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ON LEFT INVARIANT CR STRUCTURES ON $SU(2)$

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Abstract. There is a well known one–parameter family of left invariant CR structures on $SU(2) \cong S^3$. We show how purely algebraic methods can be used to explicitly compute the canonical Cartan connections associated to these structures and their curvatures. We also obtain explicit descriptions of tractor bundles and tractor connections.

1. Introduction

Three dimensional CR structures are among the examples of geometric structures for which Elie Cartan constructed an associated normal Cartan connection, see [1]. The homogeneous model for this geometry is S^3, viewed as a quotient of the semisimple group $G := PSU(2,1)$ by a parabolic subgroup P, so three dimensional CR structures form an example of a *parabolic geometry*.

This example is remarkable in many respects. On the one hand, it is sufficiently complicated to incorporate many of the features of general parabolic geometries. On the other hand, the low dimension of the group G and the fact that all important natural bundles are (either real or complex) line bundles, and hence all sections can locally be viewed as functions, simplify matters considerably. In fact, Cartan was even able to describe an algorithm for computing the essential curvature invariant of such structures. Moreover, many questions that have to be attacked using representation theory for general parabolic geometries can be easily solved directly in this case. An example for this is provided by the analysis of possible dimensions of automorphism groups in [2].

Returning to the homogeneous model $S^3 = G/P$, consider the compact subgroup $K = SU(2) \subset G$. Acting with elements of K induces a diffeomorphism $K \cong S^3$, so we can actually view the standard CR structure on S^3 as a left invariant structure on K. In this picture, the structure can be easily obtained from data on the Lie algebra $\mathfrak{k}$ of K. These data admit an evident one–parameter deformation, which gives rise to a one parameter family of left invariant CR structures on

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K. The aim of this article is to show that the canonical Cartan connections associated to these CR structures can be computed using only linear algebra. On the way, one also gets explicit formulae for their curvatures. Finally, we also describe all tractor bundles and normal tractor connections explicitly. These developments should also serve as a basis for a more detailed analysis of these structures and as a prototype for dealing with general left invariant parabolic geometries.

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2. Left invariant CR structures on $SU(2)$

2.1. **3–dimensional CR structures.** Recall that a CR structure on a 3–manifold M is given by a complex line subbundle $H \subset TM$, which defines a contact structure on M. The subbundle $H \subset TM$ is called the *CR subbundle*. Equivalently, we have a rank two subbundle $H \subset TM$ endowed with a complex structure $J : H \to H$ such that for one (or equivalently any) locally non vanishing section $\xi \in \Gamma(H)$ the vector fields ξ, $J(\xi)$ and $[\xi, J(\xi)]$ form a local frame for TM. In contrast to higher dimensions, there is no condition of partial integrability or integrability in dimension 3.

Given two CR structures, there is an evident notion of a (local) CR diffeomorphism. This is a (local) diffeomorphism f, such that for each point x the tangent map T_xf maps the CR subbundle to the CR subbundle and the restriction of T_xf to the CR subbundle is complex linear.

The basic examples for such structures are provided by generic real hypersurfaces in two dimensional complex manifolds. If $(\tilde{M}, \tilde{J})$ is a two dimensional complex manifold and $M \subset \tilde{M}$ is a real hypersurface, then for each $x \in M$ the tangent space T_xM has real dimension 3 and sits in $T_x\tilde{M}$, which is a two dimensional complex vector space. Now $H_x := T_xM \cap \tilde{J}(T_xM)$ is a complex subspace of $T_x\tilde{M}$, which evidently must have complex dimension one. By construction, the spaces H_x fit together to define a complex line subbundle $H \subset TM$, with the complex structure J given by the restriction of $\tilde{J}$. Generically, the subbundle $H \subset TM$ is maximally non–integrable, and hence defines a CR structure on M. From the construction it is clear that a biholomorphism $f : \tilde{M} \to \tilde{M}$ which preserves the hypersurface M restricts to a CR automorphism of M.

The simplest example of this situation is provided by the unit sphere $S^3 \subset \mathbb{C}^2$. For $x \in S^3$ we get $T_xS^3 = \{y \in \mathbb{C}^2 : \mathrm{Re}(\langle x, y\rangle) = 0\}$. The maximal complex subspace of this is $H_x = \{y \in \mathbb{C}^2 : \langle x, y\rangle = 0\}$. One easily verifies directly that this defines a contact structure on S^3. Hence we have obtained a CR structure on S^3, called the *standard structure*. CR structures which are locally isomorphic to the standard structure on S^3 are called *spherical*.

