ARCHIVUM MATHEMATICUM (BRNO)
Tomus 42 (2006), 397 – 408

NATURAL WEAK FACTORIZATION SYSTEMS

MARCO GRANDIS* AND WALTER THOLEN†

Dedicated to Jiří Rosický at the occasion of his sixtieth birthday

Abstract. In order to facilitate a natural choice for morphisms created by the (left or right) lifting property as used in the definition of weak factorization systems, the notion of natural weak factorization system in the category $\mathcal{K}$ is introduced, as a pair (comonad, monad) over $\mathcal{K}^{\mathbf{2}}$. The link with existing notions in terms of morphism classes is given via the respective Eilenberg–Moore categories.

1. Introduction

Weak factorization systems $(\mathcal{L}, \mathcal{R})$ play a key role for Quillen model categories, defined in terms of (cofibrations, trivial fibrations) and (trivial cofibrations, fibrations). While the two players $\mathcal{L}$ and $\mathcal{R}$ have good stability properties under *some* colimits and limits, respectively, unlike the counterparts appearing in the orthogonal factorization systems, they fail to be closed under the formation of *all* colimits and limits, coequalizers and equalizers among them. The general reason for that, of course, lies in the fact that morphisms provided by the (right or left) lifting property are not chosen naturally, even in the presence of a functorial realization for the system.

While the notion of lax factorization algebra presented in [8] leads to a natural extension of the presentation of orthogonal factorization systems as Eilenberg–Moore algebras for the "squaring monad" on CAT (see [4], [7], [10]) it does not give a remedy for the defect just described. This paper, therefore, takes a new look at what "functorial weak factorization systems" ought to be and, after a careful recollection of the existing notions, proposes to define such systems in the category $\mathcal{K}$ by a pair (comonad, monad) in $\mathcal{K}^{\mathbf{2}}$, under suitable conditions.

A first step in the passage towards a pair of morphism classes in $\mathcal{K}$ is made by considering the respective Eilenberg–Moore categories. Their (co)algebras are morphisms in $\mathcal{K}$ that come with a (co)algebraic structure, and it is that structure

* Partially supported by MIUR Research Projects.
† Partial financial assistance by NSERC is gratefully acknowledged.

that contains all information to construct “liftings” naturally. Of course, as in all Eilenberg–Moore categories, all (co)limits are now created over $\mathcal{K}^{\mathbf{2}}$. If both the comonad and monad are idempotent, the structures become properties, and the approach takes us back to the traditional presentation of orthogonal factorization systems in terms of two subclasses of mor$\mathcal{K}$.

Those results are presented in Sections 2 and 3, and examples follow in Section 4. These encompass the examples treated in [8]. Furthermore, all cofibrantly generated weak factorization systems in locally finitely-presentable categories are natural, but we must leave a presentation of the rather lengthy and cumbersome proof to a later paper.

2. Natural weak factorization systems

2.1. Morphisms $(u, v)\colon f \to g$ in the category $\mathcal{K}^{\mathbf{2}}$ are commutative squares

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd}$$

of morphisms in the category $\mathcal{K}$. The two projections give the (vertical) domain and the codomain functors dom, cod: $\mathcal{K}^{\mathbf{2}} \to \mathcal{K}$, and there is a natural transformation $\kappa\colon \mathrm{dom} \to \mathrm{cod}$ with $\kappa_f = f$. According to [8], a *functorial realization of a weak factorization system* of $\mathcal{K}$ is given by a functor $F\colon \mathcal{K}^{\mathbf{2}} \to K$ and natural transformations $\lambda\colon \mathrm{dom} \to F$, $\rho\colon F \to \mathrm{cod}$ such that

$$\begin{tikzcd} & F \arrow[dr, "\rho"] & \\ \mathrm{dom} \arrow[ur, "\lambda"] \arrow[rr, "\kappa"] & & \mathrm{cod} \end{tikzcd}$$

commutes and, for all $f \in \mathrm{ob}\mathcal{K}^{\mathbf{2}}$,

$$\lambda_f \in \mathcal{L}_F := \{g \mid \exists s : \lambda_g = s \cdot g,\ \rho_g \cdot s = 1\}\,,$$

$$\rho_f \in \mathcal{R}_F := \{g \mid \exists p : \rho_g = g \cdot p,\ p \cdot \lambda_g = 1\}\,.$$

Now, $(\mathcal{L}_F, \mathcal{R}_F)$ is indeed a weak factorization system (wfs) of $\mathcal{K}$ in the sense of [1], and any wfs $(\mathcal{L}, \mathcal{R})$ that admits a functorial realization (F, λ, ρ) (so that all properties above are satisfied, with $\mathcal{L}_F$, $\mathcal{R}_F$ traded for $\mathcal{L}$, $\mathcal{R}$) necessarily satisfies $\mathcal{L} = \mathcal{L}_F$, $\mathcal{R} = \mathcal{R}_F$ (see Theorem 2.4 of [8]). These data provide for every morphism f a commutative diagram

$$\begin{tikzcd} FLf \arrow[r, "\rho_{Lf}"] & Ff \arrow[r, "\lambda_{Rf}"] \arrow[dr, "\rho_f"] & FRf \arrow[d, "\rho_{RF}"] \\ A \arrow[u, "\lambda_{Lf}"] \arrow[ur, "\lambda_f"'] \arrow[rr, "f"'] & & B \end{tikzcd}$$

where we have written Lf for λ_f (considered as an object of $\mathcal{K}^{\mathbf{2}}$), and Rf for ρ_f, and where ρ_{Lf}, λ_{Rf} have splittings s, p with

$$\lambda_{Lf} = s \cdot \lambda_f,\ \rho_{Lf} \cdot s = 1\,,\ \rho_{Rf} = \rho_f \cdot p,\ p \cdot \lambda_{Rf} = 1\,. \tag{1}$$

Unfortunately, these splittings (that are used for constructing "liftings") need to be chosen each time and add a non–constructive aspect to the notion of functorial realization of wfs.

