equivalent to the existence of initial normal covers in the category of pointed objects of the topos. For Grothendieck toposes, actions are representable provided the topos admits an essential Boolean covering. This contains the case of Boolean toposes and toposes of presheaves. In the localic case, the representability of actions forces the topos to be *bi-Heyting*: the lattices of subobjects are both Heyting algebras and the dual of Heyting algebras.

1. Introducing the problem

Given a semi-abelian category $\mathcal{V}$ (see [3] or [12]), consider for every object $G \in \mathcal{V}$ the category $\mathsf{Pt}(G)$ of *points over* G, that is, of split epimorphisms with codomain G. The 'kernel functor'

$$\mathsf{Ker} : \mathsf{Pt}(G) \longrightarrow \mathcal{V},\ (q, s \colon A \leftrightarrows G, qs = \mathsf{id}_G) \longmapsto \mathsf{Ker}\, q$$

is monadic (see [11]); let us write T_G for the corresponding monad on $\mathcal{V}$. For every object $X \in \mathcal{V}$ we have thus a notion of G-*action on* X, namely, a structure (X, ξ) of T_G-algebra on X. This yields a functor

$$\mathsf{Act}_X : \mathcal{V}^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

mapping an object G to the set of G-actions on X. By monadicity, this functor is thus isomorphic to the functor

$$\mathsf{SplExt}_X : \mathcal{V}^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

mapping an object $G \in \mathcal{V}$ to the set of isomorphism classes of points over G with kernel X, that is, the set of isomorphism classes of split exact sequences

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "k"] & A \arrow[r, shift right, "q"'] & G \arrow[l, shift right, "s"'] \arrow[r] & 0 \end{tikzcd}$$

with prescribed kernel object X.

Definition 1.1. Given an object X of a semi-abelian category $\mathcal{V}$, we say that actions on X are representable when the functor Act_X is representable, that is equivalently, when the functor SplExt_X is representable.

The representability of the functor SplExt_X is a rather strong property, and when it holds, the representing object is often highly interesting (see [6] and [7]): for example

- the group $\mathsf{Aut}(X)$ of automorphisms of X when $\mathcal{V}$ is the category of groups;
- the Lie algebra $\mathsf{Der}(X)$ of derivations of X when $\mathcal{V}$ is the category of Lie algebras;
- the ring $\mathsf{End}(X)$ of X-linear endomorphisms of X, when $\mathcal{V}$ is the category of Boolean rings or the category of commutative von Neumann regular rings;
- the crossed module $\mathsf{Act}(X)$ of actors on X when $\mathcal{V}$ is the category of crossed modules.

Of course, in the general case, the functor SplExt_X may be representable only for certain objects X, not for all of them.

Like every contravariant set-valued functor, the functor $\mathsf{Act}_X \cong \mathsf{SplExt}_X$ is representable when its category of elements has a terminal object, that is, when there exists a terminal split exact sequence with fixed kernel object X. One is naturally tempted to compare this result with the corresponding non-split result: the existence of a *terminal short exact sequence* with kernel object X. In general, those two problems — even if they look similar — are of a totally different nature. For example in the abelian case, both problems become trivial, the first one with a positive answer (Act_X is always represented by the zero object) and the second one with a negative answer (except when X is the zero object).

Nevertheless, it was proved in [5] that

Theorem 1.2. *If the semi-abelian category $\mathcal{V}$ is arithmetical, for every object $X \in \mathcal{V}$ the following conditions are equivalent:*

(i) *there exists a terminal split exact sequence with fixed kernel X;*
(ii) *there exists a terminal short exact sequence with fixed kernel X.*

We recall that *arithmetical* means that for every object X, the lattice of equivalence relations on X is distributive. The categories of commutative von Neumann regular rings, of Boolean rings and of Heyting semi-lattices are examples of semi-abelian arithmetical categories; in the first two cases, actions are thus known to be representable (see [7]).

For every topos $\mathcal{E}$, the dual $\mathcal{E}_*^{\mathrm{op}}$ of the category $\mathcal{E}_*$ of pointed objects is semi-abelian (see [4], 5.1.8); and it is arithmetical, since lattices of equivalence relations in $\mathcal{E}_*^{\mathrm{op}}$ correspond to lattices of subobjects in $\mathcal{E}_*$, and these inherit distributivity from $\mathcal{E}$. The representability of actions in $\mathcal{E}_*^{\mathrm{op}}$ is thus equivalent to the existence in $\mathcal{E}_*$ of an initial short exact sequence with fixed cokernel object X, or equivalently again to the existence of an 'initial normal cover' of X, that is:

Definition 1.3. By an *initial normal cover* of a pointed object X in a topos $\mathcal{E}$, we mean an initial object in the full subcategory $\mathsf{Nml}(X)$ of $\mathcal{E}_*/X$ whose objects are normal epimorphisms with codomain X.

We shall give a direct topos-theoretic proof of this equivalence in section 2 below, by exploiting an adjunction between the categories of short exact sequences and of split exact sequences with given cokernel object.

The purpose of the present paper is to investigate situations where the conditions of Theorem 1.2 are satisfied for objects of $\mathcal{E}_*^{\mathrm{op}}$, where $\mathcal{E}$ is a topos. We prove that it is trivially the case when the basepoint of X is decidable in the topos $\mathcal{E}$: so it is the case for every X when the topos $\mathcal{E}$ is Boolean. In a non-Boolean $\mathcal{E}$, there may be nontrivial examples of initial normal covers, as we show by computing them explicitly in the Sierpiński topos. When $\mathcal{E}$ is a Grothendieck topos, we show that the problem may be reduced to some limit–colimit property, and we observe that this property is certainly satisfied when the topos $\mathcal{E}$ is localic and completely distributive, or when there exists an essential surjection $p\colon \mathcal{B} \to \mathcal{E}$, with $\mathcal{B}$ a Boolean Grothendieck topos. As a corollary, we obtain the existence of initial normal covers in $\mathcal{E}_*$ for every topos $\mathcal{E}$ of presheaves.

Various arguments show the relevance, for the existence of initial normal covers, of what we call *bi-Heyting toposes*, that is, toposes in which each lattice of subobjects is both a Heyting algebra and the dual of a Heyting algebra. All our examples are of this type and in the case of localic toposes, this bi-Heyting property is even necessary, as we show in the final section of the paper.

2. The categories $\mathsf{Ext}(X)$ and $\mathsf{SplExt}(X)$

From now on, $\mathcal{E}$ will always denote a topos and $\mathcal{E}_*$ its category of pointed objects. Since it will never lead to any confusion, we use the same notation $*$ for the basepoint of every pointed object of $\mathcal{E}$. And to avoid too heavy notation, we often say 'the pointed object X' to mean the object $(X, *) \in \mathcal{E}_*$ and 'the object X' to mean the object $X \in \mathcal{E}$. We reserve the notation $\amalg$ for the coproduct in the topos $\mathcal{E}$ and the notation $+$ for the coproduct in the category $\mathcal{E}_*$ of pointed objects.

By a short exact sequence in $\mathcal{E}_*$

$$1 \longrightarrow K \overset{k}{\rightarrowtail} A \overset{q}{\twoheadrightarrow} X \longrightarrow 1$$

we mean the usual conditions $k = \mathsf{Ker}\, q$ and $q = \mathsf{Coker}\, k$. The object X is then the quotient of A by the equivalence relation

$$\big((K \times K) \cup \Delta_A\big) \subseteq A \times A$$

where Δ_A denotes the diagonal of A.

We shall frequently argue informally in the internal language of $\mathcal{E}$ (see [14], section D1.2). The following characterization of normal epimorphisms will often be useful.

