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# ON COUNTABLE EXTENSIONS OF PRIMARY ABELIAN GROUPS

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Abstract. It is proved that if $A$ is an abelian $p$-group with a pure subgroup $G$ so that $A/G$ is at most countable and $G$ is either $p^{\omega+n}$-totally projective or $p^{\omega+n}$-summable, then $A$ is either $p^{\omega+n}$-totally projective or $p^{\omega+n}$-summable as well. Moreover, if in addition $G$ is nice in $A$, then $G$ being either strongly $p^{\omega+n}$-totally projective or strongly $p^{\omega+n}$-summable implies that so is $A$. This generalizes a classical result of Wallace (J. Algebra, 1971) for totally projective $p$-groups as well as continues our recent investigations in (Arch. Math. (Brno), 2005 and 2006).

Some other related results are also established.

## 1. Introduction

In theory of abelian groups there exist two problems of interest, the first of which was posed by Irwin-Cutler (see [2] or [7] for extra details) and the second one by Wallace ([14]).

**Problem 1.** Suppose that $A$ is an abelian $p$-group with a subgroup $G$ such that $A/G$ is bounded. Does it follow that $A \in \mathbf{K}$, a class of abelian groups, if and only if $G \in \mathbf{K}$?

A special case is when $G = p^n A$ for some positive integer $n$ and its generalization for $G = L$, a large subgroup of $A$ (see, for instance, [1] or [3] and [9]); notice that $A/L$ is always a direct sum of cyclic groups which may be unbounded.

**Problem 2.** Suppose that $A$ is an abelian $p$-group with a subgroup $G$ such that $A/G$ is countable. Whether $G \in \mathbf{K}$, a class of abelian groups, does imply that $A \in \mathbf{K}$?

It is worthwhile noticing that in both questions the structure of the factor-group being either bounded or countable is determinative.

Here we shall be concentrated only on the latter problem. Our further work is motivated by the following significant attainment (see [14]):

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**Theorem** (Wallace, 1971)**.** *Let $A$ be a reduced abelian $p$-group with a totally projective subgroup $G$ such that $A/G$ is countable. Then $A$ is totally projective.*

Some new achievements in this theme for other important sorts of abelian groups were obtained by us in a subsequent series of papers (e.g. [4], [5], [6]).

In what follows, all notations and notions are standard and will be in agreement with those used in [10].

## 2. Main results

The major aim here is to extend the aforementioned Wallace's theorem from [14] to two important classes of torsion abelian groups the first of which properly contains the class of totally projective groups, whereas the second one properly contains the class of summable groups.

**1. $p^{\omega+n}$-totally projective $p$-groups and related concepts; $n \in \mathbb{N} \cup \{0\}$.**

We start by some definitions in terms of $\omega$-elongations of groups which terms were introduced by Nunke (see, e.g., [13]).

**Definition 2.1.** An abelian $p$-group $A$ is an $\omega$-elongation of a totally projective $p$-group by a $p^{\omega+n}$-projective $p$-group $\iff$ $p^\omega A$ is totally projective and $A/p^\omega A$ is $p^{\omega+n}$-projective.

To facilitate the exposition, we call such groups as *$p^{\omega+n}$-totally projective*.

**Definition 2.2.** An abelian $p$-group $A$ is a strong $\omega$-elongation of a totally projective $p$-group by a $p^{\omega+n}$-projective $p$-group $\iff$ $p^\omega A$ is totally projective and $\exists$ $P \leq A[p^n] : A/(P + p^\omega A)$ is a direct sum of cycles $\iff$ $p^\omega A$ is totally projective and $A/p^\omega A$ is strongly $p^{\omega+n}$-projective.

To simplify the exposition, these groups are called by us *strongly $p^{\omega+n}$-totally projective*.

**Definition 2.3** ([11])**.** An abelian $p$-group $A$ is a separate strong $\omega$-elongation of a totally projective $p$-group by a $p^{\omega+n}$-projective $p$-group $\iff$ $p^\omega A$ is totally projective and $\exists$ $P \leq A[p^n] : P \cap p^\omega A = 0$ and $A/(P \oplus p^\omega A)$ is a direct sum of cycles $\iff$ $p^\omega A$ is totally projective and $A/p^\omega A$ is separately strongly $p^{\omega+n}$-projective.

To justify the exposition, such groups are said to be *separately strongly $p^{\omega+n}$-totally projective*.

**Definition 2.4** ([11])**.** An abelian $p$-group $A \in \zeta_n$, a class of abelian $p$-groups, $\iff$ $\exists$ $P \leq A[p^n]$, $P$ is nice in $A$, $P \cap p^\omega A = 0$ and $A/P$ is totally projective.

The following claim gives satisfactory connections between these new concepts.

