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SUBLOCALE SETS AND SUBLOCALE LATTICES

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Dedicated to Jiří Rosický on the occasion of his 60th birthday

Abstract. We present very short and simple proofs of such facts as co-frame distributivity of sublocales, zero-dimensionality of the resulting co-frames, Isbell's Density Theorem and characteristic properties of fit and subfit frames, using sublocale sets.

Introduction

There are few new results, if any, in this paper. Its purpose, instead, is to present new and very simple proofs of several known facts.

Viewing locales (frames) as generalized spaces we can choose among several approaches in representing "subspaces": sublocale maps (onto frame homomorphisms), congruences or nuclei. One that is largely neglected is that of sublocale sets, appearing as an exercise in Johnstone's "Stone spaces" [3] and not much exploited even there (see 1.3 below - one can think of them as of "left ideals" if we take the meet structure as the additive part and the Heyting operation as the multiplicative part of the frame structure).

Unlike in the other representations, both the meet and join structure of the system of sublocale sets is very simple (the meets are intersections and the joins are what is to be expected as joins of ideals). What we want to emphasize, however, is that one can prove in a very simple way several facts on sublocales (using just trivial Heyting identities, or Heyting identities that themselves require only two-line proofs). Thus we have a very short proof of the co-frame distributivity in the sublocale lattice and of the behaviour of open and closed sublocales (complementarity, generating of general sublocales by open and closed ones); the structure of the closure is so transparent that the Isbell's Density Theorem follows as an immediate observation. Furthermore, we present discussion of fitness and subfitness

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under this viewpoint; also here the proofs are very simple. In the last section we formulate a problem connected with yet another sublocale representation.

1. Preliminaries

1.1. Recall that a *frame* is a complete lattice L satisfying the distributivity law

$$\text{(Distr)} \qquad a \wedge \bigvee B = \bigvee \{a \wedge b \mid b \in B\}$$

for any $a \in L$ and $B \subseteq L$. *Frame homomorphisms* $h : L \to M$ preserve general joins and finite meets.

For instance the lattice $\Omega(X)$ of all open sets in a topological space X is a frame, and if $f : X \to Y$ is a continuous map then $\Omega(f) = (U \mapsto f^{-1}[U]) : \Omega(Y) \to \Omega(X)$ is a frame homomorphism. Furthermore, for a big class of spaces (the *sober* spaces) such $\Omega(f)$ are precisely the homomorphisms $\Omega(Y) \to \Omega(X)$ so that we can think of the category of frames as a (contravariant) extension of (a large subcategory of) the category of topological spaces.

One can make this extension covariant by considering the dual category of the category of frames and frame homomorphisms, that is, the *category of locales and locale maps*. Locales and frames are the same thing, but locale maps go in the opposite direction.

For more information about frames and locales see e.g., [3] or [5].

1.1.1. The formula (Distr) can be interpreted as that the maps $a \wedge (-) : L \to L$ preserve all suprema. Hence they have right Galois adjoints and every frame is automatically a Heyting algebra. The Heyting operation will be denoted by $\to$ and we have the standard Heyting equivalence

$$\text{(H)} \qquad a \wedge b \leq c \quad \text{iff} \quad a \leq b \to c\,.$$

1.2. Due to the contravariance of the extension of the category of spaces to that of frames it is natural to represent (generalized) subspaces of L as the *sublocale maps*, that is, the *onto* frame homomorphisms $h : L \to M$ (they are the extremal epimorphisms in the category of frames and hence the extremal monomorphisms in the category of locales).