2.2. The natural transformation λ in 2.1 is equivalently described by a functor $L\colon \mathcal{K}^{\mathbf{2}} \to \mathcal{K}^{\mathbf{2}}$ with

$$\mathrm{dom}L = \mathrm{dom}\,, \qquad \mathrm{cod}L = F\,, \qquad \kappa L = \lambda\,. \tag{2}$$

Now ρ may be described by a natural transformation $\Phi\colon L \to 1_{\mathcal{K}^{\mathbf{2}}}$ with

$$\mathrm{dom}\Phi = 1_{\mathrm{dom}}\,, \qquad \mathrm{cod}\Phi = \rho\,; \tag{3}$$

explicitly, Φ_f is the commutative square

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "\lambda_f"'] & A \arrow[d, "f"] \\ Ff \arrow[r, "\rho_f"] & B \end{tikzcd}$$

Likewise, when we present ρ as a functor $R\colon \mathcal{K}^{\mathbf{2}} \to \mathcal{K}^{\mathbf{2}}$ with

$$\mathrm{dom}R = F\,, \qquad \mathrm{cod}R = \mathrm{cod}\,, \qquad \kappa R = \rho\,, \tag{4}$$

then λ may be presented as a natural transformation $\Lambda\colon 1_{\mathcal{K}^{\mathbf{2}}} \to R$ with

$$\mathrm{dom}\Lambda = \lambda\,, \qquad \mathrm{cod}\Lambda = 1_{\mathrm{cod}}\,; \tag{5}$$

hence, Λ_f is the commutative square

$$\begin{tikzcd} A \arrow[r, "\lambda_f"] \arrow[d, "f"'] & Ff \arrow[d, "\rho_f"] \\ B \arrow[r, "1_B"] & B \end{tikzcd}$$

Now, let us suppose that there is a natural choice for the splittings s, p satisfying (1). Hence, we suppose that there are natural transformations $\sigma\colon F \to FL$, $\pi\colon FR \to F$ with

$$\lambda L = \sigma \cdot \lambda\,, \quad \rho L \cdot \sigma = 1_F\,, \qquad \rho R = p \cdot \pi\,, \quad \pi \cdot \lambda R = 1_F\,. \tag{6}$$

Equivalently, σ and π can be described by natural transformations $\Sigma\colon L \to LL$ and $\Pi\colon RR \to R$ with

$$\begin{array}{lll} \mathrm{dom}\Sigma = 1_{\mathrm{dom}}\,, & \mathrm{cod}\Sigma = \sigma\,, & \Phi L \cdot \Sigma = 1_L\,, \\ \mathrm{cod}\Pi = 1_{\mathrm{cod}}\,, & \mathrm{dom}\Pi = \pi\,, & \Pi \cdot \Lambda R = 1_R\,. \end{array} \tag{7}$$

Explicitly, Σ_f and Π_f are respectively the commutative diagrams

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "\lambda_f"'] & A \arrow[d, "\lambda_{Lf}"] \\ Ff \arrow[r, "\sigma_f"] & FLf \end{tikzcd} \qquad \begin{tikzcd} FRf \arrow[r, "\pi_f"] \arrow[d, "\rho_{Rf}"'] & Fr \arrow[d, "\rho_f"] \\ B \arrow[r, "1_B"] & B \end{tikzcd}$$

It seems natural to assume that (L, Φ, Σ) and (R, Λ, Π) actually form a comonad and a monad on $\mathcal{K}^{\mathbf{2}}$, respectively, so that in addition to (7) one has

$$\begin{aligned} L\Phi \cdot \Sigma &= 1_L\,, \qquad & \Sigma L \cdot \Sigma &= L\Sigma \cdot \Sigma\,, \\ \Pi \cdot \Lambda R &= 1_R\,, \qquad & \Pi \cdot \Pi R &= \Pi \cdot R\Pi\,. \end{aligned} \tag{8}$$

Alternatively, in addition to (6) one requires

$$\begin{aligned} F(1_a, \rho_f) \cdot \sigma_f &= 1_{Ff}\,, \qquad & \sigma_{Lf} \cdot \sigma_f &= F(1_A, \sigma_f) \cdot \sigma_f\,, \\ \pi_f \cdot F(\lambda_f, 1_B) &= 1_{Ff}\,, \qquad & \pi_f \cdot \pi_{Rf} &= \pi_f \cdot F(\sigma_F, 1_B)\,. \end{aligned} \tag{9}$$

This leads to the Definition 2.4 below, for which the following setting is considered.