Any element $A \in U(2)$ defines a biholomorphism of $\mathbb{C}^2$ which preserves the unit sphere S^3, and hence restricts to a CR automorphism of the standard CR structure on S^3. Of course, this action is transitive, so we see that S^3 with its standard structure is a homogeneous CR manifold. The group of CR automorphisms of this structure however is larger than $U(2)$. Identifying S^3 with the space those complex lines in $\mathbb{C}^3$ which are isotropic for a Hermitian inner product of signature $(2,1)$ leads to an faithful action of $G := PSU(2,1)$ on S^3 by CR automorphisms. Correspondingly, one obtains a diffeomorphism $S^3 \cong G/P$, where $P \subset G$ is the stabilizer of an isotropic line in $\mathbb{C}^3$.

2.2. **Left invariant deformations of the standard structure.** Restricting the action of $U(2)$ on S^3 further to $K := SU(2)$ we obtain a diffeomorphism $K \to S^3$, which we can use to carry over the standard CR structure to K. In this picture, multiplication from the left by any element of K is a CR automorphism, so we have constructed a left invariant CR structure on K.

It is well known that left invariant structures on a Lie group can be described in terms of the Lie algebra. Denoting by $e \in K$ the unit element and by $\mathfrak{k} = T_eK$ the Lie algebra of K, we get the fiber $H_e \subset \mathfrak{k}$ of the subbundle. This must be a complex subspace of complex dimension 1 in the real vector space $\mathfrak{k}$. By left invariance, the fiber H_g in each point $g \in K$ is spanned by the values $L_X(g)$ of the left invariant vector fields generated by elements $X \in H_e$, and the complex structure on H_g comes from the linear isomorphism $X \mapsto L_X(g)$. Explicitly, $\mathfrak{k}$ consists of all trace–free skew Hermitian 2×2–matrices, i.e.

$$\mathfrak{k} = \left\{ \begin{pmatrix} it & -\overline{z} \\ z & -it \end{pmatrix} : t \in \mathbb{R}, z \in \mathbb{C} \right\} ,$$

and we will denote elements of $\mathfrak{k}$ as pairs (it, z). Using the action on the first vector in the standard basis of $\mathbb{C}^2$ to identify K with S^3, we see that $H_e = \{(0,z) : z \in \mathbb{C}\} \subset \mathfrak{k}$. The fact that this defines a left invariant contact structure on K is then immediate from the fact that $[L_X, L_Y] = L_{[X,Y]}$ for all $X, Y \in \mathfrak{k}$ and from $\big[(0,1),(0,i)\big] = (-2i, 0)$. Indeed, the linear functional $\alpha : \mathfrak{k} \to \mathbb{R}$ defined by $\alpha(it, z) = t$ defines a left invariant contact form for the contact structure H.

Now the crucial idea is that we can leave this left invariant contact structure unchanged but deform the complex structure in the space H_e to obtain a family (H, J_λ) of left invariant CR structures on K parametrized by a positive real number λ. Namely, for $\lambda > 0$ we define $J_\lambda(e)(0, u+iv) := \big(0, i(\lambda u + i\frac{1}{\lambda}v)\big) = (0, -\frac{1}{\lambda}v + i\lambda u)$. This extends to a left invariant complex structure on the contact subbundle $H \subset TK$, which in addition induces the standard orientation. The obvious question is whether this is a true deformation of the standard CR structure, or whether one just obtains (locally) isomorphic structures.

Notice that, viewed as CR structures on S^3, the structures (H, J_λ) for $\lambda \neq 1$ are not invariant under the group $U(2)$. Indeed the element $\left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right) \in U(2)$ fixes the first vector in the standard basis. The tangent map of its action is given by $(it, z) \mapsto (it, iz)$, which is complex linear for $J_\lambda(e)$ if and only if $\lambda = 1$. Invariance under $U(2)$ would actually imply that the structure is spherical, since by a classical result of Cartan, the automorphism group of a non–spherical CR structure has dimension at most three. A simple proof of this result can be found in [2].

