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ON CORINGS AND COMODULES

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To my friend and colleague Jiří Rosický on his 60th birthday

Abstract. It is shown that the categories of R-coalgebras for a commutative unital ring R and the category of A-corings for some R-algebra A as well as their respective categories of comodules are locally presentable.

Introduction

The categories under consideration are defined as the categories of comonoids and comonoid-coactions in certain monoidally closed categories as follows:

a) Given a commutative unital ring R
- the category $\mathbf{Coalg}_R$ of R-coalgebras is the category of comonoids in $(\mathbf{Mod}_R, - \otimes_R -, R)$,
- the categories $\mathbf{Comod}_A$ and ${}_A\mathbf{Comod}$ of right resp. left A-comodules for an R-coalgebra A are the corresponding categories of right resp. left A-coactions on R-modules.

b) Given an R-algebra A,
- the category $\mathbf{Coring}_A$ of A-corings is the category of comonoids in $({}_A\mathbf{Mod}_A, - \otimes_A -, A)$, where ${}_A\mathbf{Mod}_A$ denotes the category of A, A-bimodules,
- the categories $\mathbf{Comod}_{\mathcal{C}}$ and ${}_{\mathcal{C}}\mathbf{Comod}$ of right resp. left $\mathcal{C}$-comodules for an A-coring $\mathcal{C}$ again are the respective categories of $\mathcal{C}$-coactions on left (right) A-modules.

Only scattered results are known about the structure of these categories: cocompleteness of these categories is a rather trivial fact (see Fact 2 below), cocommutative coalgebras form a cartesian closed category ([3]), $\mathbf{Comod}_A$ is locally presentable and comonadic over $\mathbf{Mod}_R$ ([11]). A first systematic approach to completeness — limited, however, to the case where the rings involved are regular — was presented in [8] using the dualized construction of colimits in varieties.

In this note we will offer a unified approach to these and many new results by considering the categories in question as subcategories of certain categories of

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functor–coalgebras $\mathbf{Coalg}F$; using methods from the theory of accessible categories (see [2], [8]) we will show first that these categories are complete and then, in a second step, that this also holds for their interesting subcategories $\mathbf{Coalg}_R$, $\mathbf{Coring}_A$, and $\mathbf{Comod}_A$. In fact we will prove even more: all categories mentioned so far are locally presentable categories.

Local presentability of $\mathbf{Coalg}_R$ generalizes Sweedler's so-called *Fundamental Theorem of Coalgebras* (see [10], [5]), which states that every coalgebra (over some field k) is a directed colimit of coalgebras whose underlying vector space is finite dimensional, hence of finitely presentable coalgebras, since the following is easy to prove:

Proposition 1. *A k-coalgebra is finitely presentable iff its underlying k-vector space is of finite dimension.*

Note, however, that neither this proposition nor Sweedler's prove generalize to arbitrary rings.

1. The categories $\mathbf{Coalg}T_I$ and $\mathbf{Coalg}M_A$

Let $(\mathbf{C}, \otimes, I)$ be any of the monoidally closed categories $(\mathbf{Mod}_R, - \otimes_R -, R)$ or $({}_A\mathbf{Mod}_A, - \otimes_A -, A)$ mentioned in the introduction. We consider the following functors:

$$\begin{array}{rccl} T_n\colon & \mathbf{C} & \longrightarrow & \mathbf{C} \\ & C & \longmapsto & \otimes^n C \end{array}$$

$$\begin{array}{rccl} T_I\colon & \mathbf{C} & \longrightarrow & \mathbf{C} \\ & C & \longmapsto & T_2 C \times I = (C \otimes C) \times I \end{array}$$

$$\begin{array}{rccl} {}_AM\colon & \mathbf{C}' & \longrightarrow & \mathbf{C}' \\ & C & \longmapsto & A \otimes C \end{array} \qquad \begin{array}{rccl} M_A\colon & \mathbf{C}' & \longrightarrow & \mathbf{C}' \\ & C & \longmapsto & C \otimes A \end{array}$$

where A is a monoid in $(\mathbf{C}, \otimes, I)$ and $\mathbf{C}' = \mathbf{C}$ in the commutative case and, for A non–commutative, $\mathbf{C}' = A$–$\mathbf{Mod}$ and $\mathbf{Mod}$–A, the categories of left and right A-modules respectively, with $- \otimes -$ the obvious bifunctor $\mathbf{C} \times \mathbf{C}' \longrightarrow \mathbf{C}'$.