Another representation of the same is provided by *frame congruences* (congruences with respect to general joins and finite meets) the translation being given by

$$h : L \to M \quad \mapsto \quad E_h = \{(x, y) \mid h(x) = h(y)\}\,,$$
$$E \quad \mapsto \quad h_E = (x \mapsto Ex) : L \to L/E\,.$$

A further one is constituted by the *nuclei*, which are maps $\nu : L \to L$ satisfying

(1) $a \leq \nu(a)$,
(2) $a \leq b \Rightarrow \nu(a) \leq \nu(b)$,
(3) $\nu\nu(a) = \nu(a)$, and
(4) $\nu(a \wedge b) = \nu(a) \wedge \nu(b)$,

the translation being (say, between congruences and nuclei)

$$\begin{array}{rcl} \nu & \mapsto & E_\nu = \{(x,y) \mid \nu(x) = \nu(y)\}\,, \\ E & \mapsto & \nu_E = (x \mapsto \bigvee Ex) : L \to L\,. \end{array}$$

Note that if ν is a nucleus then the set $\nu[L]$ is a frame, with the same meets as in L but generally with different joins.

1.3. We will be concerned with yet another representation, based on Exercise II.2.3 in [3] (from now on we will automatically use the Heyting operation as in 1.1.1 above). Define a *sublocale set* (briefly, a *sublocale*) S in a frame L as a subset $S \subseteq L$ such that

(S1) for every $A \subseteq S$, $\bigwedge A$ is in S, and
(S2) for every $s \in S$ and every $x \in L$, $x \to s$ is in S.

An easy but important fact (the subject of the exercise in question) is that the sublocales (sublocale sets) are in a natural one-to-one correspondence with the nuclei given by

$$\begin{array}{rcl} \nu & \mapsto & S_\nu = \nu[L]\,, \\ S & \mapsto & \nu_S, \quad \nu_S(x) = \bigwedge\{s \in S \mid x \le s\}\,. \end{array}$$

Hence in particular

1.3.1. *Each sublocale S $(= \nu_S[L])$ is a frame with the same meets as in L, and since the Heyting operation $\to$ depends on the meet structure only, with the same Heyting operation.*

The following well-known Heyting formulas will be used. Note that the proofs are extremely easy; we include them to avoid the impression that the gist of the very simple proofs presented later might be hidden in something deep in the computation.

1.4. Proposition. *The following hold for any frame:*

(1) $a \le b \to a$.
(2) $a \to b = 1$ *iff* $a \le b$.
(3) $a \to b = a \to (a \wedge b)$.
(4) $a \wedge (a \to b) \le b$.
(5) $1 \to a = a$.
(6) $(a \wedge b) \to c = a \to (b \to c) = b \to (a \to c)$.
(7) $a \le b \to c$ *iff* $b \le a \to c$.
(8) $a \to \bigwedge_{i \in J} b_i = \bigwedge_{i \in J}(a \to b_i)$.
(9) $(\bigvee_{i \in J} a_i) \to b = \bigwedge_{i \in J}(a_i \to b)$.
(10) $a \wedge (a \to b) = a \wedge b$.
(11) $a \wedge b = a \wedge c$ *iff* $a \to b = a \to c$.
(12) $x = (x \vee a) \wedge (a \to x)$.

Proof. (1) till (5) immediately follow from the basic Galois equivalence (H) (as for (2), $1 = a \to b$ iff $1 \le a \to b$, in the last $x \le 1 \to a$ iff $x = x \wedge 1 \le a$).

(6) is obtained by associativity of $\wedge$ confronting $x \le (a \wedge b) \to c$ with $x \le a \to (b \to c)$ and $x \le b \to (a \to c)$.
(7) follows from (6) and (2); (8) resp. (9) are then immediate consequences of (H) and (7).
(10) : By (1) and (4), $a \wedge b \le a \wedge (a \to b) \le a \wedge b$.
(11) is an immediate consequence of (3) and (10).
(12) : By (1), $x \le (x \vee a) \wedge (a \to x)$ and $(x \vee a) \wedge (a \to x) = (x \wedge (a \to x)) \vee a \wedge (a \to x) \le x$ by (3). $\square$

Note. Of course, (4) is included in (10). However, we have formulated it extra as one of the most immediate facts (the reader has certainly recognized it as the well-known *Modus Ponens*); also, it is useful to have it prepared for the half-line proof of (10).