**Claim 2.1.** Definition 2.4 $\iff$ Definition 2.3 $\Rightarrow$ Definition 2.2 $\Rightarrow$ Definition 2.1.

**Proof.** The first equivalence was shown in [11]. The second implication is straightforward, while the second one follows like this. Because $A/p^\omega A$ possesses quotient $A/p^\omega A/(P + p^\omega A)/p^\omega A \cong A/(P + p^\omega A)$ which is a direct sum of cyclic groups and also a bounded by $p^n$ subgroup $(P + p^\omega A)/p^\omega A$, the criterion of Nunke (see, for instance, [12] or [4]) applies to show that $A/p^\omega A$ is $p^{\omega+n}$-projective, indeed. $\square$

Now, we are ready to proceed by proving the following statement.

**Theorem 2.1.** *Suppose $A$ is a reduced abelian $p$-group with a pure subgroup $G$ such that $A/G$ is countable. If $G$ is $p^{\omega+n}$-totally projective, then so does $A$.*

**Proof.** Since $p^\omega G$ is totally projective and $p^\omega A/p^\omega G = p^\omega A/(G \cap p^\omega A) \cong (p^\omega A + G)/G \subseteq p^\omega(A/G) \subseteq A/G$ is countable, we employ [14] to conclude that $p^\omega A$ is totally projective.

On the other hand, the fact that $G/p^\omega G$ being $p^{\omega+n}$-projective yields the same property for $A/p^\omega A$ follows by application of ([5], Corollary). □

Before arguing the other central assertion, we need the following technicality.

**Lemma 2.1.** *Let $G$ be a nice subgroup of the abelian $p$-group $A$. Then, for each (limit) ordinal number $\alpha$, $G + p^\alpha A$ is nice in $A$.*

**Proof.** What is sufficient to be demonstrated is that, for any (limit) ordinal number $\tau$, $\cap_{\delta<\tau}(G + p^\alpha A + p^\delta A) = G + p^\alpha A + p^\tau A$ holds. In doing this, we differ two basic cases.

*Case 1.* $\alpha \leq \tau$. Consequently, $\cap_{\delta<\tau}(G+p^\alpha A+p^\delta A) = \cap_{\delta<\alpha\leq\tau}(G+p^\alpha A+p^\delta A) \cap \cap_{\alpha\leq\delta<\tau}(G+p^\alpha A+p^\delta A) = \cap_{\delta<\alpha}(G+p^\delta A) \cap \cap_{\delta<\tau}(G+p^\alpha A) = (G+p^\alpha A) \cap (G+p^\alpha A) = G+p^\alpha A = G+p^\alpha A+p^\tau A$, as required.

*Case 2.* $\alpha > \tau$. Therefore, $\cap_{\delta<\tau}(G+p^\alpha A+p^\delta A) = \cap_{\delta<\tau}(G+p^\delta A) = G+p^\tau A = G+p^\alpha A+p^\tau A$, which allows us to conclude the wanted dependence. □

We have now accumulated all the information necessary to prove the following.

**Theorem 2.2.** *Suppose $A$ is an abelian reduced $p$-group with a pure and nice subgroup $G$ so that $A/G$ is countable. If $G$ is strongly $p^{\omega+n}$-totally projective, then so is $A$. In particular, if $G$ is separately then $A$ is separately.*

**Proof.** In accordance with the corresponding definition, we write that $G/(M + p^\omega G)$ is a direct sum of cyclic groups for some existing $M \leq G[p^n]$; in addition $M \cap p^\omega G = 0$ for the situation of separately groups. It suffices to show that $A/(M + p^\omega A)$ is a direct sum of cycles, where it is obvious that $M \leq A[p^n]$.

First, we observe with the aid of the modular law from [10] that $(G+p^\omega A)/(M+p^\omega A) \cong G/[G\cap(M+p^\omega A)] = G/[M+(G\cap p^\omega A)] = G/(M+p^\omega G)$ is a direct sum of cyclic groups. Besides, $A/(M+p^\omega A)/(G+p^\omega A)/(M+p^\omega A) \cong A/(G+p^\omega A)$ is countable as an epimorphic image of the countable factor-group $A/G$.