Lemma 2.1. *Let $q\colon A \to X$ be a morphism in $\mathcal{E}_*$. Then q is a normal epimorphism if and only if it is an epimorphism and the formula*

$$(qx = qy) \Rightarrow \big((x = y) \vee (qx = * = qy)\big)$$

is valid for variables x, y of sort A.

Proof. This is just the translation into the internal language of the observation above that the kernel-pair of q is the union of the diagonal and $K \times K$. □

We recall also that a morphism $f\colon A \to B$ in $\mathcal{E}$ is epic iff it is 'surjective in the internal language', i.e., the formula $(\forall y \in B)(\exists x \in A)(fx = y)$ is satisfied. The following consequence of Lemma 2.1 will also be used frequently.

Corollary 2.2. *Let $q\colon A \twoheadrightarrow X$ be a normal epimorphism and $m\colon B \rightarrowtail A$ a monomorphism in $\mathcal{E}_*$. Then the composite qm is a normal epimorphism provided it is epimorphic.*

Proof. Given variables x, y of sort B, from $qmx = qmy$ we may deduce either $mx = my$ or $qmx = * = qmy$. But since m is monic the first alternative implies $x = y$. So the condition of 2.1 is satisfied. □

A further useful consequence of Corollary 2.2 is:

Corollary 2.3. *Suppose a pointed object X of $\mathcal{E}_*$ has an initial normal cover $q\colon A \twoheadrightarrow X$. Then no proper subobject of A which contains the basepoint can map epimorphically to X.*

Proof. Suppose $m\colon B \rightarrowtail A$ is a subobject which contains the basepoint and maps epimorphically to X. By Corollary 2.2, the composite qm is a normal epimorphism, so by the initiality of q there must exist a morphism $r\colon A \to B$ in $\mathcal{E}_*/X$, and the composite mr must be the identity. So m is epic, and hence an isomorphism. □

The dual of Corollary 2.2 is a well-known result in semi-abelian categories (see [4], 3.2.7), but it seemed worth giving a direct topos-theoretic proof of it here. Similarly, the next result is the dual of a well-known fact about semi-abelian categories (see [4], 4.2.4.2 and 4.2.5.2), but we give a topos-theoretic proof.

Lemma 2.4. *In the following diagram in $\mathcal{E}_*$, suppose the row is a short exact sequence and the square is a pushout. Then there is a canonical normal epimorphism $r\colon B \twoheadrightarrow X$ with kernel v.*

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r, "k"] \arrow[d, "f"] & A \arrow[r, "q"] \arrow[d, "u"] & X \arrow[r] & 1 \\
 & L \arrow[r, "v"] & B & &
\end{tikzcd}$$

Proof. We define r to be the unique morphism satisfying $ru = q$ and $rv = 0$ (the 'zero map' which sends everything to the basepoint). Since q factors through r, the latter is epic; we show that it satisfies the condition of 2.1. For any $x \in B$ we have

$$(\exists y \in A)(uy = x) \vee (\exists z \in L)(vz = x)\,,$$

so given $x, x' \in B$ with $rx = rx'$, we have

$$(\exists z \in L)(vz = x) \vee (\exists z' \in L)(vz' = x') \vee (\exists y, y' \in A\big((uy = x) \wedge (uy' = x')\big)\,.$$

The first two alternatives both yield $rx = * = rx'$, and the third yields $qy = qy'$ and hence $(y = y') \vee (qy = * = qy')$, from which we obtain $(x = x') \vee (rx = * = rx')$.

It remains to show that v is the kernel of r. It is certainly monic, by a well-known property of pushouts in a topos (see [2], 5.9.10 or [14], A2.4.3). And, given $x \in B$ with $rx = *$, we have either $(\exists z \in L)(vz = x)$ or $(\exists y \in A)(uy = x)$: the first alternative is what we want, and from the second one we deduce $qy = *$, so that $(\exists w \in K)(kw = y)$ and hence $x = ukw = vfw$. □

Now fix an object X of $\mathcal{E}_*$. We wish to investigate the relationship between the category $\mathsf{Ext}(X)$ of extensions of X, that is, the category whose objects are short exact sequences

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r, tail, "k"] & A \arrow[r, two heads, "q"] & X \arrow[r] & 1 \end{tikzcd}$$

and whose morphisms are commutative diagrams

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r, tail, "k"] \arrow[d, "g"] & A \arrow[r, two heads, "q"] \arrow[d, "f"] & X \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & K' \arrow[r, tail, "k'"] & A' \arrow[r, two heads, "q'"] & X \arrow[r] & 1 \end{tikzcd}$$

and the category $\mathsf{SplExt}(X)$ whose objects are those short exact sequences for which the kernel k has a retraction $s\colon A \to X$, and whose morphisms are additionally required to commute with the retractions. We note that, for a morphism (f, g) of $\mathsf{Ext}(X)$, the component g is uniquely determined by f; hence $\mathsf{Ext}(X)$ is equivalent to the full subcategory $\mathsf{Nml}(X)$ of $\mathcal{E}_*/X$ whose objects are normal epimorphisms. We shall feel free to pass back and forth between these two categories without further comment.

Our next Lemma is more conveniently stated in terms of the category $\mathsf{Nml}(X)$, rather than $\mathsf{Ext}(X)$.

Lemma 2.5. *The category* $\mathsf{Nml}(X)$ *is closed under finite limits in* $\mathcal{E}_*/X$*; in particular, it has finite limits.*

Proof. It is clear that the terminal object $\mathsf{id}_X\colon X \to X$ of $\mathcal{E}_*/X$ is a normal epimorphism. Consider next a pair of normal epimorphisms $q\colon A \twoheadrightarrow X$,$r\colon B \twoheadrightarrow X$;

their product in $\mathcal{E}_*/X$ is of course given by the pullback

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "q"] \\ B \arrow[r, "r"] & X \end{tikzcd}$$

and the latter maps epimorphically to X since epimorphisms are stable under pullback in a topos. We must verify that the composite $qf = rg$ satisfies the condition of Lemma 2.1. But if $x, y \in P$ satisfy $qfx = qfy$, then we obtain $(fx = fy) \vee (qfx = * = qfy)$ since q is normal, and similarly we have $(gx = gy) \vee (rgx = * = rgy)$. So we have

$$\big((fx = fy) \wedge (gx = gy)\big) \vee (qfx = * = qfy)\,,$$

and the first alternative implies $x = y$ since the pair (f, g) is jointly monic.

Finally, we must consider equalizers. Suppose given a diagram in $\mathcal{E}_*/X$

$$\begin{tikzcd} E \arrow[r, tail, "e"] \arrow[dr, "p"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "q"] & B \arrow[dl, "r"] \\ & X & \end{tikzcd}$$

where the row is an equalizer and q and r are normal epimorphisms. By Lemma 2.2, it suffices to prove that p is epic. But, given $x \in X$, we have $(\exists y \in A)(qy = x)$, and the pair $(fy, gy) \in B \times B$ satisfies $rfy = x = rgy$. So we have either $fy = gy$, in which case $(\exists z \in E)(ez = y)$ as required, or $rfy = * = rgy$, in which case $x = *$ and so the basepoint $* \in E$ satisfies $p* = x$. $\square$

Given an object (q, k) of $\mathsf{Ext}(X)$, let us apply the pushout construction of Lemma 2.4 in the particular case when $f = k$, so that (u, v) is the cokernel-pair of k. We shall write $D(A)$ for the common codomain of u and v in this case. Thus we have a short exact sequence

$$1 \longrightarrow A \xrightarrow{\ v\ } D(A) \xrightarrow{\ r\ } X \longrightarrow 1\ ;$$

but in this case v is easily seen to be split by the codiagonal map $\nabla\colon D(A) \to A$, i.e. the unique morphism satisfying $\nabla u = \nabla v = \mathsf{id}_A$. It is straightforward to verify that this construction defines a functor $\mathsf{Ext}(X) \to \mathsf{SplExt}(X)$, which we shall denote by D.