1.4.1. We will often use the following simple

Lemma. *Let $S \subseteq L$ be a sublocale, $s \in S$. Then for any $x \in L$, $x \to s = \nu_S(x) \to s$.*

Proof. By 1.4(7), (S2) and 1.4(7) again we have $y \le x \to s$ iff $x \le y \to s$ iff $\nu(x) \le y \to s$ iff $y \le \nu(x) \to s$. $\square$

1.4.2. Note that $(-) \to \bigwedge S$ is the pseudocomplement in S: since $0_S = \bigwedge S$ is the bottom of S, we have, for $x, y \in S$,

$$x \wedge y = 0_s \quad \text{iff} \quad x \le y \to \bigwedge S\,.$$

2. The coframe of sublocales

2.1. Obviously, arbitrary intersections of sublocales (sublocale sets) are sublocales. Thus, the sublocales of L constitute a complete lattice; it will be denoted by

$$\mathfrak{Sls}(L)\,.$$

Obviously, the least element in $\mathfrak{Sls}(L)$ is $\mathsf{O} = \{1\}$ and the largest one is L.

2.2. Proposition. *The joins in $\mathfrak{Sls}(L)$ are given by the formula*

$$\bigvee_{i \in J} S_i = \{\bigwedge A \mid A \subseteq \bigcup_{i \in J} S_i\}\,;$$

in particular, $S_1 \vee S_2 = \{a_1 \wedge a_2 \mid a_i \in S_i,\ i = 1, 2\}$.

Proof. Obviously $\{\bigwedge A \mid A \subseteq \bigcup S_i\}$ is closed under meets, and for a general $x \in L$ we have, by 1.4(8), $x \to \bigwedge A = \bigwedge_{a \in A}(x \to a) \in \{\bigwedge A \mid A \subseteq \bigcup S_i\}$; on the other hand, if $S_i \subseteq T \in \mathfrak{Sls}(L)$ for all i then trivially $\{\bigwedge A \mid A \subseteq \bigcup S_i\} \subseteq T$. $\square$

2.3. Proposition. *$\mathfrak{Sls}(L)$ is a coframe, that is, it satisfies the distributivity law*

$$A \vee (\bigcap_{i \in J} B_i) = \bigcap_{i \in J}(A \vee B_i)\,.$$

Proof. If $x \in \bigcap_{i \in J}(A \vee B_i)$ we have, for each i, an $a_i \in A$ and a $b_i \in B_i$ such that $x = a_i \wedge b_i$. Set $a = \bigwedge_{i \in J} a_i$ so that $x = a \wedge (\bigwedge_{i \in J} b_i) \le a \wedge b_i \le a_i \wedge b_i = x$

and $x = a \wedge b_i$ for all i. By 1.4(11), then, $a \to b_i$ does not depend on i; denote the common value by b. Thus, by 1.4(10), $x = a \wedge b_i = a \wedge (a \to b_i) = a \wedge b$ with $a \in A$ and $b \in \bigcap_{i \in J} B_i$ (as, for each i, $b = a \to b_i \in B_i$). Hence $\bigcap_{i \in J}(A \vee B_i) \subseteq A \vee (\bigcap_{i \in J} B_i)$, and the other inclusion is trivial. $\square$

2.4. Proposition. *A sublocale of a sublocale is a sublocale.*

Proof. Let $T \subseteq S$ resp. $S \subseteq L$ be sublocales (of S resp. L). Then T obviously satisfies (S1) in L. Let $t \in T$ and let $x \in L$ be general. Set $y = \nu_S(x)$. By Lemma 1.4.1 $x \to t = y \to t$ and as $y \in S$, $y \to t$ is in T. $\square$

3. Open and closed sublocales

3.1. For every a,

$$\mathfrak{o}(a) = \{a \to x \mid x \in L\} = \{x \mid a \to x = x\}$$

is a sublocale: it is closed under meets by 1.4(9) and if $y \in L$ then $y \to (a \to x) = a \to (y \to x)$ by 1.4(6) (the second formula follows from 1.4(6) as well: $a \to (a \to x) = (a \wedge a) \to x$). It will be referred to as an *open sublocale* of L

Further define *closed sublocales* as

$$\mathfrak{c}(a) = {\uparrow} a$$

(for the second sublocale property recall 1.4(1)).