2.3. Let $\mathsf{CAT}//\mathcal{K}$ be the 2–category whose objects are functors with values in $\mathcal{K}$, whose arrows $F\colon U \to V$ are commutative triangles

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathcal{B} \arrow[dl, "V"] \\ & \mathcal{K} & \end{tikzcd}$$

of functors, and whose 2-cells are natural transformations $\alpha\colon F \to G$ with $V\alpha = 1_U$. A monad (T, η, μ) on U in $\mathsf{CAT}//K$ is simply a monad (T, η, μ) on $\mathcal{A}$ in CAT with $UT = U$, $U\eta = 1_U$, $U\mu = 1_U$. A comonad on U in $\mathsf{CAT}//\mathcal{K}$ is described analogously.

Definition 2.4. A *natural weak factorization system (natural wfs)* in a category $\mathcal{K}$ is a pair $(\mathbb{L}, \mathbb{R})$ such that

(1) $\mathbb{L} = (L, \Phi, \Sigma)$ is a comonad on dom in $\mathsf{CAT}//\mathcal{K}$,

(2) $\mathbb{R} = (R, \Lambda, \Pi)$ is a monad on cod in $\mathsf{CAT}//\mathcal{K}$,

(3) $\mathrm{cod}L = \mathrm{dom}R$, $\mathrm{cod}\Phi = \kappa R$, $\mathrm{dom}\Lambda = \kappa L$.

From our discussion in 2.1, 2.2 one sees immediately:

Proposition 2.5.

(i) *Let us be given a functor* $F\colon \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$ *and natural transformations*

$$\lambda\colon \mathrm{dom} \to F\,, \quad \rho\colon F \to \mathrm{cod}\,, \quad \sigma\colon F \to FL\,, \quad \pi\colon FR \to F\,,$$

(*where* $L, R\colon \mathcal{K}^{\mathbf{2}} \to \mathcal{K}^{\mathbf{2}}$ *respectively represent* λ, ρ *as in* 2.2). *If* $\rho \cdot \lambda = \kappa\colon$ *dom* $\to$ *cod and* (6) *holds, then* (F, λ, ρ) *is a functorial wfs* (*with a natural choice of splittings*).

(ii) *There is a bijection between natural wfs* $(\mathbb{L},\mathbb{R}) = (L,\Phi,\Sigma;R,\Lambda,\Pi)$ *on* $\mathcal{K}$, *as in* 2.4, *and systems* $(F,\lambda,\rho,\sigma,\pi)$ *as in* (i) *which satisfy* $\rho\cdot\lambda=\kappa$ *and the conditions* (6), (9). *Given a natural wfs* $(\mathbb{L},\mathbb{R})$, *one defines the system:*

$$F := \mathrm{cod}L = \mathrm{dom}R\,,\ \ \lambda := \mathrm{dom}\Lambda\,,\ \ \rho := \mathrm{cod}\Phi\,,\ \ \sigma := \mathrm{cod}\Sigma\,,\ \ \pi := \mathrm{dom}\Pi\,.$$

Conversely, given such a system, one defines the associated natural wfs as in 2.2.

2.6. For a natural wfs $(\mathbb{L},\mathbb{R})$, let $\mathcal{L}_{\mathbb{L}}$ and $\mathcal{R}_{\mathbb{R}}$ denote the Eilenberg–Moore categories of $\mathbb{L}$ and $\mathbb{R}$, respectively. Hence, an object in $\mathcal{L}_{\mathbb{L}}$ is a pair $(f,(i,s)\colon f\to Lf)$ such that

$$\begin{tikzcd}
 & f \arrow[r, "{(i,s)}"] \arrow[dl, "1"'] \arrow[d, "{(i,s)}"] & Lf \arrow[d, "\Sigma_f"] \\
f & Lf \arrow[l, "\Phi_f"] \arrow[r, "{L(i,s)}"'] & LLf
\end{tikzcd}$$

commutes in $\mathcal{K}^{\mathbf{2}}$. Since necessarily $i=1$, with 2.5 we can simply write

$$\mathrm{ob}\mathcal{L}_{\mathbb{L}} = \{(f\colon A\to B,\ s\colon B\to Ff)\mid \lambda_f = sf, \rho_f\cdot s = 1_B, \sigma_f\cdot s = F(1_A,s)\cdot s\}\,;$$

a morphism $(u,v)\colon (f,s)\to(g,t)$ in $\mathcal{L}_{\mathbb{L}}$ is a morphism $(u,v)\colon f\to g$ in $\mathcal{K}^{\mathbf{2}}$ which satisfies $t\cdot v = F(u,v)\cdot s$. Hence, objects in $\mathcal{L}_{\mathbb{L}}$ are, in the setting of 2.1, simply morphisms of $\mathcal{L}_F$ (see 2.1) that come with a *given* splitting s which, in addition, must be compatible with the (co)multiplicative structure of $\mathbb{L}$; morphisms of $\mathcal{L}_{\mathbb{L}}$ must respect the given splittings.

Similarly one obtains

$$\mathrm{ob}\mathcal{R}_{\mathbb{R}} = \{(f\colon A\to B,\ p\colon A\to Ff)\mid \rho_f = fp,\ p\lambda_f = 1_A,\ p\pi_f = p\cdot F(p,1_B)\}\,,$$

with morphisms $(u,v)\colon (f,p)\to(g,q)$ in $\mathcal{R}_{\mathbb{R}}$ satisfying $u\cdot p = q\cdot F(u,v)$.