Theorem 2.6. *The functor D just defined is left adjoint to the forgetful functor $U\colon \mathsf{SplExt}(X) \to \mathsf{Ext}(X)$.*

Proof. Consider a morphism

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r, tail, "k"] \arrow[d, "g"] & A \arrow[r, two heads, "q"] \arrow[d, "f"] & X \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & K' \arrow[r, tail, "k'"] & A' \arrow[r, two heads, "q'"] & X \arrow[r] & 1
\end{tikzcd}$$

where k' has a retraction s. We claim that there is a unique morphism

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r, shift right, tail, "v"'] \arrow[d, "g'"] & D(A) \arrow[l, shift right, "\nabla"'] \arrow[r, two heads, "r"] \arrow[d, "f'"] & X \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & K' \arrow[r, shift right, tail, "k'"'] & A' \arrow[l, shift right, "s"'] \arrow[r, two heads, "q'"] & X \arrow[r] & 1
\end{tikzcd}$$

in $\mathsf{SplExt}(X)$ such that $f'u = f$. Indeed, commutativity of f' and g' with the leftward arrows forces $g' = g'\nabla u = sf'u = sf$, and commutativity with the rightward arrows then forces $f'v = k'g' = k'sf$, so that f' is uniquely determined. But it is readily verified that if we set $g' = sf$ and f' to be the morphism induced by the pair $(f, k'sf)$, then the diagram does commute.

This establishes a bijection between morphisms $(q, k) \to U(q', k', s)$ and morphisms $D(q, k) \to (q', k', s)$; the verification that it is natural is easy. □

Corollary 2.7. *If* $\mathsf{Ext}(X)$ *has an initial object, then so does* $\mathsf{SplExt}(X)$.

Proof. Left adjoints preserve initial objects when they exist. □

To obtain the converse to Corollary 2.7, we need to establish a further fact about the adjunction $(D \dashv U)$.

Proposition 2.8. *The unit of the adjunction of Theorem* 2.6 *is a cartesian natural transformation, i.e. its naturality squares are pullbacks.*

Proof. It is easy to see that the component of the unit at an object (q, k) is given by the diagram

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r, tail, "k"] \arrow[d, "k"] & A \arrow[r, two heads, "q"] \arrow[d, "u"] & X \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & A \arrow[r, tail, "v"] & D(A) \arrow[r, two heads, "r"] & X \arrow[r] & 1
\end{tikzcd}$$

So suppose we are given a morphism $(f, g)\colon (q, k) \to (q', k')$ in $\mathsf{Ext}(X)$. It is clear that the square

$$\begin{tikzcd} K \arrow[r, "g"] \arrow[d, tail, "k"] & K' \arrow[d, tail, "k'"] \\ A \arrow[r, "f"] & A' \end{tikzcd}$$

is a pullback, since k and k' are respectively the pullbacks of the basepoint of X along $q = q'f$ and q'. We need to show that the square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"] & A' \arrow[d, "u'"] \\ D(A) \arrow[r, "D(f)"] & D(A') \end{tikzcd}$$

is also a pullback. For this, we argue in the internal lanuguage. Suppose given variables $x \in D(A)$ and $y' \in A'$ satisfying $D(f)x = u'y'$. We have

$$(\exists y \in A)(uy = x) \vee (\exists z \in A)(vz = x)\,,$$

as noted in the proof of Lemma 2.4. From the first alternative we deduce $u'fy = D(f)uy = u'y'$, whence $fy = y'$ since u' is monic, so we have found an element of A mapping to both x and y'. From the second, we obtain $v'fz = D(f)x = u'y'$, so since the pushout square

$$\begin{tikzcd} K' \arrow[r, tail, "k'"] \arrow[d, tail, "k'"] & A' \arrow[d, tail, "u'"] \\ A' \arrow[r, tail, "v'"] & D(A') \end{tikzcd}$$

is also a pullback, we obtain $(\exists w' \in K')(k'w' = fz = y')$. Then since the first square above is a pullback we have $(\exists w \in K)\big((gw = w') \wedge (kw = z)\big)$, so that $uz = ukw = vkw = vz = x$. Thus the second alternative reduces to the first. □

The following result from general category theory may well be known, but we have not been able to find a reference for it

Lemma 2.9. *Let $(D\colon \mathcal{C} \to \mathcal{D} \dashv U\colon \mathcal{D} \to \mathcal{C})$ be an adjoint pair of functors. Suppose also that $\mathcal{C}$ has finite limits, and that the unit of the adjunction is cartesian. Then $\mathcal{C}$ is equivalent to a (full) coreflective subcategory of $\mathcal{D}/DT$, where T is the terminal object of $\mathcal{C}$.*

Proof. D clearly induces a functor $\widetilde{D}\colon \mathcal{C} \cong \mathcal{C}/T \to \mathcal{D}/DT$. Given an arbitrary object $(B \to DT)$ of $\mathcal{D}/DT$, we may form the pullback

$$\begin{tikzcd} VB \arrow[r] \arrow[d] & UB \arrow[d] \\ T \arrow[r] & UDT \end{tikzcd}$$

in $\mathcal{C}$, where the bottom arrow is the unit of $(D \dashv U)$. It is straightforward to verify that V defines a functor $\mathcal{D}/DT \to \mathcal{C}$, right adjoint to $\widetilde{D}$; and the condition that the unit of $(D \dashv U)$ is cartesian says precisely that the unit of $(\widetilde{D} \dashv V)$ is an isomorphism, i.e. that $\widetilde{D}$ is full and faithful. So it induces an equivalence between $\mathcal{C}$ and its image, which is a coreflective subcategory of $\mathcal{D}/DT$. □

Corollary 2.10. *If* $\mathsf{SplExt}(X)$ *has an initial object, then so does* $\mathsf{Ext}(X)$.

Proof. The adjunction $(D \dashv U)$ of Theorem 2.6 satisfies the hypotheses of Lemma 2.9, by Lemma 2.5 and Proposition 2.8. But $\mathsf{SplExt}(X)/DT$ clearly inherits an initial object from $\mathsf{SplExt}(X)$, and so does any coreflective subcategory of it. □

3. The decidable case

In this section we investigate the 'trivial' case when the terminal object of $\mathsf{Ext}(X)$ is also initial. Our first significant observation is:

Proposition 3.1. *In the category $\mathcal{E}_*$ of pointed objects of a topos, if a normal epimorphism $q\colon A \twoheadrightarrow X$ admits a section, this section is necessarily unique.*

Proof. If s and t are two sections of q, then for every $x \in X$ we have

$$\begin{aligned}(qsx = x = qtx) &\Rightarrow (sx = tx) \vee (x = qsx = qtx = *)\\ &\Rightarrow (sx = tx) \vee (sx = * = tx)\\ &\Rightarrow (sx = tx)\end{aligned}$$

which proves $s = t$. □

Once again, Proposition 3.1 is the dual of a result which holds in any arithmetical semi-abelian category, see [9], Theorem 3.16. However, it seemed worth giving a topos-theoretic proof.

Theorem 3.2. *In the category $\mathcal{E}_*$ of pointed objects of a topos $\mathcal{E}$, the following conditions are equivalent for a pointed object X:*

(i) *the basepoint $*\colon 1 \to X$ is decidable, that is complemented as a subobject of X;*
(ii) *every normal epimorphism with codomain X admits a section;*
(iii) *the terminal object $(1 \to X \to X)$ of $\mathsf{Ext}(X)$ is also initial.*

Proof. First assume condition (i) and consider a normal epimorphism $q\colon A \twoheadrightarrow X$. We have $X \cong \{*\} \amalg X'$ for some X'; thus taking the inverse images along q, we obtain $A \cong K \amalg A'$ for some A'. But the restriction of q to A' is an isomorphism $A' \to X'$ by normality of q. Taking the inverse of this isomorphism on X' and sending the basepoint of X to that of A yields the required section.