On the other hand, by the usage of the modular law from [10], for every natural $k$, we compute that $[(G+p^\omega A)/(M+p^\omega A)] \cap p^k(A/(M+p^\omega A)) = [(G+p^\omega A)/(M+p^\omega A)] \cap [(p^k A + M)/(M + p^\omega A)] = [(G + p^\omega A) \cap (M + p^k A)]/(M + p^\omega A) = [p^\omega A+(G\cap(M+p^k A))]/(M+p^\omega A) = (p^\omega A+M+G\cap p^k A)/(M+p^\omega A) = (p^\omega A+M+p^k G)/(M+p^\omega A) = p^k((G+p^\omega A)/(M+p^\omega A))$, hence $(G+p^\omega A)/(M+p^\omega A)$ is pure in $A/(M + p^\omega A)$. But Lemma 2.1 enables us to deduce that $G + p^\omega A$ is nice in $A$ whence $p^\omega(A/(G + p^\omega A)) = (p^\omega A + G)/(p^\omega A + G) = 0$ and thereby $A/(G+p^\omega A) \cong A/(M+p^\omega A)/(G+p^\omega A)/(M+p^\omega A)$ is separable. Thus a lemma from ([4], p. 271) works to infer that $A/(M + p^\omega A)$ is separable. Therefore, a consequence of the alluded to above result due to Wallace [14] is applicable to get that $A/(M+p^\omega A)$ is a direct sum of cyclic groups, as desired.

As a final point, that $G$ being separately, i.e. $M \cap p^\omega G = 0$, implies the same property for $A$, i.e. $M \cap p^\omega A = 0$, follows via the simple fact that $M \cap p^\omega A = M \cap G \cap p^\omega A = M \cap p^\omega G$. □

## 2. $p^{\omega+n}$-summable $p$-groups and related concepts; $n \in \mathbb{N} \cup \{0\}$.

Imitating [11], we begin with some parallel definitions to these from section 1.

**Definition 2.5.** An abelian $p$-group $A$ is an $\omega$-elongation of a summable $p$-group by a $p^{\omega+n}$-projective $p$-group $\iff$ $p^\omega A$ is summable and $A/p^\omega A$ is $p^{\omega+n}$-projective.

For simplicity of the exposition, we call these groups as $p^{\omega+n}$*-summable*.

**Definition 2.6.** An abelian $p$-group $A$ is a strong $\omega$-elongation of a summable $p$-group by a $p^{\omega+n}$-projective $p$-group $\iff$ $p^\omega A$ is summable and $\exists\ P \leq A[p^n]$ : $A/(P+p^\omega A)$ is a direct sum of cycles $\iff$ $p^\omega A$ is summable and $A/p^\omega A$ is strongly $p^{\omega+n}$-projective.

For facilitating of the exposition, such groups are called *strongly* $p^{\omega+n}$*-summable*.

**Definition 2.7.** An abelian $p$-group $A$ is a separate strong $\omega$-elongation of a summable $p$-group by a $p^{\omega+n}$-projective $p$-group $\iff$ $p^\omega A$ is summable and $\exists$ $P \leq A[p^n] : P \cap p^\omega A = 0$ and $A/(P \oplus p^\omega A)$ is a direct sum of cycles $\iff$ $p^\omega A$ is summable and $A/p^\omega A$ is separately strongly $p^{\omega+n}$-projective.

For justifying of the exposition, these groups are said to be *separately strongly* $p^{\omega+n}$*-summable*.

**Definition 2.8.** An abelian $p$-group $A \in \eta_n$, a class of abelian $p$-groups, $\iff$ $\exists$ $P \leq A[p^n]$, $P$ is nice in $A$, $P \cap p^\omega A = 0$ and $A/P$ is summable.

The following claim assures comprehensive relations between these nomenclatures.

**Claim 2.2.**
(*) Definition 2.7 $\Rightarrow$ Definition 2.8.
(**) Definition 2.7 $\Rightarrow$ Definition 2.6 $\Rightarrow$ Definition 2.5.

**Proof.** We foremost deal with the first point. Since $A/(P \oplus p^\omega A)$ is a direct sum of cyclic groups, it follows that $P \oplus p^\omega A$ is nice in $A$ whence, referring to ([11], Lemma 1), $P$ is so in $A$. Moreover, $p^\omega(A/P) = (p^\omega A \oplus P)/P \cong p^\omega A$ is summable and $A/(P \oplus p^\omega A) \cong A/P/(P \oplus p^\omega A)/P = A/P/p^\omega(A/P)$ is a direct sum of cyclic groups. Finally, we employ a group attainment from [8] to infer the desired implication.

Turning to the second point, we observe that the first implication is trivial while the second one follows as in Claim 2.1. □

It is well to note that since by a result of Nunke [12] each totally projective group of length not exceeding $\Omega$, the first uncountable ordinal, is a direct sum of countable groups, hence it is summable, we derive that if $A$ is an abelian $p$-group with $length(A) \leq \Omega$ then $A$ being $p^{\omega+n}$-totally projective yields that $A$ is $p^{\omega+n}$-summable and so the same inclusion is valid for the other sorts of groups previously defined in two ways as above. Moreover, similar kinds of such elongations of groups can be defined provided $p^{\omega}A$ is a $\Sigma$-group or $\sigma$-summable or a $Q$-group or (weakly) $\omega_1$-separable. Nevertheless, the proofs of the corresponding theorems are analogous as to the above presented by exploiting the results from [6]. Likewise, a discussion for the truthfulness of equivalence in (*), similar to that in Claim 2.1, may be initiated.