Corollary 2.7. *Let* $(\mathbb{L},\mathbb{R})$ *be a natural wfs of* K. *Then, in the notation of* 2.1 *and* 2.2, *every morphism* $f\colon A\to B$ *factors as*

$$\begin{tikzcd}
 & Ff \arrow[dr, "\rho_f"] & \\
A \arrow[ur, "\lambda_f"] \arrow[rr, "f"] & & B
\end{tikzcd}$$

with $(\lambda_f,\sigma_f)\in\mathcal{L}_{\mathbb{L}}$ *and* $(\rho_f,\pi_f)\in\mathcal{R}_{\mathbb{R}}$. *Furthermore, for all* $(f,s)\in\mathcal{L}_{\mathbb{L}}$ *and* $(g,q)\in\mathcal{R}_{\mathbb{R}}$ *and* $(u,v)\colon f\to g$ *in* $\mathcal{K}^{\mathbf{2}}$, *there is a naturally chosen diagonal morphism* w *as in*

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\
B \arrow[r, "v"'] \arrow[ur, dashed, "w"] & D
\end{tikzcd}$$

namely: $w = q\cdot F(u,v)\cdot s$. *If* $\mathcal{K}$ *has colimits* (*resp. limits*) *of a given type, then* $\mathcal{L}_{\mathbb{L}}$ (*resp.* $\mathcal{R}_{\mathbb{R}}$) *also has them, formed as in* $\mathcal{K}^2$.

Proof. The forgetful functors $\mathcal{L}_{\mathbb{L}}\to\mathcal{K}^{\mathbf{2}}$ and $\mathcal{R}_{\mathbb{R}}\to K^{\mathbf{2}}$ create colimits and limits, respectively. □

3. Orthogonal factorization systems

3.1. Recall that an *orthogonal factorization system* in a category $\mathcal{K}$ may be given by a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms of $\mathcal{K}$, both closed under composition with isomorphisms, such that $\mathcal{K} = \mathcal{R} \cdot \mathcal{L}$, and for all $f \in \mathcal{L}, g \in \mathcal{R}$ and $(u, v)\colon f \to g$ in $K^{\mathbf{2}}$ there is a unique morphism w with $wf = u$ and $gw = v$.

Equivalently, it may be described by a functor $F\colon K^{\mathbf{2}} \to K$ with $FE = 1_K$ and

$$F(1, f) \in \mathcal{L}_F^1 := \{g \mid F(g, 1) \text{ iso}\},$$
$$F(f, 1) \in \mathcal{R}_F^1 := \{g \mid F(1, g) \text{ iso}\}$$

(see Theorem A of [7]); here $E\colon \mathcal{K} \to \mathcal{K}^{\mathbf{2}}$ is the full embedding with $A \mapsto 1_A$, and the morphisms $(1, f)$, $(f, 1)$ stem from the *generic factorization*

$$\begin{tikzcd}
& f \arrow[dr, "{(f,1_B)}"] & \\
1_A \arrow[ur, "{(1_A,f)}"] \arrow[rr, "{Ef=(f,f)}"'] & & 1_B
\end{tikzcd}$$

in $\mathcal{K}^{\mathbf{2}}$, for every $f\colon A \to B$ in $\mathcal{K}$.

Note that such a functor F gives rise to a natural wfs, with $\lambda_f = F(1, f)$, $\rho_f = F(f, 1)$ and σ_f, π_f both isomorphisms. Since orthogonal factorization systems are weak factorization systems ([2], [1]), one has $\mathcal{L}_F^1 = \mathcal{L}_F$ and $\mathcal{R}_F^1 = \mathcal{R}_F$.

Theorem 3.2. *Orthogonal factorization systems of $\mathcal{K}$ are equivalently described as those natural wfs $(\mathbb{L}, \mathbb{R})$ for which $\mathbb{L}$ and $\mathbb{R}$ are idempotent. In this case the Eilenberg–Moore categories $\mathcal{L}_{\mathbb{L}}$ and $\mathcal{R}_{\mathbb{R}}$ are equivalent to $\mathcal{L}_F$ and $\mathcal{R}_F$, considered as full coreflective and reflective subcategories of $\mathcal{K}^{\mathbf{2}}$, with F as in* 2.5.

Proof. It is clear that an orthogonal factorization system gives rise to a natural wfs $(\mathbb{L}, \mathbb{R})$ with $\mathbb{L}$, $\mathbb{R}$ idempotent, see 3.1. Conversely, having such a natural wfs, in the notation on 2.5 one has σ, π iso. In order to be able to apply Theorem A of [7], we just have to show $FE = 1_{\mathcal{K}}$; in fact, it is sufficient to show $FE \cong 1_{\mathcal{K}}$ (see 2.2 of [7]). Hence, we must show that for every object A in $\mathcal{K}$, the morphisms $l = \lambda_{1_A}$ and $r = \rho_{1_A}$ are isos (see 2.6 of [8]).