Now assume (ii). By Proposition 3.1, every normal epimorphism $q\colon A \twoheadrightarrow X$ admits a *unique* section; equivalently, there is a unique morphism $\mathsf{id}_X \to q$ in $\mathsf{Nml}(X)$. So (iii) holds.

Finally, assume (iii). Consider the morphism

$$q = (0, \mathsf{id}_X)\colon A = 1 \amalg X \longrightarrow X$$

where A is made into a pointed object by taking the added singleton as basepoint. The morphism q is trivially an epimorphism, since its second component is so. And it is normal because, given $x, y \in A$ with $qx = qy$, we have

$$\big((x \in X) \wedge (y \in X)\big) \vee (x = *) \vee (y = *)$$

(where $*$ denotes the basepoint of A). From the first alternative, we deduce $x = y$; from the other two we obtain $qx = * = qy$ (where $*$ now denotes the basepoint of X). By assumption, this epimorphism admits a (unique) section $s\colon X \to A$. Pulling back the coproduct decomposition of A along s, we obtain $X \cong s^{-1}\{*\} \amalg s^{-1}X$. Since s is monic, the first summand must be a subobject of 1; but it contains the basepoint of X (since s preserves the basepoint), and so must be exactly the subobject $\{*\}$ of X. So the latter is complemented. □

Corollary 3.3. *Let $\mathcal{E}$ be a Boolean topos. In the category $\mathcal{E}_*$ of pointed objects of $\mathcal{E}$, every object is its own initial normal cover.* □

Next, we establish the 'idempotency' of the initial normal cover process in a topos:

Lemma 3.4. *Let $\mathcal{E}$ be a topos. In the category $\mathcal{E}_*$ of pointed objects of $\mathcal{E}$, the composite of two normal epimorphisms is still a normal epimorphism.*

Proof. Consider a composable pair

$$A \overset{f}{\twoheadrightarrow} B \overset{g}{\twoheadrightarrow} C$$

of normal epimorphisms. The composite gf is certainly an epimorphism; moreover, given $x, y \in A$ we have

$$\begin{aligned} gfx = gfy &\Rightarrow (fx = fy) \vee (gfx = * = gfy) \\ &\Rightarrow (x = y) \vee (fx = * = fy) \vee (gfx = * = gfy) \\ &\Rightarrow (x = y) \vee (gfx = * = gfy)\,, \end{aligned}$$

so that gf is normal by Lemma 2.1. □

It is proved in [10] that the dual of a topos $\mathcal{E}$ is strongly protomodular in the sense of [8], which implies easily the strong protomodularity of $\mathcal{E}_*^{\mathrm{op}}$. This strong protomodularity means that 'some' composites of normal monomorphisms are still normal (see [4]): Lemma 3.4 reinforces this statement by showing that in $\mathcal{E}_*^{\mathrm{op}}$ *all* composites of normal epimorphisms are normal.

Proposition 3.5. *Let $\mathcal{E}$ be a topos. If a pointed object X admits an initial normal cover $\chi\colon [X] \twoheadrightarrow X$, then the basepoint of $[X]$ is decidable and therefore $[X]$ is its own initial normal cover.*

Proof. If $\theta\colon Y \twoheadrightarrow [X]$ is a normal epimorphism, the composite $\chi\theta$ is a normal epimorphism by Lemma 3.4. By initiality of χ, we get a unique α such that $(\chi\theta)\alpha = \chi$. But then $\theta\alpha = \mathsf{id}_{[X]}$ by initiality of χ. Thus θ has the section α and we conclude by Theorem 3.2. $\square$

At this point, it seems appropriate to give an example of a topos in which initial normal covers exist but are not all trivial. Perhaps the simplest example of a non-Boolean topos is the *Sierpiński topos*, that is the category $[\mathbf{2}, \mathbf{Set}]$ of diagrams of shape $(\bullet \to \bullet)$ in $\mathbf{Set}$. We shall see later (Corollary 6.6) that all pointed objects have initial normal covers in any topos of presheaves; so this applies in particular to the Sierpiński topos. But the explicit calculation of the form of an initial normal cover is of some interest.

Example 3.6. In the Sierpiński topos, the initial normal cover of a pointed object $(\alpha\colon X_0 \to X_1)$ is given by

$$\chi\colon ([\alpha]\colon [X]_0 \to [X]_1) \longrightarrow (\alpha\colon X_0 \to X_1)$$

where $[X]_0 = X_0$ (and χ_0 is the identity map), $[X]_1 = X_1 + \mathsf{Ker}\,\alpha$ (here $+$, as usual, denotes coproduct of pointed sets), χ_1 sends each element of X_1 to itself and each element of $\mathsf{Ker}\,\alpha$ to the basepoint, and

$$\begin{aligned} [\alpha](x) &= x \in \mathsf{Ker}\,\alpha && \text{if } \alpha(x) = *; \\ &= \alpha(x) \in Y && \text{if } \alpha(x) \neq *. \end{aligned}$$

for $x \in X_0$.

It is trivial that χ is a morphism of $\mathcal{E}$, i.e. that $\alpha\chi_0 = \chi_1[\alpha]$. The kernel of χ is simply the pointed object $(\{*\} \rightarrowtail \mathsf{Ker}\,\alpha)$ and χ is indeed the quotient which identifies to the basepoint all the elements in this kernel and leaves the rest unchanged. Thus χ is a normal epimorphism.

Now consider another normal epimorphism between pointed objects

$$(q_0, q_1)\colon (\beta\colon A_0 \to A_1) \longrightarrow (\alpha\colon X_0 \to X_1)\,.$$

We must prove the existence of a unique morphism of pointed objects

$$(f_0, f_1)\colon ([\alpha]\colon [X]_0 \to [X]_1) \longrightarrow (\beta\colon A_0 \to A_1)$$

such that $qf = \chi$. Let us write respectively s_0, s_1 for the unique sections of q_0, q_1 in $\mathbf{Set}_*$ (see Theorem 3.2). Since χ_0 is the identity, we have necessarily $f_0 = s_0$. Since χ_1 is the identity on X_1, we get as well that f_1 must agree with s_1 on this subset of $[X]_1$. And the equality $\beta f_0 = f_1[\alpha]$ forces finally $f_1(x) = \beta s_0(x)$ for $x \in \mathsf{Ker}\,\alpha \subseteq [X]_1$. This proves the uniqueness of f. And it is trivial to observe that f defined in this way is indeed a morphism of $\mathcal{E}_*$ and $qf = \chi$.

Of course, the non-decidability of objects in the Sierpiński topos arises from the fact that distinct elements of X_0 may have the same image in X_1 (cf. [14], A1.4.16). The construction of $[X]$ above has the effect of 'pulling apart' X_1 just sufficiently to make the basepoint decidable; this is what we should have expected from Proposition 3.5.

Similar calculations may be performed in other toposes of presheaves on simple categories: for example, in the topos $[\mathbf{3}, \mathbf{Set}]$ of diagrams of shape $(\bullet \to \bullet \to \bullet)$, we find that the initial normal cover of a pointed object

$$X_0 \xrightarrow{\alpha} X_1 \xrightarrow{\beta} X_2$$

has the form

$$X_0 \longrightarrow X_1 + \mathsf{Ker}\,\alpha \longrightarrow X_2 + \mathsf{Ker}\,\beta + \mathsf{Ker}\,\alpha\,.$$

And if one considers more involved examples, like presheaves on a poset having infinite ascending or descending chains, or having 'diamonds', it is quite easy in each case to describe explicitly the initial normal cover of a pointed object. One uses if necessary a coequalizer to force the commutativity of a 'diamond', or limits and colimits to take care of the infinite chains.