3.2. Proposition. *$\mathfrak{c}(a)$ and $\mathfrak{o}(a)$ are complements to each other in $\mathfrak{Sls}(L)$.*

Proof. If $x \in \mathfrak{c}(a) \cap \mathfrak{o}(a)$ then $a \le x = a \to y$. Thus $a \le a \to y$, that is, $a \le y$, and $x = a \to y = 1$. Hence $\mathfrak{c}(a) \cap \mathfrak{o}(a) = \mathsf{O}$.

On the other hand, for any $x \in L$, $x = (x \vee a) \wedge (a \to x) \in \mathfrak{c}(a) \vee \mathfrak{o}(a)$, by 1.4(12). $\square$

3.3. Proposition. *Let S be a sublocale. Then*

$$S = \bigcap \{\mathfrak{c}(x) \vee \mathfrak{o}(y) \mid \nu(x) = \nu(y)\}\,.$$

Proof. Let $a \in S$ and let $\nu(x) = \nu(y)$. Then, by 1.4.1 $x \to a = \nu(x) \to a = \nu(y) \to a = y \to a$ and hence, by 1.4(12), $a = (a \vee x) \wedge (x \to a) = (a \vee x) \wedge (y \to a) \in \mathfrak{c}(x) \vee \mathfrak{o}(y)$.

Conversely, let a be in $\bigcap \{\mathfrak{c}(x) \vee \mathfrak{o}(y) \mid \nu(x) = \nu(y)\}$. Then in particular $a \in \mathfrak{c}(\nu(a)) \vee \mathfrak{o}(a)$. Hence, $a = y \wedge (a \to z)$ for some $y \ge \nu(a)$ $(\ge a)$ and $z \in L$. Then $1 = a \to a = a \to (y \wedge (a \to z)) = (a \to y) \wedge (a \to (a \to z)) = 1 \wedge (a \to z) = a \to z$ by 1.4(8), (2) and (6), and hence $a = y \wedge (a \to z) = y \ge \nu(a)$. Thus, $a = \nu(a) \in S$. $\square$

3.4. The *closure* $\overline{S}$ of a sublocale $S \subseteq L$, that is, the smallest closed sublocale containing S is obviously given by the formula

$$\overline{S} = {\uparrow} \bigwedge S\,.$$

We have:

Proposition.
(1) $\overline{\mathsf{O}} = \mathsf{O}$, $S \subseteq \overline{S}$ *and* $\overline{\overline{S}} = \overline{S}$.
(2) $\overline{S \vee T} = \overline{S} \vee \overline{T}$.

Proof. (1) is trivial.
(2) Trivially $\overline{S} \vee \overline{T} \subseteq \overline{S \vee T}$. On the other hand, $\bigwedge(S \vee T) = \bigwedge S \vee \bigwedge T \in \overline{S} \vee \overline{T}$ and hence $\overline{S \vee T} = \uparrow\bigwedge(S \vee T) \subseteq \overline{S} \vee \overline{T}$. □

4. Isbell's density theorem. Boolean sublocales

4.1. By 3.4, a sublocale S is dense in L, that is, $\overline{S} = L$, if and only if $0 \in S$. Taking into account that $\mathfrak{b}(0) = \{x \to 0 \mid x \in L\}$ is a sublocale (it is closed under meets by 1.4(9), and $y \to (x \to 0) = (x \wedge y) \to 0$ by 1.4(6)) and that it is, by (S2), the smallest sublocale containing 0, we immediately obtain

Proposition (Isbell's Density Theorem)**.** *Each frame has a smallest dense sublocale, namely* $\mathfrak{b}(0)$.