3. The canonical Cartan connections

3.1. **Three dimensional CR structures and Cartan geometries.** Three dimensional CR structures can be equivalently described as Cartan geometries, which in particular implies that the curvature gives a complete obstruction to being spherical. We first have to describe the group $G = PSU(2,1)$ and its Lie algebra $\mathfrak{g} = \mathfrak{su}(2,1)$ in a bit more detail. Consider the Hermitian form on $\mathbb{C}^3$ defined by

$$\big((z_0,z_1,z_2),(w_0,w_1,w_2)\big) \mapsto z_0\overline{w_2} + z_2\overline{w_0} + z_1\overline{w_1}\,.$$

Then the first and last vector in the standard basis are isotropic, while the second one has positive length, so this form has signature $(2,1)$. A direct computation shows that for this form we get

$$\mathfrak{g} = \left\{ \begin{pmatrix} \alpha+i\beta & w & i\psi \\ x & -2i\beta & -\overline{w} \\ i\varphi & -\overline{x} & -\alpha+i\beta \end{pmatrix} : \alpha,\beta,\varphi,\psi \in \mathbb{R}, x,w \in \mathbb{C} \right\}$$

We obtain a grading $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ of $\mathfrak{g}$ by

$$\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{pmatrix}.$$

The associated filtration is defined by $\mathfrak{g}^i = \mathfrak{g}_i \oplus \dots \oplus \mathfrak{g}_2$, so we have

$$\mathfrak{g} = \mathfrak{g}^{-2} \supset \mathfrak{g}^{-1} \supset \dots \supset \mathfrak{g}^{2}\,,$$

and $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$. The parabolic subgroup $P \subset G$ is the stabilizer of an isotropic line, for which we take the line generated by the first basis vector. The Lie algebra of P then evidently is given by $\mathfrak{p} = \mathfrak{g}^0 = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. In particular the filtration $\{\mathfrak{g}^i\}$ is invariant under the adjoint actions of $\mathfrak{p}$ and P.

Definition. (1) A *Cartan geometry* of type (G,P) on a smooth manifold M is a principal P–bundle $p : \mathcal{G} \to M$ together with a one form $\omega \in \Omega^1(\mathcal{G},\mathfrak{g})$ such that

- $(r^g)^*\omega = \mathrm{Ad}(g)^{-1} \circ \omega$ for all $g \in P$, where r^g denotes the principal right action of g.
- $\omega(\zeta_A) = A$ for all $A \in \mathfrak{p}$, where ζ_A denotes the fundamental vector field with generator A.
- $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$.

(2) A *morphism* between two Cartan geometries $(\mathcal{G} \to M, \omega)$ and $(\tilde{\mathcal{G}} \to \tilde{M}, \tilde{\omega})$ is a principal bundle homomorphism $\Phi : \mathcal{G} \to \tilde{\mathcal{G}}$ such that $\Phi^*\tilde{\omega} = \omega$. Note that since both ω and $\tilde{\omega}$ are bijective on each tangent space, this implies that Φ is a local diffeomorphism.

(3) The *homogeneous model* of the geometry is the principal bundle $G \to G/P$ together with the left Maurer–Cartan form ω^{MC}.

Given a Cartan geometry $(p : \mathcal{G} \to M, \omega)$ of type (G,P) on M, we can form the associated bundle $\mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$. The map $\mathcal{G} \times (\mathfrak{g}/\mathfrak{p}) \to TM$ given by $(u,X) \mapsto T_up \cdot \big(\omega(u)^{-1}(X)\big)$ descends to an isomorphism $\mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p}) \cong TM$. Now $\mathfrak{g}/\mathfrak{p}$ contains the P–invariant subspace $\mathfrak{g}^{-1}/\mathfrak{p}$, so this gives rise to a subbundle $H \subset TM$. Moreover,

$\mathfrak{g}^{-1}/\mathfrak{p} \cong \mathbb{C}$ and since P consists of complex matrices, this complex structure is invariant under the adjoint action of P. Therefore, it makes the associated bundle $H = \mathcal{G} \times_P (\mathfrak{g}^{-1}/\mathfrak{p})$ into a complex line bundle. If H is a contact structure, then we obtain a three dimensional CR structure on M.

3.2. **Regularity and normality.** To characterize when H is a contact structure, we need the curvature $\kappa \in \Omega^2(\mathcal{G}, \mathfrak{g})$ of ω. This is defined by $\kappa(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$. In the case of the homogeneous model, κ vanishes identically by the Maurer–Cartan equation. Conversely, it can be shown that any Cartan geometry with vanishing curvature is locally isomorphic to the homogeneous model. Now we call a Cartan geometry of type (G, P) *regular* if and only if $\kappa(\xi, \eta)$ has values in $\mathfrak{g}^{-1} \subset \mathfrak{g}$ provided that $\omega(\xi)$ and $\omega(\eta)$ have values in $\mathfrak{g}^{-1}$.