However, it should be noted that in these examples, the construction of the initial normal cover introduces 'at the lower level' the elements of $\mathsf{Ker}\,\alpha$, which 'live at the upper level'. Such a construction seems to be opposite to what the constructions in the internal logic of the topos generally do. Therefore, it seems unlikely that the construction of initial normal covers – when these exist – could be handled in the internal logic of the topos.

4. The functor Ext_X

In this section we present an alternative approach to the problem of finding an initial normal cover of a pointed object X, via the functor which to a pointed object K assigns the set of isomorphism classes of short exact sequences with kernel K and cokernel X. In order to show that these isomorphism classes form a set, we need the following lemma.

Lemma 4.1. *Suppose given a short exact sequence*

$$1 \longrightarrow K \xrightarrow{k} A \xrightarrow{q} X \longrightarrow 1$$

in $\mathcal{E}_*$, for some topos $\mathcal{E}$. Then the morphism

$$(q, \Omega^k\{\})\colon A \longrightarrow X \times \Omega^K$$

is a monomorphism.

Proof. As usual, we argue in the internal language of $\mathcal{E}$. Suppose given $x, x' \in A$ satisfying $(qx = qx') \wedge (\Omega^k\{x\} = \Omega^k\{x'\})$. From the first equation we deduce $(x = x') \vee (qx = * = qx')$; but the latter alternative implies $(\exists y, y' \in K)\big((ky = x) \wedge (ky' = x')\big)$. Now we have

$$\Omega^k\{x\} = \Omega^k(\exists k)\{y\} = \{y\}$$

since k is monic (cf. [14], A2.2.5), and similarly $\Omega^k\{x'\} = \{y'\}$. So we deduce $\{y\} = \{y'\}$; but the singleton map $\{\}: K \to \Omega^K$ is also monic ([14], A2.2.3), so this implies $y = y'$ and hence $x = x'$. □

Corollary 4.2. *Let $\mathcal{E}$ be a locally small topos and $\mathcal{E}_*$ its category of pointed objects. For every pointed object $X \in \mathcal{E}_*$, there exists a covariant functor*

$$\mathsf{Ext}_X : \mathcal{E}_* \longrightarrow \mathbf{Set}$$

mapping a pointed object K to the set of isomorphism classes of short exact sequences with prescribed kernel K and prescribed cokernel X.

Proof. We make Ext_X into a covariant functor by means of the 'pushout construction' of Lemma 2.4: given $(q, k) \in \mathsf{Ext}_X(K)$ and $f: K \to L$, we define $\mathsf{Ext}_X(f)(q, k)$ to be (r, v) in the notation of that Lemma. It is easy to verify that this construction is well-defined up to isomorphism and functorial. Lemma 4.1 ensures that it takes values in **Set**, since any locally small topos is well-powered (isomorphism classes of subobjects of an object B correspond bijectively to morphisms $B \to \Omega$). □

The key observation linking the functor Ext_X to the existence of initial normal covers is the following.

Proposition 4.3. *Let $\mathcal{E}$ be a locally small topos. A pointed object $X \in \mathcal{E}_*$ admits an initial normal cover if and only if the functor Ext_X is representable.*

Proof. A set-valued functor is representable if and only if its category of elements has an initial object. But the category of elements of Ext_X is exactly the category $\mathsf{Ext}(X)$, which as we have already noted is equivalent to $\mathsf{Nml}(X)$. And the existence of an initial object in this last category means precisely the existence of an initial normal cover of X. □

For a Grothendieck topos, the problem can be further reduced, using classical arguments:

Corollary 4.4. *Let $\mathcal{E}$ be a Grothendieck topos. A pointed object $X \in \mathcal{E}_*$ admits an initial normal cover if and only if the functor Ext_X preserves limits.*

Proof. We use the 'representability version' of the Special Adjoint Functor Theorem (see [15], corollary to V.8.2). For this we need to know that $\mathcal{E}_*$ is complete and well-powered, and has a small coseparating family. The first two conditions are trivially satisfied. The topos $\mathcal{E}$ itself has a coseparating family (in fact a single coseparator G; see [14], B3.1.13), thus the set of pointed objects obtained by equipping G with all its possible basepoints constitutes a coseparating family in $\mathcal{E}_*$. □

In fact Corollary 4.4 can be further improved. By a slight modification of (the dual of) a result from [7], one can show that the functor Ext_X always preserves equalizers of cokernel-pairs. But a functor on a finitely complete Barr-exact Mal'cev category which preserves finite coproducts and coequalizers of kernel-pairs preserves all finite limits [7]. Applying the dual of this result to Ext_X, we deduce:

Corollary 4.5. *Let $\mathcal{E}$ be a Grothendieck topos. A pointed object $X \in \mathcal{E}_*$ admits an initial normal cover if and only if the functor* Ext_X *preserves products.*

We omit the details of the proof, since we shall not use this result.

We conclude this section with a slight digression, inspired by an observation in the proof of Corollary 4.4. We know that every Grothendieck topos $\mathcal{E}$ has a single coseparator, but, in order to get a coseparating family for $\mathcal{E}_*$, we have to equip this object with all its possible basepoints — which will in general produce many non-isomorphic objects. Is it possible to find a single coseparator for $\mathcal{E}_*$? The following result, which extends that of [1], shows that the answer is 'yes' at least when $\mathcal{E}$ is localic.

Proposition 4.6. *For a localic topos $\mathcal{E}$, $(\Omega, \top)$ is a coseparator in the corresponding category $\mathcal{E}_*$ of pointed objects.*

Proof. Let $\mathcal{E}$ be the category of sheaves on the frame L. Given a sheaf F and two elements $a \neq b \in F(u)$ for some $u \in L$, we consider the following truth values (= elements of L)

- $\alpha = [\![a = *]\!]$, the truth value of $a = *$;
- $\beta = [\![b = *]\!]$, the truth value of $b = *$;
- $\delta = [\![a = b]\!]$, the truth value of $a = b$.

We cannot have both
$$\alpha \vee \delta = u, \ \ \beta \vee \delta = u$$
because then a and b would have equal restrictions on the two pieces of the covering
$$u = (\alpha \vee \delta) \wedge (\beta \vee \delta) = (\alpha \wedge \beta) \vee \delta$$
and thus a, b would be equal. Let us assume that $\alpha \vee \delta \neq u$.

Consider now the subsheaf $S \subseteq F$ generated by $* \in F(1)$ and $b \in F(u)$. In terms of truth values we have
$$[\![a \in S]\!] = [\![(a = *) \vee (a = b)]\!] = [\![a = *]\!] \vee [\![a = b]\!] = \alpha \vee \delta \,.$$
Since $\alpha \vee \delta \neq u$, $a \notin S(u)$ while $b \in S(U)$. Thus S is a pointed subobject of F which contains a but not b. Therefore its characteristic mapping
$$\varphi \colon (F, *) \longrightarrow (\Omega, \top)$$
is a morphism of pointed objects which separates a and b. □

5. Generalized pullbacks

Given an arbitrary familly of morphisms $(q_i \colon A_i \to X)_{i \in I}$ in a complete category $\mathcal{E}$, the product of the q_i in $\mathcal{E}/X$ is an object P equipped with morphisms $p_i \colon P \to A_i$ for all $i \in I$, such that the composite $q = q_i p_i$ is independent of i, and satisfying the appropriate universal property. We shall call this morphism q the *generalized pullback* of the morphisms q_i (it is also sometimes called a *wide pullback*).

We may now state a necessary condition for the existence of initial normal covers.