Then $\mathbf{Coalg}_R$ and $\mathbf{Coring}_A$, respectively, are the full subcategories of $\mathbf{Coalg}T_I$ (w.r.t. the appropriately chosen $(C, \otimes, I)$ — see above) spanned by those T_I-coalgebras $\mathbb{C} = (C, C \xrightarrow{\langle m,e \rangle} (C \otimes C) \times I)$ which make the following diagrams commute

$$(1) \qquad \begin{tikzcd} C \arrow[r, "m"] \arrow[d, "m"'] & C \otimes C \arrow[d, "m \otimes \mathsf{id}_C"] \\ C \otimes C \arrow[r, "\mathsf{id}_C \otimes m"'] & C \otimes C \otimes C \end{tikzcd}$$

$$(2)\quad \begin{tikzcd} C \arrow[d, "m"'] \arrow[dr, "r_C"] & \\ C \otimes C \arrow[r, "\mathsf{id}_C \otimes e"'] & C \otimes I \end{tikzcd} \qquad \text{and} \qquad (3)\quad \begin{tikzcd} C \arrow[r, "m"] \arrow[dr, "l_C"'] & C \otimes C \arrow[d, "e \otimes \mathsf{id}_C"] \\ & I \otimes C \end{tikzcd}$$

Similarly, the various categories of comodules are subcategories of $\mathbf{Coalg}M_A$ and $\mathbf{Coalg}_A M$, respectively, defined by the obvious diagrammatic axioms.

We will need the following results, which are easy to prove (see [1], [8]).

Fact 2.

1. *For each functor $F\colon \mathbf{C} \longrightarrow \mathbf{C}$ the category $\mathbf{Coalg}F$ is cocomplete provided the category $\mathbf{C}$ is so.*
2. *The categories of comonoids and comonoid–coactions are closed in their respective functor–categories under colimits.*

Clearly M_A is accessible since it is a left adjoint. Also, if F is an accessible endofunctor on a category $\mathbf{A}$ with biproducts, then $F \times A = F + A$ is accessible for each object A in $\mathbf{A}$. Thus, T_I is accessible by the following fact:

Lemma 3. *Let $(\mathbf{C}, - \otimes -, I)$ be a monoidally closed category and $F\colon \mathbf{C} \longrightarrow \mathbf{C}$ a finitary functor. Then $\hat{F}\colon \mathbf{C} \longrightarrow \mathbf{C}$ with $\hat{F}(C) = C \otimes FC$ is finitary. In particular, the functor $T_n\colon \mathbf{C} \longrightarrow \mathbf{C}$ with $T_n C = \otimes^n C$ preserves directed limits.*

Proof. If $D\colon \mathbf{I} = (I, \leq) \longrightarrow \mathbf{C}$ is a directed diagram in $\mathbf{C}$ with colimit $D_i \xrightarrow{d_i} C$ the colimit of the diagram $\tilde{D}\colon \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{C}$ with $\tilde{D}(i,j) = D_i \otimes FD_j$ can be computed as $D_i \otimes FD_j \xrightarrow{d_i \otimes Fd_j} C \otimes C$ since F and each $X \otimes -$ and $- \otimes Y$ preserve (directed) colimits. Finally, the diagram $\hat{F} \circ D$ is a cofinal subdiagram of $\tilde{D}$. □

Remark 4. Since the monoidal categories under consideration are varieties also T_I preserves directed colimits (see also [8]). As a consequence of these observations we obtain that the underlying functors $|-|$ of $\mathbf{Coalg}T_I$ and $\mathbf{Coalg}M_A$ into $\mathbf{C}$ and $\mathbf{C}'$, respectively, have right adjoints and thus are comonadic (see [1]); their domains are also accessible by the following observation.