Suppose that this condition is satisfied and that ξ and η are local lifts of vector fields $\underline{\xi}$ and $\underline{\eta}$ on M. Then the fact that $\omega(\xi)$ and $\omega(\eta)$ have values in $\mathfrak{g}^{-1}$ exactly means that $\underline{\xi}$ and $\underline{\eta}$ are sections of $H \subset TM$. By definition of the curvature and the assumptions we see that $-\omega\big([\xi, \eta]\big) + \big[\omega(\xi), \omega(\eta)\big]$ has values in $\mathfrak{g}^{-1}$. Since $[\xi, \eta]$ lifts $[\underline{\xi}, \underline{\eta}]$, we see that the latter bracket cannot have values in H unless $\big[\omega(\xi), \omega(\eta)\big]$ has values in $\mathfrak{g}^{-1}$. One immediately verifies that the bracket in $\mathfrak{g}$ induces a non–degenerate map $\mathfrak{g}^{-1}/\mathfrak{p} \times \mathfrak{g}^{-1}/\mathfrak{p} \to \mathfrak{g}/\mathfrak{g}^{-1}$. Hence we see that regularity of the Cartan geometry ensures that we obtain an underlying CR structure.

It is a general theorem, that any three dimensional CR structure arises as the underlying structure of a Cartan geometry of type (G, P). However, there are many non–isomorphic Cartan geometries having the same underlying CR structure. To get rid of this freedom, one has to put an additional normalization condition on (the curvature of) the Cartan connection ω. Under this additional condition, the Cartan geometry is then uniquely determined up to isomorphism. See [3] for a discussion of all these issues and [6] for proofs, both in the realm of general parabolic geometries.

We will not need the detailed form of the normalization condition, but only some of its consequences. These follow from the fact that one may relate the values of the curvature of a regular normal Cartan geometry to certain explicitly computable Lie algebra cohomology groups. In the case of three dimensional CR structures, these conditions imply that $\kappa(\xi, \eta)$ has values in $\mathfrak{g}^1 \subset \mathfrak{g}$ for all ξ and η. Moreover, if both $\omega(\xi)$ and $\omega(\eta)$ have values in $\mathfrak{g}^{-1}$, then $\kappa(\xi, \eta)$ has to vanish identically. Finally, projecting the values of κ to $\mathfrak{g}^1/\mathfrak{g}^2 \cong \mathfrak{g}_1$, one obtains the *harmonic curvature*, which still is a complete obstruction to the CR structure being spherical.

3.3. **The case of left invariant structures.** Let us now consider one of the left invariant CR structures (H, J_λ) on $K = SU(2)$. As an ansatz, we use the trivial principal P–bundle $\mathcal{G} := K \times P$. For $X \in \mathfrak{k}$, define $\hat{L}_X := (L_X, 0) \in \mathfrak{X}(\mathcal{G})$. The second part of the ansatz is that $\omega(\hat{L}_X)$ is constant along $K \times \{e\}$. The motivation for this ansatz is as follows. For any $k \in K$, the left translation by k defines a CR automorphism of K, which leaves each L_X invariant. These automorphisms lift to the canonical principal bundle in a way compatible with the canonical Cartan connection. Fixing an identification of the fiber of the Cartan bundle over $e \in K$ with P, we can use these lifts to trivialize the Cartan bundle and in such a way that $\omega(\hat{L}_X)$ is constant along $K \times \{e\}$.

Consider a linear map $\varphi : \mathfrak{k} \to \mathfrak{g}$ such that the composition with the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{p}$ with φ is a linear isomorphism. Any tangent vector in $(k, g) \in K \times P$ can be uniquely written as $\big(L_X(k), L_A(g)\big)$ for some $X \in \mathfrak{k}$ and $A \in \mathfrak{p}$. Hence we can define $\omega \in \Omega^1(K \times P, \mathfrak{g})$ by

$$\omega\big(L_X(k), L_A(g)\big) := \mathrm{Ad}(g^{-1})\big(\varphi(X)\big) + A\,.$$

By the assumption on φ this defines a linear isomorphism on each tangent space, and using that the principal right action is just multiplication from the right in the second factor and that $\zeta_A = (0, L_A)$ for each $A \in \mathfrak{p}$, one immediately verifies that this defines a Cartan connection.

We can also immediately compute the curvature κ of this connection. Since κ is horizontal and P–equivariant, it suffices to compute $\kappa(\hat{L}_X, \hat{L}_Y)(k, e)$ for all $X, Y \in \mathfrak{k}$ and $k \in K$. Now by definition,

$$\kappa(\hat{L}_X, \hat{L}_Y) = d\omega(\hat{L}_X, \hat{L}_Y) + \big[\omega(\hat{L}_X), \omega(\hat{L}_Y)\big] = -\omega\big([\hat{L}_X, \hat{L}_Y]\big) + \big[\omega(\hat{L}_X), \omega(\hat{L}_Y)\big]\,.$$

Using $[L_X, L_Y] = L_{[X,Y]}$ we see that, along $K \times \{e\}$, the function $\kappa(\hat{L}_X, \hat{L}_Y)$ is constant and equal to

$$\big[\varphi(X), \varphi(Y)\big] - \varphi\big([X, Y]\big)\,.$$

Hence the curvature exactly expresses the obstruction against φ being a homomorphism of Lie algebras.