Proposition 5.1. *Let $\mathcal{E}$ be a complete topos. If a pointed object $X \in \mathcal{E}_*$ admits an initial normal cover, then in $\mathcal{E}_*$, the generalized pullback of every family of normal epimorphisms with codomain X is still an epimorphism.*

Proof. Let $\chi \colon [X] \twoheadrightarrow X$ be the initial cover of X. Given a family $(q_i \colon A_i \to X \mid i \in I)$ of normal epimorphisms, χ factors through each of the q_i, and hence through their generalized pullback $q \colon P \to X$. But χ is an epimorphism, so q must be epic. □

For a Grothendieck topos, we have a sufficient condition which appears very similar.

Proposition 5.2. *Let $\mathcal{E}$ be a Grothendieck topos. Let $X \in \mathcal{E}_*$ be a pointed object such that the generalized pullback of every family of normal epimorphisms with codomain X is still a normal epimorphism. Then X admits an initial normal cover.*

Proof. This time we use the General Adjoint Functor Theorem in its 'initial-object' form ([15], Theorem V 6.1). The hypothesis says that $\mathsf{Nml}(X)$ is closed under arbitrary products in $\mathcal{E}_*/X$; but we already know it is closed under equalizers, by Lemma 2.5, and hence it is complete. It is locally small since $\mathcal{E}$ is, so it remains only to verify the solution-set condition. For this we use a local presentability argument, as follows.

Any Grothendieck topos is locally presentable ([14], D2.3.7), so we may choose a regular cardinal κ such that both X and the terminal object of $\mathcal{E}$ are κ-presentable. Now, given any normal epimorphism $q \colon A \twoheadrightarrow X$, we may express A as an epimorphic image of a coproduct $\coprod_{i \in I} G_i$ of members of some separating set for $\mathcal{E}$. The union of the images of the composites $G_i \to A \to X$ is the whole of X; so by κ-presentability we can find a subset $I' \subseteq I$ of cardinality less than κ such that the union of the images of the $G_i \to X$ with $i \in I'$ is still the whole of X, and additionally such that the union of the images of the $G_i \to A$ with $i \in I'$ contains the basepoint of A. Now let $A' \rightarrowtail A$ be the union of the images of the $G_i \to A$ for $i \in I'$. Then the composite $A' \rightarrowtail A \twoheadrightarrow X$ is epimorphic, so by Corollary 2.2 it is a normal epimorphism. Thus we may obtain our solution set for $\mathsf{Nml}(X)$ by taking a representative set of quotients of coproducts of fewer than κ generators, and equipping them with all possible choices of basepoints and normal epimorphisms to X. □

An alternative proof of Proposition 5.2 may be given using Corollary 4.5; we omit the details.

Propositions 5.1 and 5.2 thus very clearly delimit the problem. The necessary and sufficient condition for an object $X \in \mathcal{E}_*$ to admit an initial normal cover is 'squeezed' between the condition that any generalized pullback of normal epimorphisms with codomain X is a normal epimorphism, and the condition that any such generalized pullback should be simply epimorphic.

We do not know any actual instance where the necessary condition holds but the sufficient one fails. However, in order to prove them equivalent, we need to assume a rather strong additional property of our topos $\mathcal{E}$, which we introduce in the next section.

6. Bi-Heyting toposes

Definition 6.1. A topos $\mathcal{E}$ is called *bi-Heyting* when the duals of its Heyting algebras of subobjects are again Heyting algebras.

In other words, a topos is bi-Heyting when the union with a subobject admits a left adjoint. In particular, in the Grothendieck case, or more generally in the presence of arbitrary intersections:

Lemma 6.2. *A Grothendieck topos $\mathcal{E}$ is bi-Heyting when finite unions distribute over arbitrary intersections:*

$$S \cup \Big(\bigcap_{i \in I} T_i \Big) = \bigcap_{i \in I} (S \cup T_i)\,.$$

Obviously Boolean toposes are bi-Heyting, since the dual of a Boolean algebra is again a Boolean algebra. For future reference, we also note:

Lemma 6.3. *Let $\mathcal{E}$ be a Grothendieck topos, and $\{G_i \mid i \in I\}$ a separating set of objects of $\mathcal{E}$. Then $\mathcal{E}$ is bi-Heyting if and only if each G_i has bi-Heyting subobject lattice.*

Proof. For an arbitrary object X of $\mathcal{E}$, we have an epimorphism

$$\coprod_{j \in J} G_{f(j)} \twoheadrightarrow X$$

for some set J and function $f \colon J \to I$. Pulling back along this epimorphism yields an injection

$$\mathrm{Sub}(X) \rightarrowtail \prod_{j \in J} \mathrm{Sub}(G_{f(j)})$$

which preserves arbitrary unions and intersections; so the domain of this map inherits the distributive law of Lemma 6.2 from its codomain. $\square$

Proposition 6.4. *Let $\mathcal{E}$ be a bi-Heyting Grothendieck topos. Given a family of normal epimorphisms $(q_i \colon A_i \twoheadrightarrow X \mid i \in I)$ between pointed objects and their generalized pullback $q \colon P \to X$, the following conditions are equivalent:*

(i) *q is an epimorphism;*
(ii) *q is a normal epimorphism.*

In particular, $X \in \mathcal{E}_*$ admits an initial normal cover if and only if these equivalent conditions are satisfied.

Proof. Of course it suffices to prove that (i) implies (ii). For this we again use the criterion of Lemma 2.1: given $x, y \in P$ with $qx = qy$, we have for each $i \in I$

$$(p_i x = p_i y) \vee (q_i p_i x = * = q_i p_i y)$$

(where p_i denotes the projection $P \to A_i$). But the second alternative is independent of i, so by the infinite distributive law of Lemma 6.2 we obtain

$$\Big(\bigwedge_{i\in I}(p_i x = p_i y)\Big) \vee (qx = * = qy)\,.$$

And the first alternative here is equivalent to $x = y$, since the family $(p_i \mid i \in I)$ is jointly monic. $\square$

It appeared that in Proposition 6.4 we did not use the full strength of the bi-Heyting axiom, but only the special case

$$u \vee \Big(\bigwedge_{i\in I} v_i\Big) = \bigwedge_{i\in I}(u \vee v_i), \text{ provided } \forall i,j \in I \ u \vee v_i = u \vee v_j\,.$$

However, this particular case is equivalent to the general one. Given arbitrary elements u and $(v_i \mid i \in I)$ of a distributive complete lattice, let us write $w = \bigwedge_{i\in I}(u \vee v_i)$ and set $v_i' = v_i \wedge w$ for all i. (Note that $u = u \wedge w$, since $u \leq w$.) Then $u \vee v_i' = (u \vee v_i) \wedge w = w$ for all i, but

$$u \vee \Big(\bigwedge_{i\in I} v_i'\Big) = \Big(u \vee \Big(\bigwedge_{i\in I} v_i\Big)\Big) \wedge w = u \vee \Big(\bigwedge_{i\in I} v_i\Big)$$

since $u \vee \bigwedge v_i \leq \bigwedge(u \vee v_i)$ is true in any complete lattice. Thus the particular case of the distributive law, applied to u and the v_i', yields the general case for u and the v_i.

Next, we show that possession of universal normal covers 'descends' in a suitable sense along essential surjections. Recall that a geometric morphism $f\colon \mathcal{F} \to \mathcal{E}$ is said to be *surjective* if the inverse image functor $f^*\colon \mathcal{E} \to \mathcal{F}$ is faithful, and *essential* if f^* has a left adjoint $f_!$ (as well as its usual right adjoint f_*).

Proposition 6.5. *Let $f\colon \mathcal{F} \to \mathcal{E}$ be an esssential surjective morphism between Grothendieck toposes. Then*

(i) *if $\mathcal{F}$ is bi-Heyting, so is $\mathcal{E}$;*
(ii) *if $\mathcal{F}$ is bi-Heyting and pointed objects of $\mathcal{F}$ have initial normal covers, then the same is true in $\mathcal{E}$.*

Proof. (i) For every object X of $\mathcal{E}$, applying f^* to subobjects of X yields a mapping $\mathrm{Sub}(X) \to \mathrm{Sub}(f^*X)$ which is injective since f^* is faithful, and preserves arbitrary meets and joins, since they can be defined in terms of limits and colimits, and f^* preserves these. So $\mathrm{Sub}(X)$ inherits the infinite distributive law of Lemma 6.2 from $\mathrm{Sub}(f^*X)$.