4.2. For the same reasons as above

$$\mathfrak{b}(c) = \{x \to c \mid x \in L\}$$

is a sublocale and we obtain

Proposition. *Let $c \in L$. Then $\mathfrak{b}(c)$ is the smallest sublocale of L containing c. For any sublocale $S \subseteq L$, $\mathfrak{b}(\bigwedge S)$ is the smallest sublocale dense in S (that is, such that its closure contains S).*

4.3. Here is another feature of the sublocales $\mathfrak{b}(c)$.

Proposition. *A sublocale $S \subseteq L$ is a Boolean algebra iff $S = \mathfrak{b}(c)$ for some $c \in L$.*

Proof. First, observe that, by 1.4(9) (and (5)), $\bigwedge \mathfrak{b}(c) = (\bigvee L) \to c = 1 \to c = c$. Thus,

$$c \text{ is the bottom of the frame } \mathfrak{b}(c).$$

To show that $\mathfrak{b}(c)$ is Boolean we will show that for any $x \in \mathfrak{b}(c)$, $x \to c$ is its complement in $\mathfrak{b}(c)$. First, by 1.4(10), $x \wedge (x \to c) = x \wedge c = c$. Second, let $y = z \to c \in \mathfrak{b}(c)$ and let $x \leq y$ and $x \to c \leq y$. By the former and 1.4(7), $z \leq x \to c$ and combining this with the latter, $z \leq z \to c$, hence $z = z \wedge z \leq c$ and by 1.4(2) $y = 1$.

Now let S be Boolean. Set $c = \bigwedge S$. Then by 1.4.2, $(-) \to c$ is the pseudocomplement in S and since pseudocomplements in Boolean algebras are complements, we have for any $x \in S$, $x = (x \to c) \to c \in \mathfrak{b}(c)$. On the other hand $\mathfrak{b}(c) \subseteq S$ by 4.2 since $c = \bigwedge S \in S$. □

5. Fitness

5.1. A frame L is said to be *fit* if

$$\text{(Fit)} \qquad a \not\leq b \quad \Rightarrow \quad \exists c,\ a \vee c = 1 \ \text{ and } \ c \to b \neq b$$

(see [2], 2.2 – this definition was presented there as an equivalent characterization, the original definition of this property is what we will have as (4) in 5.2 below).

5.2. For a sublocale S of a general L define

$$S' = \downarrow(S \smallsetminus \{1\})\ (= \{x \in L \mid \nu_S(x) \neq 1\})\,.$$

5.3. Lemma. *For any sublocale S and any $c \in L$, $S \subseteq \mathfrak{o}(c)$ iff $\nu_S(c) = 1$.*

Proof. If $\nu_S(c) = 1$ then for $s \in S$, by 1.4.1 and 1.4(5), $c \to s = \nu_S(c) \to s = 1 \to s = s$. If $s = \nu_S(c) \neq 1$ then $c \leq s$ and $c \to s = 1 \neq s$, by 1.4(2). □

5.4. Proposition. *The following statements about a frame L are equivalent:*

1. *L is fit.*
2. *For any sublocales S and T of L, $S' = T' \ \Rightarrow\ S = T$.*
3. *Congruences in L coincide iff the respective classes of the top element do.*
4. *For each sublocale S,*
$$S = \bigcap\{\mathfrak{o}(x) \mid \nu_S(x) = 1\}\,.$$
5. *Each sublocale is an intersection of open sublocales.*
6. *Each closed sublocale is an intersection of open sublocales.*

Proof. (1)⇒(2): Let $S' = T'$ and let $b \in T$, $b \neq 1$. Set $a = \nu_S(b)$.