To this end, one considers the morphism $(l, 1_A)\colon 1_A \to r$ in $\mathcal{K}^{\mathbf{2}}$ and notices that $F(l, 1_A)$ is iso since π_{1_A} is iso (by (9) of 2.2). Now we factor $(l, 1_A)$ as

$$1_A \xrightarrow{(1_A, l)} l \xrightarrow{(l, r)} r\,.$$

By idempotency of $\mathbb{R}$ and $\mathbb{L}$, we may assume $\rho_l = 1_B$ and $\lambda_r = 1_B$ (with $B = F1_A$), so that an application of F to the factorization of $(l, 1_A)$ leads to the following commutative diagram

$$\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "l"'] & A \arrow[r, "l"] \arrow[d, "l"] \arrow[dr, phantom, "\boxed{1}"] & B \arrow[d, "1_B"] \\
B \arrow[r, "F(1_A,l)"] \arrow[d, "r"'] \arrow[dr, phantom, "\boxed{2}"] & B \arrow[r, "F(l,r)"] \arrow[d, "1_B"] & B \arrow[d, "r"] \\
A \arrow[r, "l"'] & B \arrow[r, "r"'] & A
\end{tikzcd}$$

Now one has:

$$\begin{aligned}
l \cdot r &= F(l,r) \cdot l \cdot r && \text{(commutativity of } \boxed{1}\text{)} \\
&= F(l,r) \cdot F(1_A, l) && \text{(commutativity of } \boxed{2}\text{)} \\
&= F(l, 1_A) && \text{(since } r \cdot l = 1_A\text{)}
\end{aligned}$$

Hence, $l \cdot r$ is an isomorphism, and then both l, r must be isos. □

Remarks 3.3. Here are three related open problems.
1. The proof of 3.2 uses idempotency of both players of the natural wfs $(\mathbb{L}, \mathbb{R})$. We do not know whether idempotency of one of them implies idempotency of the other.
2. In a wfs $(\mathcal{L}, \mathcal{R})$, one class determines the other. We do not know if, for a natural wfs $(\mathbb{L}, \mathbb{R})$, $\mathbb{L}$ and $\mathbb{R}$ determine each other.
3. Are there distinct natural wfs inducing the same functorial wfs, that is: do there exist distinct natural wfs $(\mathbb{L}, \mathbb{R})$, $(\mathbb{L}', \mathbb{R}')$ with (in the notation of 2.5) $F = F'$, $\lambda = \lambda'$, $\rho = \rho'$, but $\sigma \neq \sigma'$ or $\pi \neq \pi'$?

4. Examples

4.1. In a category $\mathcal{K}$ with binary products, every map $f\colon X \to Y$ has a well-known *graph-factorization*

$$f = \rho_f \cdot \lambda_f = p_2\langle 1, f\rangle \colon X \to X \times Y \to Y\,, \tag{10}$$

where p_2 is the second projection of the cartesian product.

Dually, in a category $\mathcal{K}$ with binary sums, a map $f\colon X \to Y$ has a *cograph-factorization*

$$f = \rho_f \cdot \lambda_f = [f, 1] i_1 \colon X \to X + Y \to Y\,, \tag{11}$$

where i_1 is the first injection of the sum and $\rho_f = [f,1]\colon X + Y \to Y$ has co-components $\rho_f \cdot i_1 = f$, $\rho_f \cdot i_2 = 1_Y$.

Plainly, both factorizations are functorial. Furthermore, they can be made into natural wfs, by dual procedures: below, we describe the second, which, when $\mathcal{K}$ is lextensive [3], leads to the weak factorization system (coproduct injections, split epimorphisms), recently considered in [9].

For the first, it is well-known that, when $\mathcal{K} = \mathsf{Set}$, $\mathcal{L}_F$ coincides with the class of split monos, which amounts to the injective mappings except the empty embeddings in non-empty sets, while $\mathcal{R}_F$ contains all the surjective mappings and empty inclusions.

Proposition 4.2 (The cograph factorization)**.** *Let $\mathcal{K}$ be a category with binary sums. The cograph factorization of a map, recalled above in* (11)*, can be made into a natural wfs, so that, if $\mathcal{K}$ is lextensive, the maps of $\mathcal{L}_F, \mathcal{R}_F$ can be characterised as coproduct injections and split epimorphisms, respectively.*

Proof. The cograph factorization is functorial, with

$$\begin{aligned} &F\colon \mathcal{K}^{\mathbf{2}} \to \mathcal{K}\,, \\ &F(f\colon X \to Y) = X + Y\,, \qquad F\big((u,v)\colon f \to g\big) = u + v\,, \end{aligned} \tag{12}$$

and the natural transformations λ, ρ defined above, in (11). In order to make it into a natural wfs, let us define the following two natural transformations, related with the factorization of $Lf = i_1$ and $Rf = [f, 1]$ displayed in the diagram below

$$\begin{aligned} &\sigma\colon F \to FL\,, \qquad \sigma_f = [i_1, i_3]\colon X + Y \to X + X + Y\,, \\ &\pi\colon FR \to F\,, \qquad \pi_f = [i_1, i_2, i_2]\colon X + Y + Y \to X + Y\,, \end{aligned} \tag{13}$$

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "i_1"'] & X \arrow[d, "i_1"] \\ X+Y \arrow[r, "\sigma_f"] \arrow[d, "1"'] & X+X+Y \arrow[d, "{[i_1,i_1,i_2]}"] \\ X+Y \arrow[r, equal] & X+Y \end{tikzcd} \qquad \begin{tikzcd} X+Y \arrow[r, equal] \arrow[d, "{[i_1,i_2]}"'] & X+Y \arrow[d, "1"] \\ X+Y+Y \arrow[r, "\pi_f"] \arrow[d, "{[f,1,1]}"'] & X+Y \arrow[d, "{[f,1]}"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

The last axiom (9) is easily verified.