Recall that for functors $F, G\colon \mathbf{K} \longrightarrow \mathbf{L}$ the *inserter of F and G* is the full subcategory $\mathbf{Ins}(F, G)$ of the comma category $F \downarrow G$ spanned by all arrows $FK \longrightarrow GK$ ([2, 2.71]). Since $\mathbf{Coalg}F = \mathbf{Ins}(\mathsf{id}_{\mathbf{C}}, F)$ it follows from [2, 2.72] and the remark above that the categories $\mathbf{Coalg}T_I$ and $\mathbf{Coalg}M_A$ are accessible. Since any cocomplete accessible category is locally presentable, we obtain

Proposition 5. *The categories $\mathbf{Coalg}T_I$ and $\mathbf{Coalg}M_A$ are locally presentable.*

Remark 6. There is no reason to assume that limits in these categories are respected by their obvious underlying functors $|-|$ into $\mathbf{Mod}_R$. Consult [8] or [11] for how to possibly describe these limits.

2. The categories $\mathbf{Coalg}_R$ and $\mathbf{Comod}_A$

The defining axioms for R-coalgebras, i.e., the commutativity of the diagrams (1), (2), and (3) above can be interpreted as follows:

Denote by φ and ψ the natural transformations

$$\begin{array}{rcl} \varphi & : & |-| \longrightarrow T_3 \circ |-| \\ \varphi_{\mathbb{C}} & = & C \xrightarrow{m} C \otimes C \xrightarrow{m \otimes \mathsf{id}_C} C \otimes C \otimes C \end{array}$$

and

$$\begin{array}{rcl} \psi & : & |-| \longrightarrow T_3 \circ |-| \\ \psi_{\mathbb{C}} & = & C \xrightarrow{m} C \otimes C \xrightarrow{\mathsf{id}_C \otimes m} C \otimes C \otimes C \end{array}$$

(Naturality of φ and ψ is a consequence of functoriality of $-\otimes-$ and the definition of coalgebra homomorphism.)

Lemma 7. *$\mathbb{C} = (C, \langle m, e\rangle)$ satisfies (1) iff $\varphi_{\mathbb{C}} = \psi_{\mathbb{C}}$.*

Similarly,

$$\begin{array}{rcl} \varrho & : & |-| \longrightarrow |-| \otimes R \\ \varrho_{\mathbb{C}} & = & C \xrightarrow{m} C \otimes C \xrightarrow{\mathsf{id}_C \otimes e} C \otimes R \end{array}$$

and

$$\begin{array}{rcl} \lambda & : & |-| \longrightarrow R \otimes |-| \\ \lambda_{\mathbb{C}} & = & C \xrightarrow{m} C \otimes C \xrightarrow{e \otimes \mathsf{id}_C} R \otimes C \end{array}$$

are natural transformations and the following obviously hold

Lemma 8.

1. *$\mathbb{C} = (C, \langle m, e\rangle)$ satisfies (2) iff $\varrho_{\mathbb{C}} = r_{|\mathbb{C}|}$.*
2. *$\mathbb{C} = (C, \langle m, e\rangle)$ satisfies (3) iff $\lambda_{\mathbb{C}} = l_{|\mathbb{C}|}$.*

Recall now that (see [2, 2.76]), for accessible functors $F^t, G^t \colon \mathbf{K} \longrightarrow \mathbf{L}_t$ and families of natural transformations $\mu^t, \nu^t \colon F^t \longrightarrow G^t$ $(t \in T)$ the *equifier* of (μ^t) and (ν^t) is the full subcategory $\mathbf{Eq}(\mu^t, \nu^t)$ of $\mathbf{K}$ spanned by all K in $\mathbf{K}$ with $\mu^t_K = \nu^t_K$ for all $t \in T$, and that this subcategory is accessible.