Suppose $a \vee c = 1$ and $b \vee c \leq a_1 \in S$. Then $a_1 \geq a \vee c = 1$ so that $b \vee c \notin S' = T'$. By 1.4(12), however, $(b \vee c) \wedge (c \to b) \leq b$ and hence $b \vee c \leq (c \to b) \to b \in T$ so that $(c \to b) \to b = 1$ and $c \to b = b$ by 1.4(1) and (2). Therefore, by (Fit), $a \leq b$ and hence $b = \nu_S(b) \in S$.
(2)⇔(3): $S' = L \smallsetminus \nu_S^{-1}(1)$ so that $S' = T'$ iff $\nu_S^{-1}(1) = \nu_T^{-1}(1)$ and since $xE_S y \equiv \nu_S(x) = \nu_S(y)$ this happens iff $E_S 1 = E_T 1$. Now recall that $S \mapsto E_S$ is a one-one correspondence between sublocales and congruences.
(2)⇒(4): The inclusion $\subseteq$ follows from 5.3. On the other hand, if $a \in T = \bigcap\{\mathfrak{o}(x) \mid \nu_S(x) = 1\}$ we have $x \to a = a$ whenever $\nu_S(x) = 1$. Thus, if $\nu_S(a) = 1$ we have $a = a \to a = 1$. Hence $T \smallsetminus \{1\} \subseteq S'$, consequently $T' \subseteq S' \subseteq T'$, and by (2), $S = T$.
(4)⇒(5)⇒(6) is trivial.
(6)⇒(1): By 5.3, if $\uparrow a$ is an intersection of some open sublocales then it is the intersection of all open sublocales $\mathfrak{o}(c)$ with $\nu_{\uparrow a}(c) = 1$. As $\nu_{\uparrow a}(c) = \bigwedge\{s \mid a \leq s,\ c \leq s\} = a \vee c$, we have $\mathfrak{c}(a) = \uparrow a = \bigcap\{\mathfrak{o}(c) \mid a \vee c = 1\}$ and hence, if $c \to b = b$ for all c such that $a \vee c = 1$ then $a \leq b$. □

6. Subfitness

6.1. A frame L is said to be *subfit* (*conjunctive* in [10]) if

$$\text{(Sfit)} \qquad a \nleq b \quad \Rightarrow \quad \exists c, \quad a \vee c = 1 \neq b \vee c\,.$$

6.2. Lemma. $\mathfrak{c}(b) \subseteq \mathfrak{o}(a)$ *iff* $a \vee b = 1$.

Proof. If $a \vee b = 1$ and $x \geq b$ then by 1.4(5), (9) and (2) $x = (a \vee b) \to x = (a \to x) \wedge (b \to x) = a \to x$ and $x \in \mathfrak{o}(a)$. If $\mathfrak{c}(b) = \mathfrak{o}(a)$ we have $a \vee b = \bigwedge\{x \in \mathfrak{c}(b) \mid a \leq x\} \geq \bigwedge\{x \in \mathfrak{o}(a) \mid a \leq x\} = 1$ since if $a \leq a \to y$ then $1 \wedge a = a \wedge a \leq y$ and hence $1 \leq a \to y$. $\square$

6.3. Proposition. *The following statements about a frame L are equivalent:*

(1) *L is subfit.*
(2) *For a sublocale $S \subseteq L$, $S \smallsetminus \{1\}$ is cofinal in $L \smallsetminus \{1\}$ only if $S = L$.*
(3) *Any congruence E on L such that $E1 = \{1\}$ is trivial.*
(4) *If $S \neq L$ for a sublocale $S \subseteq L$ then there is a closed $\mathfrak{c}(x) \neq \mathsf{O}$ such that $S \cap \mathfrak{c}(x) = \mathsf{O}$.*
(5) *For each open sublocale $\mathfrak{o}(a)$,*