(ii) By Proposition 6.4, it suffices to show that $\mathcal{E}$ inherits the property that an arbitrary generalized pullback of normal epimorphisms is epimorphic. But f^* preserves normal epimorphisms (since the property of being a normal epimorphism is expressible by geometric sequents in the internal language of a topos) and arbitrary limits; and it reflects epimorphisms because it is faithful. So $\mathcal{E}$ inherits this property from $\mathcal{F}$. $\square$

Note that we could have stated Proposition 6.5(ii) in 'local' form: given f as in the statement, if ($\mathcal{F}$ is bi-Heyting and) X is an object of $\mathcal{E}_*$ such that f^*X has a universal normal cover in $\mathcal{F}_*$, then X has one in $\mathcal{E}_*$.

Corollary 6.6. *For any small category $\mathcal{C}$, the topos $\mathcal{E} = [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ of presheaves on $\mathcal{C}$ is bi-Heyting, and every object of $\mathcal{E}_*$ has an initial normal cover.*

Proof. The inclusion $\mathcal{C}_0 \to \mathcal{C}$, where $\mathcal{C}_0$ denotes the discrete category with the same objects as $\mathcal{C}$ (or simply the set of objects of $\mathcal{C}$) induces an essential surjection

$$\mathbf{Set}/\mathcal{C}_0 \simeq [\mathcal{C}_0^{\mathrm{op}}, \mathbf{Set}] \longrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]\,.$$

But $\mathbf{Set}/\mathcal{C}_0$ is Boolean, so it is bi-Heyting; and its category of pointed objects has initial normal covers by Corollary 3.3. □

It would of course be easy to prove Corollary 6.6 directly from Proposition 6.4, using the facts that unions, intersections, generalized pullbacks and epimorphisms are all computed 'pointwise' in a presheaf topos.

Another class of Grothendieck toposes where initial normal covers exist may be obtained using a strengthening of the bi-Heyting condition. Let us recall that a complete lattice L is *completely distributive* when, given any doubly-indexed family $(x_{i,j} \mid i \in I, j \in J_i)$ of elements of L, one has

$$\bigwedge_{i\in I}\Big(\bigvee_{j\in J_i} x_{i,j}\Big) = \bigvee_{\phi\in F}\Big(\bigwedge_{i\in I} x_{i,\phi(i)}\Big)$$

where F denotes the set of choice functions ϕ for the family of sets $(J_i \mid i \in I)$, that is functions such that $\phi(i) \in J_i$ for all $i \in I$.

Definition 6.7. We call a Grothendieck topos *completely distributive* when its lattices of subobjects are completely distributive.

It is clear that any completely distributive topos is bi-Heyting. Also, the analogues of Lemma 6.3 and Proposition 6.5(i) both hold for complete distributivity, with the same proofs as for the bi-Heyting property.

The following result is probably known, but we did not find an explicit reference for it.

Proposition 6.8. *In a completely distributive localic topos, the generalized pullback of a family of arbitrary epimorphisms is still an epimorphism.*

Proof. Let $(q_i\colon A_i \twoheadrightarrow X \mid i \in I)$ be a family of epimorphisms in a completely distributive localic topos $\mathcal{E}$. By working in the slice category $\mathcal{E}/X$, we may reduce to the case when X is the terminal object 1. Suppose $\mathcal{E}$ is the topos of sheaves on a frame L; then the assertion that $A_i \to 1$ is epimorphic means that the set $J_i = \{u \in L \mid A_i(u) \neq \emptyset\}$ is a covering sieve on the top element of L, i.e. that $\bigvee J_i = 1$. So we have $\bigwedge_{i\in I}(\bigvee J_i) = 1$, and hence by complete distributivity

$$\bigvee_{\phi\in F}\Big(\bigwedge_{i\in I} \phi(i)\Big) = 1\,,$$

where F is as before the set of choice functions for $(J_i \mid i \in I)$. But for every $\phi \in F$, if we write u_ϕ for $\bigwedge_{i \in I} \phi(i)$, we have $A_i(u_\phi) \neq \emptyset$ for all i, and hence

$$\Big(\prod_{i \in I} A_i\Big)(u_\phi) = \prod_{i \in I} \Big(A_i(u_\phi)\Big) \neq \emptyset\,,$$

so this implies that $\prod_{i \in I} A_i \to 1$ is epimorphic. □

Corollary 6.9. *Let $\mathcal{E}$ be a completely distributive étendue. Then $\mathcal{E}$ is a bi-Heyting topos in which every pointed object $X \in \mathcal{E}_*$ admits an initial normal cover.*

Proof. We recall that a Grothendieck topos $\mathcal{E}$ is called an étendue if there is an object A of $\mathcal{E}$ such that $A \to 1$ is epic and the topos $\mathcal{E}/A$ is localic. In this event, the induced morphism $\mathcal{E}/A \to \mathcal{E}$ is essential and surjective; and $\mathcal{E}/A$ inherits complete distributivity from $\mathcal{E}$. So it satisfies the conclusion of Proposition 6.8; but since, as we already remarked, complete distributivity implies the bi-Heyting property, this suffices by Proposition 6.4 for the existence of initial normal covers in $(\mathcal{E}/A)_*$. Their existence in $\mathcal{E}_*$ then follows from Proposition 6.5. □

The localic assumption is essential to the proof of Proposition 6.8. If G is a topological group, then the topos $\mathbf{Cont}(G)$ of continuous G-sets (cf. [14], A2.1.6) is completely distributive, since its subobject lattices are complete atomic Boolean algebras. But if G has an infinite family of open normal subgroups $(H_i \mid i \in I)$ whose intersection is not open, then the transitive G-sets G/H_i (are continuous and) map epimorphically to 1 in $\mathbf{Cont}(G)$, but their product in this category is empty. We note that this topos is Boolean, so it does not provide a counterexample to the assertion that every completely distributive Grothendieck topos has initial normal covers for all its pointed objects; indeed, we do not know any example of a completely distributive Grothendieck topos where this property fails. Note also that presheaf toposes are completely distributive, by the same argument which shows that they satisfy the bi-Heyting property.

Finally, let us remark that neither of the two classes of toposes for which we have been able to show that all pointed objects have initial normal covers — completely distributive étendues, and toposes admitting an essential surjection from a Boolean topos — is included in the other. Indeed, a complete Boolean algebra is completely distributive if and only if it is atomic (see [13], VII.1.16), so the topos of sheaves on an atomless complete Boolean algebra provides a counterexample to one inclusion. For the other, we have:

Example 6.10. Let X be the topological space whose points are those of the half-open interval $[0, 1) \subseteq \mathbb{R}$, and whose open sets are the intervals $[0, a)$ for $0 \leq a \leq 1$ (the case $a = 0$ being interpreted as the empty set). Let $\mathcal{E}$ be the topos of sheaves on X. Then $\mathcal{E}$ is localic and completely distributive (the latter because the subobject lattice of any representable functor is totally ordered). However, we claim that $\mathcal{E}$ does not admit any essential geometric morphism from a non-degenerate Boolean topos.

To prove this, suppose given an essential morphism $f\colon \mathcal{B} \to \mathcal{E}$ with $\mathcal{B}$ Boolean. There is no loss of generality in supposing that $\mathcal{B}$ is localic, since we can replace it by its localic reflection: the latter is still Boolean (since it is equivalent to a full subcategory of $\mathcal{B}$ closed under subobjects, cf. [14], A4.6.6) and the factorization of f through the reflection morphism (which exists because $\mathcal{E}$ is localic) is still essential. From now on, therefore, we assume that $\mathcal{B}$ is localic.