It remains to express the fact that the Cartan connection ω induces the "right" underlying CR structure in terms of the linear map φ. Returning to the notation of 2.2, we denote elements $X \in \mathfrak{k}$ as pairs (it, z) for $t \in \mathbb{R}$ and $z \in \mathbb{C}$. Then $L_{(it,z)}$ lies in the contact subbundle H if and only if $t = 0$, so to get the right contact subbundle, $\varphi(it, z)$ must lie in the subspace $\mathfrak{g}^{-1} \subset \mathfrak{g}$ if and only if $t = 0$. Given this, we get an induced linear map $\mathfrak{k} \supset H_e \to \mathfrak{g}^{-1}$. Composing with the natural projection, we get a linear isomorphism $\mathfrak{k} \to \mathfrak{g}^{-1}/\mathfrak{p} \cong \mathbb{C}$. The condition that we get the induced complex structure J_λ exactly means that via this isomorphism the (fixed) standard complex structure on $\mathfrak{g}^{-1}/\mathfrak{p}$ induces the complex structure $J_\lambda(e)$ on H_e.

Having all this at hand, we can prove the main technical result of this article:

Theorem. *For fixed $\lambda > 0$, the linear map $\varphi_\lambda : \mathfrak{k} \to \mathfrak{g}$ defined by*

$$\varphi_\lambda(it, u + iv) := \begin{pmatrix} \frac{1+\lambda^2}{4\lambda} it & -\frac{5-3\lambda^2}{4\sqrt{\lambda}} u - \frac{3-5\lambda^2}{4\lambda\sqrt{\lambda}} iv & \frac{-15+34\lambda^2-15\lambda^4}{16\lambda^2} it \\ \sqrt{\lambda} u + \frac{1}{\sqrt{\lambda}} iv & -\frac{1+\lambda^2}{2\lambda} it & \frac{5-3\lambda^2}{4\sqrt{\lambda}} u - \frac{3-5\lambda^2}{4\lambda\sqrt{\lambda}} iv \\ it & -\sqrt{\lambda} u + \frac{1}{\sqrt{\lambda}} iv & \frac{1+\lambda^2}{4\lambda} it \end{pmatrix}$$

induces a linear isomorphism $\mathfrak{k} \to \mathfrak{g}/\mathfrak{p}$. It has the property that $\varphi_\lambda(H_e) \subset \mathfrak{g}^{-1}$, and via the induced isomorphism $H_e \to \mathfrak{g}^{-1}/\mathfrak{p}$ the induced complex structure on H_e is $J_\lambda(e)$. Finally, the map $\kappa_\lambda : \mathfrak{k} \times \mathfrak{k} \to \mathfrak{g}$ defined by

$$\kappa_\lambda(X, Y) := [\varphi_\lambda(X), \varphi_\lambda(Y)] - \varphi_\lambda([X, Y])$$

has values in $\mathfrak{g}^1$ *and vanishes on* $H_e \times H_e$.

Explicitly, κ_λ *satisfies*

$$\kappa_\lambda\big((it,0),(0,u+iv)\big) = \begin{pmatrix} 0 & -\frac{3t(\lambda^4-1)}{2\lambda^2\sqrt{\lambda}}(v-i\lambda u) & 0 \\ 0 & 0 & \frac{3t(\lambda^4-1)}{2\lambda^2\sqrt{\lambda}}(v+i\lambda u) \\ 0 & 0 & 0 \end{pmatrix}$$

and this completely determines κ_λ.