The following assertion enlarges ([11], Theorem 1) that is the corresponding affirmation for separately strongly $p^{\omega+n}$-totally projective $p$-groups; however we emphasize that it follows via the same idea of proof.

**Proposition 2.1.** *Let $A = B \oplus C$ be an abelian $p$-group. Then $A$ is separately strongly $p^{\omega+n}$-summable $\iff$ $B$ and $C$ are separately strongly $p^{\omega+n}$-summable.*

**Proof.** "$\Rightarrow$" Set $G_1 = B \cap (P \oplus p^{\omega}A)$ and $G_2 = C \cap (P \oplus p^{\omega}A)$, where $P$ is defined as in Definition 2.3. Hence $G_1 \leq B$ and $B/G_1 \cong (B + P + p^{\omega}A)/(P + p^{\omega}A) \subseteq A/(P + p^{\omega}A)$ is a direct sum of cyclic groups. Observe that $G_1 \cap G_2 = 0$ and thus $G = G_1 \oplus G_2$ satisfies $p^{\omega}A \leq G \leq P + p^{\omega}A$, so $G = P' \oplus p^{\omega}A$ for $P' = P \cap G$. Furthermore, $p^{\omega}A = p^{\omega}B \oplus p^{\omega}C$, whence $G_1 = P_1 \oplus p^{\omega}B$ with $P_1 = G_1 \cap (P' \oplus p^{\omega}C)$. It is obvious that $P_1 \leq B, p^nP_1 = 0$ and $P_1 \cap p^{\omega}B = 0$. Moreover, $p^{\omega}B$ is summable as being a direct factor of the summable group $p^{\omega}A$. Therefore, by definition, $B$ is $p^{\omega+n}$-summable.

By symmetry, one can conclude that $C$ belongs to this class of groups too.

"$\Leftarrow$" Write that $p^{\omega}B$ and $p^{\omega}C$ are summable groups and that there exist $P \leq B[p^n]$ and $M \leq C[p^n]$ without elements of infinite height, as calculated in $B$ and $C$ respectively, with the properties that $B/(P \oplus p^{\omega}B)$ and $C/(M \oplus p^{\omega}C)$ are direct sums of cycles. Therefore, $p^{\omega}A = p^{\omega}B \oplus p^{\omega}C$ is also summable (see [10]). On the other hand, $P \cap p^{\omega}A = P \cap p^{\omega}B = 0$ as well as $M \cap p^{\omega}A = M \cap p^{\omega}C = 0$. Consequently, $(P \oplus M) \cap p^{\omega}A = (P \oplus M) \cap (p^{\omega}B \oplus p^{\omega}C) = (P \cap p^{\omega}B) \oplus (M \cap p^{\omega}C) = 0$ and $P \oplus M \leq A[p^n]$. Furthermore, we observe that $A/[(P \oplus M) \oplus p^{\omega}A] = (B \oplus C)/[P \oplus p^{\omega}B \oplus M \oplus p^{\omega}C] \cong [B/(P \oplus p^{\omega}B)] \oplus [C/(M \oplus p^{\omega}C)]$ is a direct sum of cyclic groups too, and thereby we are finished. $\square$

**Remark 2.1.** By the same manner we may illustrate that the proposition holds valid for other sorts of groups defined as above.

Now, we are in a position to prove the following statement.

**Theorem 2.3.** *Suppose $A$ is a reduced abelian $p$-group with a pure subgroup $G$ such that $A/G$ is countable. If $G$ is $p^{\omega+n}$-summable, then so does $A$.*

**Proof.** By appealing to the same reasoning as in Theorem 2.1, combined with [6], one may infer that the assertion follows. $\square$

We have now at our disposal all the machinery necessary to argue the following.

**Theorem 2.4.** *Suppose $A$ is an abelian reduced $p$-group with a pure and nice subgroup $G$ so that $A/G$ is countable. If $G$ is strongly $p^{\omega+n}$-summable, then so is $A$. In particular, if $G$ is separately then $A$ is separately.*

**Proof.** It follows by the usage of the same idea as in Theorem 2.2 along with [6]. □

The following comment is valuable.

**Remark 2.2.** Because it is a straightforward argument that each strongly $p^{\omega+n}$-totally projective $p$-group, respectively each strongly $p^{\omega+n}$-summable $p$-group, of length not exceeding $\omega + n$ is $p^{\omega+n}$-projective, as an immediate corollary we may obtain Theorem 5 of [4].

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