Let now $\mathcal{K}$ be lextensive [3], and let us proceed to characterise the sets $\mathcal{L}_F, \mathcal{R}_F$. In the left diagram below

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[d, equal] & & Y \arrow[d, equal] \arrow[dl, dashed, "s"'] \\ X \arrow[r, "i_1"'] & X+Y \arrow[r, "{[f,1]}"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "g"] \arrow[d, equal] & Y_1 \arrow[d, equal] \arrow[dl, dashed, "s_1"'] \\ X \arrow[r, "g"'] & Y \end{tikzcd} \qquad \begin{tikzcd} & Y_2 \arrow[d, "j_2"] \arrow[dl, dashed, "s_2"'] \\ Y \arrow[r, equal] & Y \end{tikzcd} \tag{14}$$

the morphism s decomposes as a sum $s_1 + s_2\colon Y_1 + Y_2 \to X + Y$, and f is the composition of a map $g\colon X \to Y_1$ with the injection; but this g is an isomorphism, since the previous diagram restricts to the central diagram above; thus, $f\colon X \to Y$ is a coproduct-injection. One easily sees that an $\mathbb{L}$-coalgebra is precisely a pair (f, s) as above, since the last condition, $\sigma_f \cdot s = F(1_X, s) \cdot s$, is automatically satisfied. Moreover, taking into account the right diagram above, s is determined by the injections $f\colon X \to Y$ and $j_2\colon Y_2 \to Y$. Therefore, an $\mathbb{L}$-coalgebra can be equivalently described as a pair of maps $(f\colon X \to Y, f'\colon X' \to Y)$ which are the injections of a sum-decomposition of Y.

Finally, in the diagram

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[d, equal] & & Y \arrow[d, equal] \\
X \arrow[r, "i_1"'] & X+Y \arrow[ul, dashed, "t"'] \arrow[r, "{[f,1]}"'] & Y
\end{tikzcd}
\tag{15}
$$

the map t must be of the form $[1_X, t']$, with $ft' = 1_Y$, whence $\mathcal{R}_F$ coincides with the set of split epis (and this holds in an arbitrary category $\mathcal{K}$). Again, an $\mathbb{R}$-algebra is just such a pair (f, t), which amounts to a splitting $ft' = 1_Y$. □

4.3. We consider now two dual factorizations of a functor, for the category CAT (cf. [6], I, 1.11)

(a) First, we can factor an arbitrary functor $f\colon X \to Y$ through the comma category $Ff = (f \downarrow Y)$, via a *left* adjoint right inverse i and a functor q

$$
\begin{tikzcd}
X \arrow[r, shift left, "i"] & (f \downarrow Y) \arrow[l, shift left, dashed, "p"] \arrow[r, "q"] & Y
\end{tikzcd}
\qquad qi = f \qquad (i \dashv p)\,,
\tag{16}
$$

$$i(x) = (x, fx; 1\colon fx \to fx)\,, \qquad q(x, y; b\colon fx \to y) = y\,, \tag{17}$$

$$
\begin{aligned}
&p(x, y; b\colon fx \to y) = x\,, \quad \eta\colon 1_X = pi\,, \\
&\varepsilon\colon ip \to 1_{Ff}\,, \quad \varepsilon_{(x,y;\,b)} = (1_x, b)\colon (x, fx; 1_{fx}) \to (x, y; b)\,.
\end{aligned}
\tag{18}
$$

(b) Dually, we can also factor an arbitrary functor $g\colon Y \to X$ through the comma category $Gg = (X \downarrow g)$, via a *right* adjoint right inverse j and a functor p

$$
\begin{tikzcd}
Y \arrow[r, shift left, "j"] & (X \downarrow g) \arrow[l, shift left, dashed, "q"] \arrow[r, "p"] & X
\end{tikzcd}
\qquad pj = g \qquad (q \dashv j)\,,
\tag{19}
$$

$$j(y) = (gy, y; 1\colon gy \to gy)\,, \qquad p(x, y; b\colon x \to gy) = x\,, \tag{20}$$

$$
\begin{aligned}
&q(x, y; b\colon x \to gy) = y\,, \quad \varepsilon\colon qj = 1_Y\,, \\
&\eta\colon 1_{Gg} \to jq, \quad \eta_{(x,y;\,b)} = (b, 1_y)\colon (x, y; b) \to (gy, y; 1_{gy})\,.
\end{aligned}
\tag{21}
$$

We prove below that these factorizations can be made into natural wfs. If $f \dashv g$, then we can identify $(f \downarrow Y)$ with $(X \downarrow g)$, and find a factorization of adjunctions; the latter is not functorial on the category of adjunctions, but on a suitable double category of functors and adjunctions (see [5], 3.5) in a sense which will be dealt with in a sequel.

4.4. Let us construct a natural wfs for the factorization 4.3(a), through $(f \downarrow Y)$ (the other can be obtained by duality). With the previous notation, we have a functor

$$
\begin{aligned}
&F\colon \mathsf{CAT}^{\mathbf{2}} \to \mathsf{CAT}\,, \qquad F(f\colon X \to Y) = (f \downarrow Y)\,, \\
&F((u, v)\colon f \to g)\colon Ff \to Fg\,, \\
&F(u, v)(x, y; b\colon fx \to y) = (ux, vy; vb\colon vfx = gux \to vy)\,,
\end{aligned}
\tag{22}
$$

and a functorial factorization, defined by the natural transformations:

$$
\begin{aligned}
&\lambda\colon \mathrm{dom} \to F\,, && \lambda_f = i\colon X \to (f \downarrow Y)\,,\\
&\rho\colon F \to \mathrm{cod}\,, && \rho_f = q\colon (f \downarrow Y) \to Y\,,\\
&\rho\lambda = \kappa\colon \mathrm{dom} \to \mathrm{cod}\,.
\end{aligned} \tag{23}
$$