Proof. From the definition of φ_λ it is evident that it induces a linear isomorphism $\mathfrak{k} \to \mathfrak{g}/\mathfrak{p}$ and that it maps elements of H_e, which are characterized by $it = 0$ to $\mathfrak{g}^{-1}$. Then the isomorphism $H_e \to \mathfrak{g}^{-1}/\mathfrak{p}$ is given by $u + iv \mapsto \sqrt{\lambda}u + i\frac{1}{\sqrt{\lambda}}v$, so the complex structure on $\mathfrak{g}^{-1}/\mathfrak{p}$ evidently induces $J_\lambda(e)$ on H_e. It is then straightforward but tedious to check that κ_λ has values in $\mathfrak{g}^1$ and vanishes on $H_e \times H_e$ as well as the explicit formula. That this expression determines κ_λ follows since by skew symmetry and vanishing of κ_λ on $H_e \times H_e$ we obtain

$$\kappa_\lambda\big((it,z),(it',z')\big) = \kappa_\lambda\big((it,0),(0,z')\big) - \kappa_\lambda\big((it',0),(0,z)\big). \qquad \square$$

3.4. **Digression: How to get the formula for φ_λ.** The result of Theorem 3.3 is all that is needed in the sequel. Since the proof does not explain how the formula for φ_λ was obtained (although really doing the computation gives some hints), we will briefly discuss this. As a spin off, this will show that φ_λ is essentially uniquely determined by the four properties listed in Theorem 3.3. The main point is that there is some evident non–uniqueness around, and dealing with this is the key step to determine φ_λ. Recall that for any element $g \in P$, the adjoint action $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ preserves the filtration. In particular, it preserves $\mathfrak{g}^1$ and $\mathfrak{g}^{-1}$ as well as $\mathfrak{p}$ and therefore induces a linear isomorphisms on $\mathfrak{g}/\mathfrak{p}$ and $\mathfrak{g}^{-1}/\mathfrak{p}$. One immediately checks that the second of these isomorphisms is complex linear. From these observations it is evident, that if $\varphi : \mathfrak{k} \to \mathfrak{g}$ is a linear map which satisfies the four properties of Theorem 3.3 and $g \in P$ is arbitrary, then also $\mathrm{Ad}(g) \circ \varphi$ has these properties.

To deal with this freedom, we need a bit more information on the group P. Note first that there is a subgroup $G_0 \subset P$ consisting of all $g \in P$ for which $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ even preserves the grading. It is a general result (see [6, Proposition 2.10]) that G_0 has Lie algebra $\mathfrak{g}_0$ and any $g \in G$ can be uniquely written in the form $g = g_0 \exp(Z_1)\exp(Z_2)$ for $g_0 \in G_0$ and $Z_i \in \mathfrak{g}_i$. For our choice of G and P, one immediately verifies that the (complex) linear automorphism on $\mathfrak{g}^{-1}/\mathfrak{p}$ induced by $\mathrm{Ad}(g)$ depends only on g_0 and one obtains an isomorphism $G_0 \to \mathbb{C} \setminus \{0\}$ in this way.

But now any linear isomorphism $H_e \to \mathfrak{g}^{-1}/\mathfrak{p}$, which induces $J_\lambda(e)$ on H_e can be written (identifying $\mathfrak{g}^{-1}/\mathfrak{p}$ with the matrix component in the first column of the second row) as the composition of a complex linear automorphism of $\mathfrak{g}^{-1}/\mathfrak{p}$ with $u + iv \mapsto \sqrt{\lambda}u + \frac{1}{\sqrt{\lambda}}iv$.

Hence if we want φ to induce a linear isomorphism $\mathfrak{k} \to \mathfrak{g}/\mathfrak{p}$, map $H_e \to \mathfrak{g}^{-1}$, and induce $J_\lambda(e)$, then we may assume the lower two rows of the first column of $\varphi(it, u+$

iv) have the form $\begin{pmatrix} \sqrt{\lambda}u + \frac{1}{\sqrt{\lambda}}iv + tz_0 \\ ist \end{pmatrix}$ for some $z_0 \in \mathbb{C}$ and some $s \in \mathbb{R} \setminus \{0\}$. (Of course, this also determines the second component in the last row.) Making this ansatz also reduces the freedom to composition with $\mathrm{Ad}\big(\exp(Z_1)\exp(Z_2)\big)$. Having made this ansatz, one can already compute the $\mathfrak{g}_{-2}$–component of the restriction of κ to $H_e \times H_e$ and vanishing of this forces $s = 1$.