Now $\mathcal{E}$ is locally connected (indeed, totally connected in the sense of [14], C3.6.16, since every nonempty open set in X contains the point 0) and hence the unique geometric morphism $\mathcal{E} \to \mathbf{Set}$ is essential by [14], C1.5.9. Hence also the morphism $\mathcal{B} \to \mathbf{Set}$ is essential, i.e. $\mathcal{B}$ is locally connected. But a Boolean localic topos can be represented as sheaves for the canonical coverage on a complete Boolean algebra; and such a topos is locally connected iff the Boolean algebra is atomic, since the only connected elements of a Boolean algebra are atoms. Thus $\mathcal{B}$ is actually of the form $\mathbf{Set}/A$, where A is the set of atoms. In particular, each $a \in A$ defines an essential point of $a\colon \mathbf{Set} \to \mathcal{B}$, and hence by composition an essential point $fa\colon \mathbf{Set} \to \mathcal{E}$. But $\mathcal{E}$ has no essential points, since an essential point p of a spatial topos has a smallest open neighbourhood (the image of $p_!1 \to 1$ in the topos) and no point of X has this property. Thus we conclude that A is the empty set, and $\mathcal{B}$ is degenerate; in particular, f is not surjective.

7. Is the bi-Heyting axiom necessary?

The previous section has underlined the role that the bi-Heyting property can play in the existence of initial normal covers: in fact, as we have seen, it forces our necessary condition to become sufficient. But is the bi-Heyting property itself necessary, or even, necessary and sufficient? Here is a partial answer concerning the possible necessity of the bi-Heyting axiom.

Proposition 7.1. *Let $\mathcal{E}$ be a topos such that all pointed objects of $\mathcal{E}$ have initial normal covers. Then the subobjects of 1 in $\mathcal{E}$ constitute a bi-Heyting algebra.*

Proof. In the lattice of subobjects of 1, we must prove that given two subobjects U, V, there exists a subobject $U \setminus V$ such that for every subobject W

$$W \vee V \geq U \text{ iff } W \geq U \setminus V.$$

Notice at once that it suffices to check this property when $U \geq V$; indeed if that is done and arbitrary U, V are given, it suffices to define

$$U \setminus V \ =_{\text{def}}\ (U \vee V) \setminus V\,.$$

So consider two subobjects U, V of 1 with $U \geq V$. The following pushout of monomorphisms is thus also a pullback, by [14], A2.4.3.

$$\begin{tikzcd} V \arrow[r, tail] \arrow[d, tail] & 1 \arrow[d, tail, "v"] \\ U \arrow[r, tail, "u"] & X \end{tikzcd}$$

The domain of the initial normal cover of the pointed object (X, v) has a decidable basepoint by Proposition 3.5, thus it has the form $p\colon 1 \amalg A \twoheadrightarrow X$, with the first summand of $1 \amalg A$ mapped to v. It is tempting to conjecture that A should be a subobject of 1, in which case one could easily prove that it had the required property of a co-implication $U \setminus V$; however, there seems to be no reason in general to suppose that $A \to 1$ is monic. Nevertheless, we may show that the support σA of A (that is, the image of $A \to 1$) provides the required co-implication.

We observe first that $\sigma A \leq U$: for we may regard $1 \amalg (A \times U)$ as a subobject of $1 \amalg A$ (via the monic projection $A \times U \rightarrowtail A \times 1 \cong A$), and it still maps epimorphically to X since its image contains both the subobjects v and u. So by Corollary 2.3 it must be the whole of $1 \amalg A$; in other words, $A \times U \rightarrowtail A$ is an isomorphism. Hence $\sigma A = \sigma(A \times U) = \sigma A \wedge U$.

Moreover, since the pullback of p along $u \colon U \rightarrowtail X$ is epic, and the pullback of $1 \rightarrowtail 1 \amalg A \twoheadrightarrow X$ along u is precisely $V \rightarrowtail U$, we must have $\sigma A \vee V = U$. To show that $\sigma A = U \setminus V$, we must prove that it is the smallest subobject of 1 with this property.

Suppose $W \rightarrowtail 1$ also satisfies $W \vee V = U$. As before, we may regard $1 \amalg (A \times W)$ as a subobject of $1 \amalg A$; let $q \colon 1 \amalg (A \times W) \to X$ be the restriction of p to this subobject. We claim that q is an epimorphism. Its image clearly contains the point v, so its intersection with the subobject u certainly contains $V \rightarrowtail U$. But

$$V \vee \sigma(A \times W) = V \vee (\sigma A \wedge W) = (V \vee \sigma A) \wedge (V \vee W) = U\,.$$

Thus by another application of Corollary 2.3, we deduce that $A \times W$ is the whole of A, and hence that $\sigma A \leq W$. □

Corollary 7.2. *If initial normal covers exist in a localic topos $\mathcal{E}$, this topos is necessarily a bi-Heyting one.*

Proof. If Sub(1) is a bi-Heyting algebra, then so is Sub(U) for any subobject $U \rightarrowtail 1$. So the result follows from Lemma 6.3. □

Thus we see that the possession of initial normal covers for pointed objects is a rather rare property of localic toposes. In particular, if X is any Hausdorff space in which not every intersection of open sets is open, one can show that the distributive law of Lemma 6.2 fails in the lattice of open sets of X, and so the topos of sheaves on X does not have initial normal covers.

References

[1] Borceux, F., *When is Ω a cogenerator in a topos?*, Cahiers Topol. Géom. Diff. **16** (1975), 3–15.

[2] Borceux, F., *Handbook of Categorical Algebra 3: Categories of Sheaves*, Encyclopaedia Math. Appl. **52** (1994).

[3] Borceux, F., *A survey of semi-abelian categories*, In: Galois theory, Hopf Algebras, and Semi-abelian Categories, Fields Inst. Commun. **43** (2004), 27–60.

[4] Borceux, F. and Bourn, D., *Mal'cev, Protomodular, Homological and Semi-abelian Categories*, Math. Appl. **566** (2004).

[5] Borceux, F. and Bourn, D., *Split extension classifier and centrality*, to appear in the *Proceedings of the Streetfest 2005.*

[6] Borceux, F., Janelidze, G. and Kelly, G. M., *Internal object actions*, Comment. Math. Univ. Carolin. **46** (2005), 235–255.

[7] Borceux, F., Janelidze, G. and Kelly, G. M., *On the representability of actions in a semi-abelian category*, Theory Appl. Categ. **14** (2005), 244–286.

[8] Bourn, D., *Normal functors and strong protomodularity*, Theory Appl. Categ. **7** (2000), 206–218.

[9] Bourn, D., *A categorical genealogy for the congruence distributive property*, Theory Appl. Categ. **8** (2001), 391–407.

[10] Bourn, D., *Protomodular aspects of the dual of a topos*, Adv. Math. **187** (2004), 240–255.

[11] Bourn, D. and Janelidze, G., *Protomodularity, descent and semi-direct products*, Theory Appl. Categ. **4** (1998), 37–46.

[12] Janelidze, G., Márki, L. and Tholen, W., *Semi-abelian categories*, J. Pure Appl. Alg. **168** (2002), 367–386.

[13] Johnstone, P. T., *Stone Spaces*, Cambridge Stud. Adv. Math. No. 3 (1982).

[14] Johnstone, P. T., *Sketches of an Elephant: a Topos Theory Compendium*, volumes 1–2, Oxford Logic Guides **43–44** (2002).

[15] Mac Lane, S., *Categories for the Working Mathematician*, Graduate Texts in Math. No. **5** (1971; revised edition 1998).

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