$$\mathfrak{o}(a) = \bigvee\{\mathfrak{c}(x) \mid x \vee a = 1\}\,.$$

(6) *Each open sublocale is a join of closed sublocales.*

Proof. (1)⇒(2): Let $b \in L$ and $a = \nu_S(b)$. If $a \vee c = 1$ we have $\nu_S(b \vee c) \geq a \vee c = 1$ and hence $b \vee c = 1$. Thus, $a \leq b$, that is, $b \in S$.
(2)⇔(3): $S \smallsetminus \{1\}$ is cofinal in $L \smallsetminus \{1\}$ iff $\nu_S^{-1}(\{1\}) = \{1\}$ which can be rewritten into the congruence condition (3) analogously as in (2)⇔(3) of Proposition 5.4.
(2)⇔(4): (4) is just an immediate reformulation of (2).
(5)⇔(6) follows immediately from Lemma 6.2.
(4)⇒(5): Set $S = \bigvee\{\mathfrak{c}(x) \mid x \vee a = 1\}$ and suppose $\mathfrak{c}(y) \cap (\mathfrak{c}(a) \vee S) = \mathsf{O}$. Then, $\mathfrak{c}(y) \cap \mathfrak{c}(a) = \mathsf{O}$ and, by complementarity, $\mathfrak{c}(y) \subseteq \mathfrak{o}(a)$. Therefore, by 6.2, $y \vee a = 1$, and we conclude that $\mathfrak{c}(y) \subseteq S$ and finally $\mathfrak{c}(y) = \mathfrak{c}(y) \cap (\mathfrak{c}(a) \vee S) = \mathsf{O}$. Thus by (4), $\mathfrak{c}(a) \vee S = L$ and by complementarity again $\mathfrak{o}(a) \subseteq S$ ($\subseteq \mathfrak{o}(a)$ by 6.2).
(5)⇒(1): If $a \nleq b$ we have $\mathfrak{c}(b) \nsubseteq \mathfrak{c}(a)$ (as $b \in \mathfrak{c}(b) \smallsetminus \mathfrak{c}(a)$) and hence $\mathfrak{o}(a) \nsubseteq \mathfrak{o}(b)$. Thus there is a c such that $c \vee a = 1$ and $\mathfrak{c}(c) \nsubseteq \mathfrak{o}(b)$, that is, $c \vee b \neq 1$. $\square$

6.4. Proposition. *Each complemented sublocale of a subfit locale is subfit.*

Proof. Let T be the complement of a sublocale $S \subseteq L$ and let $S_0 \subset S$ be a sublocale of S. Thus $S_0 \vee T \neq L$ (else $S_0 \supseteq S$) and hence, by 6.3.(4), $(S_0 \vee T) \cap \uparrow x = \mathsf{O}$ for some $x \neq 1$. Thus, $S_0 \cap \uparrow x = \mathsf{O}$ and as also $T \cap \uparrow x = \mathsf{O}$, $\uparrow x \subseteq S$, which means that $\uparrow x \neq \mathsf{O}$ is a closed sublocale of S that does not intersect S_0. $\square$

7. A problem: What do the open and closed sublocales have in common?

7.1. Yet another representation of sublocales can be obtained from the Priestley duality ([6], [7]). In this duality, distributive lattices are put to a dual correspondence with Priestley spaces, that is, compact ordered topological spaces $(X, \tau, \leq)$ such that incomparable elements can be separated by clopen down-sets. The counterparts of frames have the extra property that the closures of open up-sets are open ([8]).

7.2. The sublocale maps (onto frame homomorphism) are dually represented as embeddings of special subspaces. Namely, they appear as closed subsets $Y \subseteq (X, \tau, \leq)$ such that

$$\text{for every open down-set } U, \quad \overline{Y \cap U} = Y \cap \overline{U}$$

(see [9]). Thus for instance each clopen subset represents a sublocale, and, moreover, a complemented one. In particular.

- open sublocales are represented as clopen up-sets, and
- closed sublocales are represented as clopen down-sets.

Indeed one has

Fact. *Clopen subsets Y of a Priestley space $(X, \tau, \leq)$ (corresponding to a frame) represent precisely the subspaces obtained from finitely many closed and open ones by taking unions and intersections.*

Proof. The system of clopen up-sets and down-sets constitutes a subbasis of the topology τ. Since Y is open, it can be written as $\bigcup_{i \in J}(U_i \cap V_i)$ with U_i clopen up-sets and V_i clopen down-sets. Since Y is closed and hence compact, this cover contains a finite subcover. □

7.3. Here is a problem, somewhat vaguely formulated: Is there a naturally defined class of sublocale sets in the sense of 1.3 that would contain the open and closed sublocales in a similar way as the system of clopen subsets contains the clopen up-sets and down-sets in the Priestley representation?

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