Theorem 9. *The categories $\mathbf{Coalg}_R$ and $\mathbf{Coring}_A$ are locally presentable categories.*

Proof. By Lemmas 7 and 8 the category of comonoids in $\mathbb{C}$ is the equifier of the three pairs $(\varphi, \psi), (\lambda, l_{|-|}), (\varrho, r_{|-|})$ of natural transformations. Since all categories and functors under consideration are accessible, it is accessible as well. Moreover the categories under consideration are closed under colimits in their respective $\mathbf{Coalg}T_I$ by Fact 2 and hence cocomplete. Now the same argument used in the proof of Proposition 5 gives the result. $\square$

In a completely analogous way one obtains

Theorem 10. *The categories $\mathbf{Comod}_A$, ${}_A\mathbf{Comod}$, $\mathbf{Comod}_C$ and ${}_C\mathbf{Comod}$ are locally presentable categories and therefore have all limits.*

We now get, as simple corollaries,

Proposition 11.

1. $\mathbf{Coalg}_I$ *is coreflective in* $\mathbf{Coalg}T_I$.
2. $\mathbf{Comod}_A$ *is coreflective in* $\mathbf{Coalg}M_A$.

Proof. The proof is the same in both cases: the embedding of the respective subcategory preserves colimits and both subcategories, being locally presentable, are co-wellpowered and have a generator. Now apply the (dual of the) Special Adjoint Functor Theorem. □

Theorem 12.

1. $\mathbf{Coalg}_R$ *is comonadic over* $\mathbf{Mod}_R$,
2. $\mathbf{Coring}_A$ *is comonadic over* ${}_A\mathbf{Mod}_A$,
3. $\mathbf{Comod}_A$ *is comonadic over* $\mathbf{Mod}_R$, *and*
4. $\mathbf{Comod}_\mathcal{C}$ *is comonadic over* $\mathbf{Mod}$–A.

Proof. The respective underlying functors have right adjoints by Remark 4 and Proposition 11. They also create split equalizers by Remark 6 and because all of these categories are closed in their respective categories of functor–coalgebras w.r.t. subobjects carried by split monos (see [8] or Fact 17 below). □

Remark 13. The existence of cofree comodules certainly can be obtained directly. The argument given in [6] generalizes to our somewhat more general situation. See also [11]. Note also that the existence of a cofree coalgebra is known (see [3] for the cocommutative case with an argument similar or ours, and [10] with an explicit construction via the tensor algebra for the case of a field, which however generalizes to rings).

Generalizing a result in [1] we might reformulate the statement of the last theorem as follows

Theorem 14. *All categories* $\mathbf{Coalg}_R$, $\mathbf{Coring}_A$, $\mathbf{Comod}_A$, *and* $\mathbf{Comod}_\mathcal{C}$ *respectively, are covarieties.*

Remark 15. Obviously, the the above results can be extended to the categories of cocommuative coalgebras.

Problem 16. It is not clear that the kernel of a morphism in the categories $\mathbf{Coalg}_R$ or $\mathbf{Comod}_A$ is a subobject in the sense of say [4], i.e., whether its carrier map is injective. It has been shown in [8] that each injective homomorphism is a strong monomorphism, but it is clear that the converse doesn't hold: the categories under consideration, being locally presentable, carry an (epi, strong mono)-factorization structure (see [2]) but don't allow for image-factorizations of morphisms (see [8]). It thus would be interesting to characterize the injective homomorphism in these categories categorically and to describe the strong monomorphisms explicitely. If $\mathbf{Ker}f$ could be shown to be a subcoalgebra of f's domain, it would be an easy consequence to prove that it is the largest subcoalgebra contained in the $\mathbf{Mod}_R$-kernel of f. This is the case, if the ring R is regular (see [8]).