Next we observe taking the bracket with a nonzero element of $\mathfrak{g}_{-2}$ induces a linear isomorphism $\mathfrak{g}_1 \to \mathfrak{g}_{-1}$. Using this, we see that composing with $\mathrm{Ad}\big(\exp(Z_1)\big)$ for an appropriate choice of Z_1 we can require $z_0 = 0$ in the above ansatz, and this reduces the freedom to composition with $\mathrm{Ad}\big(\exp(Z_2)\big)$. To get rid of this freedom, we observe that bracketing with a nonzero element of $\mathfrak{g}_{-2}$ induces a linear isomorphism from $\mathfrak{g}_2$ to the (one–dimensional) space of real diagonal matrices contained in $\mathfrak{g}$. Hence we can eliminate all the freedom of composition with $\mathrm{Ad}(g)$ by the ansatz that the first column of $\varphi(it, u+iv)$ has the form $\begin{pmatrix} uz_0 + vz_1 + ist \\ \sqrt{\lambda}u + \frac{1}{\sqrt{\lambda}}iv \\ it \end{pmatrix}$ for elements $z_0, z_1 \in \mathbb{C}$ and $s \in \mathbb{R}$. Having made this ansatz, one can compute the complete $\mathfrak{g}_{-2}$ component of κ and the $\mathfrak{g}_{-1}$ component of the restriction to $H_e \times H_e$, and vanishing of these forces $z_0 = z_1 = 0$.

Now one can, step by step, take ansatzes for the remaining components of $\varphi(it, u + iv)$ and determine components of κ. In the end, one finds out that the conditions on κ in Theorem 3.3 are sufficient to uniquely pin down the formula for φ_λ.

3.5. **The canonical Cartan connections.** It is now easy to show that the map φ_λ from Theorem 3.3 leads to the canonical Cartan connection for (K, H, J_λ).

Corollary. (1) *For some $\lambda > 0$, consider the left invariant CR structure (H, J_λ) on $K = SU(2)$ from 2.2. Then the regular normal parabolic geometry associated to this structure is $(K \times P \to K, \omega_\lambda)$, where*

$$\omega_\lambda\big(L_X(k), L_A(g)\big) = \mathrm{Ad}(g^{-1})\big(\varphi_\lambda(X)\big) + A$$

with $\varphi_\lambda : \mathfrak{k} \to \mathfrak{g}$ the map from Theorem 3.3.
(2) *The CR structure (H, J_λ) is spherical if and only if $\lambda = 1$, i.e. if and only if it equals the standard structure.*

Proof. (1) From 3.3 we know that the formula for ω_λ defines a Cartan connection on the trivial bundle $K \times P$. The conditions on φ_λ in Theorem 3.3 which do not involve κ_λ exactly say the this Cartan connection induces the CR structure (H, J_λ) on K. Hence to prove (1), it remains to show that ω_λ is normal. The formula for κ_λ in Theorem 3.3 gives us the restriction of the curvature of ω_λ to $K \times \{e\}$. By equivariancy of the normalization condition it suffices to show normality of this restriction in order to prove that ω_λ is normal. Since κ_λ has values in $\mathfrak{g}^1$ and the restriction to $H_e \times H_e$ vanishes, it is homogeneous of degree ≥ 4, and the component of degree 4 maps $(\mathfrak{k}/H_e) \times H_e$ to $\mathfrak{g}_1$. Identifying $\mathfrak{g}_1$ with the component in the second column of the first row of a matrix, this component

is complex linear in the second variable (with respect to J_λ). It is well known (and easy to see) that the one dimensional space of such maps exactly constitutes the harmonic part in degree 4, so in particular such maps lie in the kernel of the Kostant codifferential. Since maps of homogeneity ≥ 5 automatically have that property, normality follows.
(2) This is now evident since κ_λ vanishes if and only if $\lambda^4 = 1$. $\square$

Notice that we can use the same construction replacing G by the three–fold covering $SU(2,1)$ and P by the stabilizer of a line in that group. Such an extension is necessary for example if one wants to have a standard tractor bundle, compare with [7]. In the picture of CR geometry, such an extension is associated to the choice of a third root of a certain complex line bundle. In our case, this bundle is trivial, so this poses no problem.

3.6. **Tractors and tractor calculus.** As an indication how the description of the canonical Cartan connection in Corollary 3.5 can be used further, we discuss tractor bundles and compute normal tractor connections. We will work here in the setting that $G = SU(2,1)$ and $P \subset G$ is the stabilizer of a line. Recall that for a representation V of the group G, one obtains a tractor bundle by restricting the representation to $P \subset G$ and forming the associated bundle to the canonical Cartan bundle. While sections of these bundles are unusual geometric objects, they have the advantage that they carry canonical linear connections induced by the canonical Cartan connection.

Proposition. *For some $\lambda > 0$ consider the left invariant CR structure (H, J_λ) on $K = SU(2)$ from 2.2, and let V be a representation of $G = SU(2,1)$. Then the associated tractor bundle $\mathcal{T} \to K$ is canonically trivial, so $\Gamma(\mathcal{T}) \cong C^\infty(K, V)$. In this identification, the tractor connection $\nabla^{\mathcal{T}}$ is determined by*

$$\nabla^{\mathcal{T}}_{L_X} f = L_X \cdot f + \rho\big(\varphi_\lambda(X)\big) \circ f\,,$$

for $f : K \to V$, where $L_X \in \mathfrak{X}(K)$ denotes the left invariant vector field generated by $X \in \mathfrak{k}$ and $\rho : \mathfrak{g} \to L(V,V)$ is the derivative of the representation of G on V.