First, the functor $Lf = i\colon X \to (f \downarrow Y)$ factors as follows through the comma category $FLf = (Lf \downarrow (f \downarrow Y))$, whose general object is of type $(x', x, y; a\colon x' \to x; b\colon fx \to y)$

$$
\begin{aligned}
&Lf = (RLf)\cdot(LLf)\colon X \to (Lf \downarrow (f \downarrow Y)) \to (f \downarrow Y)\,,\\
&LLf(x) = (x, x, fx; 1_x, 1_{fx})\,,\\
&RLf(x', x, y; a\colon x' \to x; b\colon fx \to y) = (x, y; b\colon fx \to y)\,.
\end{aligned} \tag{24}
$$

The natural transformation σ, related with the previous factorization of Lf, is defined as:

$$
\begin{aligned}
&\sigma\colon F \to FL\,, \qquad \sigma_f\colon (f \downarrow Y) \to \big(Lf \downarrow (f \downarrow Y)\big)\,,\\
&\sigma_f(x, y; b\colon fx \to y) = (x, x, y; 1_x, b\colon fx \to y)\,,
\end{aligned} \tag{25}
$$

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "Lf=i"'] & X \arrow[d, "LLf"] \\
(f \downarrow Y) \arrow[r, "\sigma_f"] \arrow[d, "1"'] & (Lf \downarrow (f \downarrow Y)) \arrow[d, "RLf"] \\
(f \downarrow Y) \arrow[r, equal] & (f \downarrow Y)
\end{tikzcd}
$$

Second, we factor $Rf = q\colon (f \downarrow Y) \to Y$ through $FRf = (q \downarrow Y)$, whose general object is of type $(x, y, b\colon fx \to y; y'; b'\colon y \to y')$,

$$
\begin{aligned}
&q = Rf = (RRf)\cdot(LRf)\colon (f \downarrow Y) \to (q \downarrow Y) \to Y\,,\\
&(LRf)(x, y; b\colon fx \to y) = (x, y, b\colon fx \to y; y, 1\colon y \to y)\,,\\
&(RRf) = Rq\colon (q \downarrow Y) \to Y\,,\\
&(RRf)(x, y, b\colon fx \to y; y', b'\colon y \to y') = y'\,,
\end{aligned} \tag{26}
$$

and we define the natural transformation π, related with the previous factorization of Rf

$$
\begin{aligned}
&\pi\colon FR \to F\,, \qquad \pi_f\colon (q \downarrow Y) \to (f \downarrow Y)\,,\\
&(\pi_f)(x, y, b\colon fx \to y; y', b'\colon y \to y') = (x, y'; b'b\colon fx \to y')\,,
\end{aligned} \tag{27}
$$

$$
\begin{tikzcd}
(f \downarrow Y) \arrow[r, equal] \arrow[d, "LRf"'] & (f \downarrow Y) \arrow[d, "1"] \\
(q \downarrow Y) \arrow[r, "\pi_f"] \arrow[d, "RRf"'] & (f \downarrow Y) \arrow[d, "q"] \\
Y \arrow[r, equal] & Y
\end{tikzcd}
$$

Verifying the remaining axioms is straightforward, if long.

Proposition 4.5. *This structure defines a natural wfs.*

Proof. We will use various comma categories, among which

$$C = (f \downarrow Y)\,, \qquad C' = (g \downarrow Y')\,,$$
$$C'' = (Lf \downarrow C)\,, \qquad C''' = (LLf \downarrow C'')\,, \tag{28}$$

writing their projections as follows

$$\begin{array}{lll} p\colon C \to X\,, & q\colon C \to Y\,, & \omega\colon C \to Y^{\mathbf{2}}\,, \\ p'\colon C' \to X'\,, & q'\colon C' \to Y'\,, & \omega'\colon C' \to Y'^{\mathbf{2}}\,, \\ p''\colon C'' \to X''\,, & q''\colon C'' \to C\,, & \omega''\colon C'' \to C^{\mathbf{2}}\,, \\ P\colon C''' \to X\,, & Q\colon C''' \to C''\,, & \Omega\colon C''' \to C''^{\mathbf{2}}\,. \end{array} \tag{29}$$

Computing the components of the transformations introduced above (in 4.4) quickly becomes heavy and confusing. Therefore, let us note that the functor F defined above is determined as follows by the projections of $C' = (g \downarrow Y')$:

$$\begin{array}{ll} p' \cdot F(u,v) = up\colon (f \downarrow Y) \to X'\,, & q' \cdot F(u,v) = vq\colon (f \downarrow Y) \to Y'\,, \\ \omega' \cdot F(u,v) = v\omega\colon vfp \to vq & (vfp = gup)\,. \end{array} \tag{30}$$

Again, the natural transformation σ_f is determined by the projections of $C'' = \big(Lf \downarrow (f \downarrow Y)\big)$:

$$\begin{array}{ll} p'' \cdot \sigma_f = p\colon (f \downarrow Y) \to X\,, & q'' \cdot \sigma_f = 1\colon (f \downarrow Y) \to (f \downarrow Y)\,, \\ \multicolumn{2}{l}{(\omega'' \cdot \sigma_f)(x, y; b\colon fx \to y) = (1_x, b)\colon (x, fx; 1) \to (x, y; b)\,,} \end{array} \tag{31}$$

and also the last equation can be made free of components, rewriting it as:

$$p \cdot \omega''\sigma_f = 1_p\,, \qquad q \cdot \omega''\sigma_f = \omega\colon fp \to q\,, \tag{32}$$

(which amounts to using the 2-dimensional universal property of the comma $(f \downarrow Y)$).