3. Purity

It is easy to see that $\mathbf{Coalg}_R$, $\mathbf{Coring}_A$, and $\mathbf{Comod}_A$ are closed in their respective categories $\mathbf{Coalg}T_I$ and $\mathbf{Coalg}M_A$ w.r.t. subobjects whose underlying embedding in $\mathbf{Mod}_R$ splits (see [8]). In fact, the proof of this statement given in [8] shows more:

Given a homomorphism $m\colon \mathbb{C} \longrightarrow \mathbb{D}$ in $\mathbf{Coalg}T_I$ and $\mathbf{Coalg}M_A$, respectively, then $\mathbb{C}$ is a comonoid and an A-coaction, respectively, provided that $\mathbb{D}$ is and $m \otimes m \otimes m$ and $m \otimes \mathsf{id}$ are monomorphisms in $\mathbf{C}$. In the commutative case this clearly holds, provided that m is a pure homomorphism. We thus have

Fact 17. *$\mathbf{Coalg}_R$ and $\mathbf{Comod}_A$ are closed in $\mathbf{Coalg}T_R$ and $\mathbf{Coalg}M_A$, respectively, under subobjects carried by pure R-linear maps.*

The categorical concept of λ-purity (λ a regular cardinal) as presented, e.g., in [2] generalizes the notion of a pure module homomorphism in the sense that an $\aleph_0$-pure morphism in $\mathbf{Mod}_R$ is simply a pure homomorphism, provided R is a PID. We do not know whether this fact has appeared in print elsewhere but believe it must be well known: an argument would be a straightforward generalization of the proof given for [9, 61.11], considering finitely generated submodules instead of single generated ones.

Also in the non-commutative case the notion of $\aleph_0$-purity can be exploited: Since the functor $C \otimes -$ is left adjoint it preserves (directed) colimits and finitely presentable objects, hence $\aleph_0$-pure morphisms by [2, 2.38] which are (regular) monomorphisms. Thus the closure–statement of Fact 17 holds also in the non-commutative case.

Since the underlying functors $\mathbf{Coalg}F \longrightarrow \mathbf{C}$ ($F = T_I$ or $F = M_A$) are left adjoints and $\mathbf{Coalg}F$ is a λ-presentable category for some λ they preserve λ-pure morphisms by the same argument as above, so that we can deduce

Proposition 18. *Each of the categories $\mathbf{Coalg}_R$, $\mathbf{Coring}_A$, $\mathbf{Comod}_A$, and $\mathbf{Comod}_{\mathcal{C}}$ is closed in its respective category of functor coalgebras under λ-pure subobjects for a suitable λ.*

Proof. Let $\mathbf{Coalg}F$ be λ-presentable and $\mathbb{C}$ a λ-pure subobject of $\mathbb{D}$, $\mathbb{D}$ in the subcategory under consideration. Then, in $\mathbf{C}$, the embedding $C \hookrightarrow D$ is λ-pure, thus $\aleph_0$-pure. Now the claim follows from the above observations. □

Remark 19. The proposition above allows for an alternative proof of Theorem 9 and Theorem 10. Accessibility of our subcategories is in view of [2, 2.36] an immediate consequence of Proposition 18 since they are clearly closed under colimits (see [8]).

Let us finally relate $\aleph_0$-purity in ${}_A\mathbf{Mod}_A$ with purity in ${}_A\mathbf{Mod}$ and $\mathbf{Mod}_A$.

Proposition 20. *If f is an $\aleph_0$-pure morphism in ${}_A\mathbf{Mod}_A$, then f is pure in ${}_A\mathbf{Mod}$ and in $\mathbf{Mod}_A$.*

Proof. By [2, 2.30] f is a directed colimit of split monomorphisms in ${}_A\mathbf{Mod}_A$, hence it is a directed colimit of split monomorphisms in each of the categories $\mathbf{Mod}_A$ and ${}_A\mathbf{Mod}$ as well and therefore pure in both categories again by [2, 2.30]. □

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