Proof. Since the canonical Cartan bundle is trivial, so is the associated bundle $\mathcal{T}$. Explicitly, the identification $\Gamma(\mathcal{T}) \to C^\infty(K, V)$ is given by restricting the P–equivariant function $\mathcal{G} = K \times P \to V$ corresponding to a section to the subset $K \times \{e\}$. In terms of equivariant functions, the tractor connection can be easily described explicitly, see [5, Section 3]: For the equivariant map $h : \mathcal{G} \to V$ corresponding to $s \in \Gamma(\mathcal{T})$, a vector field ξ on K and a lift $\tilde{\xi} \in \mathfrak{X}(\mathcal{G})$ of ξ, the covariant derivative $\nabla^{\mathcal{T}}_\xi s$ is represented by the function $\tilde{\xi} \cdot h + \rho\big(\omega(\xi)\big) \circ h$.

Putting $\xi = L_X$, we can use $(L_X, 0)$ for $\tilde{\xi}$. This has the particular advantage that its flow leaves the subset $K \times \{e\} \subset K \times P$ invariant. Therefore, putting $f := h|_{K\times\{e\}}$ we see that $((L_X, 0) \cdot h)|_{K\times\{e\}} = L_X \cdot f$. For the second term, restriction to $K \times \{e\}$ makes no problems anyhow, so the formula for $\nabla^{\mathcal{T}}$ follows. $\square$

As a concrete example, let us describe how the three dimensional family of infinitesimal automorphisms corresponding to the left translations by elements of K

are represented within adjoint tractors. This means that we consider the representation $V = \mathfrak{g}$, and the resulting tractor bundle is the adjoint tractor bundle $\mathcal{A}$. The canonical Cartan connection induces an isomorphism between $\Gamma(\mathcal{A})$ and the space of right invariant vector fields on $\mathcal{G}$, see [4]. Infinitesimal automorphisms of a Cartan geometry are described by such vector fields, and they are characterized by a simple differential equation, see [4, Proposition 3.2]. We will verify this equation for the three dimensional family of infinitesimal automorphisms corresponding to left translations on K.

The construction of the canonical Cartan connection on $\mathcal{G} = K \times P$ for the left invariant CR structure (H, J_λ) on K shows that for each $k' \in K$ the map $(k, g) \mapsto (k'k, g)$ is the lift of the left translation by k' to an automorphism of the parabolic geometry $(\mathcal{G}, \omega_\lambda)$. The infinitesimal generators of this three parameter group of automorphisms are of course the vector fields $(R_X, 0)$ for $X \in \mathfrak{k}$, where R_X denotes the right invariant vector field. Let $s_X \in \Gamma(\mathcal{A})$ be the corresponding section, i.e. the smooth equivariant function corresponding to s_X is $\omega_\lambda\big((R_X, 0)\big)$. Since $R_X(k) = L_{\mathrm{Ad}(k^{-1})X}(k)$ we see that the smooth function $f_X : K \to \mathfrak{g}$ corresponding to s_X is given by $f_X(k) = \varphi_\lambda(\mathrm{Ad}(k^{-1})X)$. From the proposition above, we conclude that $\nabla^{\mathcal{A}}_{L_Y} s_X$ corresponds to the function

$$L_Y \cdot \varphi_\lambda\big(\mathrm{Ad}(k^{-1})X\big) + \big[\varphi_\lambda(Y), \varphi_\lambda(\mathrm{Ad}(k^{-1})X)\big] \,.$$

Now the first term can be computed as

$$\tfrac{d}{dt}|_{t=0}\varphi_\lambda\big(\mathrm{Ad}\left(\exp(-tY)\right)\mathrm{Ad}(k^{-1})X\big) = -\varphi_\lambda\big([Y, \mathrm{Ad}(k^{-1})X]\big) \,.$$

Hence we see that $\nabla^{\mathcal{A}}_{L_Y} s_X$ corresponds to the function $\kappa_\lambda\big(Y, \mathrm{Ad}(k^{-1})(X)\big)$ which represents the curvature of ω_λ evaluated on the vector fields L_Y and R_X. This is exactly the infinitesimal automorphism equation from [4, Proposition 3.2].

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