Now, to test the condition $F(1_X, \rho_f) \cdot \sigma_f = 1_{Ff}$ we use the projections p, q, ω of $(f \downarrow Y)$, together with the characterisation (30) of the functors of type $F(u,v)$

$$\begin{aligned} p \cdot F(1_X, \rho_f)\sigma_f &= p''\sigma_f = p = p \cdot 1_{Ff}\,, \\ q \cdot F(1_X, \rho_f)\sigma_f &= \rho_f q''\sigma_f = q = q \cdot 1_{Ff}\,, \\ \omega \cdot F(1_X, \rho_f)\sigma_f &= \rho_f \omega''\sigma_f = q\omega''\sigma_f = \omega = \omega \cdot 1_{Ff}\,. \end{aligned}$$

Similarly, to verify that $\sigma_{Lf} \cdot \sigma_f = F(1_X, \sigma_f) \cdot \sigma_f\colon Ff \to FLLf$, we use the projections P, Q, Ω of $C''' = FLLf = \big(LLf \downarrow (Lf \downarrow (f \downarrow Y))\big)$, further replacing Ω with its projections $p''\Omega, q''\Omega$ (as in (32))

$$\begin{aligned} &P \cdot F(1_X, \sigma_f)\sigma_f = p''\sigma_f = P \cdot \sigma_{Lf}\sigma_f\,, \\ &Q \cdot F(1_X, \sigma_f)\sigma_f = \sigma_f q''\sigma_f = \sigma_f = Q \cdot \sigma_{Lf}\sigma_f\,, \\ &p''\Omega \cdot F(1_X, \sigma_f)\sigma_f = p''\sigma_f\omega''\sigma_f = p\omega''\sigma_f = 1_p = 1_{p''}\sigma_f = p''\Omega \cdot \sigma_{Lf}\sigma_f\,, \\ &q''\Omega \cdot F(1_X, \sigma_f)\sigma_f = q''\sigma_f\omega''\sigma_f = \omega''\sigma_f = q''\Omega \cdot \sigma_{Lf}\sigma_f\,. \end{aligned}$$

We end with verifying the remaining two conditions of (9) on π. For the first:

$$\pi_f \cdot (F(\lambda_f, 1_Y)(x,y;b) = (\pi_f)\big(\lambda_f(x), y, b\big) = (\pi_f)(x, fx, 1_{fx}, y, b)$$
$$= (x,y;b)\,.$$

For the second, after computing

$$\pi_q \colon (Rq \downarrow Y) \to (q \downarrow Y)\,,$$
$$(\pi_q)(x,y,b\colon fx \to y; y', b'\colon y \to y'; y'', b''\colon y' \to y'')$$
$$= (x,y,b\colon fx \to y; y'', b''b'\colon y \to y'')\,,$$

we have (working on components):

$$(\pi_f \cdot \pi_q)(x,y,b;y',b';y'',b'') = (\pi_f)(x,y,b;y'',b''b')$$
$$= (x, y''; b''b'b\colon fx \to y'')\,,$$
$$\pi_f \cdot F(\pi_f, 1_Y)(x,y,b;y',b';y'',b'') = (\pi_f)\big(\pi_f(x,y,b;y',b'), y'', b''\big)$$
$$= (\pi_f)(x, y'; b'b, y'', b'') = (x, y''; b''b'b\colon fx \to y'')\,. \qquad \square$$

References

[1] Adámek, J., Herrlich, H., Rosický, J. and Tholen, W., *Weak factorization systems and topological functors*, Appl. Categorical Structures **10** (2002), 237–249.

[2] Adámek, J., Herrlich, H., Strecker, G. E., *Abstract and Concrete Categories*, Wiley (New York 1990).

[3] Carboni, A. and Janelidze, G., *Decidable (= separable) objects and morphisms in lextensive categories*, J. Pure Appl. Algebra **110** (1996), 219–240.

[4] Coppey, L., *Algèbres de decompositions et précatégories*, Diagrammes 4 (Suppl.) (1980).

[5] Grandis, M. and Paré, R., *Limits in double categories*, Cah. Topol. Géom. Différ. Catég. **40** (1999), 162–220.

[6] Gray, J. W., *Formal category theory: adjointness for 2-categories*, Lecture Notes in Math. Vol. 391, Springer-Verlag (Berlin 1974).

[7] Korostenski, M., Tholen, W., *Factorization systems as Eilenberg–Moore algebras*, J. Pure Appl. Algebra **85** (1993), 57–72.

[8] Rosický, J., Tholen, W., *Lax factorization algebras*, J. Pure Appl. Algebra **175** (2002), 355–382.

[9] Rosický, J., Tholen, W., *Factorization, fibration and torsion*, preprint (York University 2006).

[10] Rosebrugh, R. and Wood, R. J., *Coherence for factorization algebras*, Theory Appl. Categories **10** (2002), 134–147.

Dipartimento di Matematica, Università di Genova
Via Dodecaneso 35, 16146-Genova, Italy
E-mail: grandis@dima.unige.it

Department of Mathematics and Statistics, York University
Toronto ON, M3J 1P3, Canada
E-mail: tholen@mathstat.yorku.ca