ARCHIVUM MATHEMATICUM (BRNO)
Tomus 43 (2007), 133 – 155

ON THE BEHAVIOR OF THE SOLUTIONS TO AUTONOMOUS LINEAR DIFFERENCE EQUATIONS WITH CONTINUOUS VARIABLE

Ch. G. Philos and I. K. Purnaras

Abstract. Autonomous linear neutral delay and, especially, (non-neutral) delay difference equations with continuous variable are considered, and some new results on the behavior of the solutions are established. The results are obtained by the use of appropriate positive roots of the corresponding characteristic equation.

1. Introduction

During the last few years, a number of articles has been appeared in the literature, which are motivated by the old but very interesting papers by Driver [7, 8] and Driver, Sasser and Slater [10] dealing with the asymptotic behavior and the stability of the solutions of delay differential equations. See [2, 4, 5, 12, 13, 17-21, 27-41, 44]. These articles are concerned with the asymptotic behavior (and, more general, the behavior) and the stability for delay differential equations, neutral delay differential equations and (neutral or non-neutral) integrodifferential equations with unbounded delay as well as for delay difference equations (with discrete or continuous variable), neutral delay difference equations and (neutral or non-neutral) Volterra difference equations with infinite delay. In the above list of articles, there are only three of them dealing with difference equations with continuous variable; see [31, 44] and the last section of [33]. For some related results, the reader is referred to [3, 9, 14, 15, 26, 42, 43]. In the present paper, we continue the study in [17-21, 27-41] to difference equations with continuous variable.

In the last two decades, the study of difference equations has attracted significant interest by many researchers. This is due, in a large part, to the rapidly increasing number of applications of the theory of difference equations to various fields of applied sciences and technology. For the basic theory of difference equations, we refer to the books by Agarwal [1], Elaydi [11], Kelley and Peterson [16],

2000 *Mathematics Subject Classification*: 39A10, 39A11.
Key words and phrases: difference equation with continuous variable, delay difference equation, neutral delay difference equation, behavior of solutions, characteristic equation.
Received July 11, 2006.

Lakshmikantham and Trigiante [24], Mickens [25], and Sharkovsky, Maistrenko and Romanenko [45].

Difference equations with continuous variable are difference equations in which the unknown function is a function of a continuous variable. We may also refer to such equations as *difference equations with continuous time*. Note here that the term "difference equation" is usually used for difference equations with discrete variable. Difference equations with continuous variable appear as natural descriptions of observed evolution phenomena in many branches of the natural sciences (see, for example, the book [45]; see, also, the paper [22]). For some results on the oscillation of difference equations with continuous variable, we refer to [6, 23, 46-48] (and the references cited therein).

In this paper, we are concerned with the behavior of the solutions of autonomous linear difference equations with continuous variable. The general case of neutral delay difference equations with continuous variable is considered in Section 2, while Section 3 is devoted to the special case of (non-neutral) delay difference equations with continuous variable. Our results will be obtained via an appropriate positive root of the corresponding characteristic equation or by the use of two suitable distinct roots of the characteristic equation. Note that the main result in [31], applied to the unforced case, as well as Theorem III in [33] are essentially included (as particular cases) in the results of the present paper.

2. Autonomous linear neutral delay difference equations with continuous variable

Consider the neutral delay difference equation with continuous variable

$$\Delta\Big[x(t)+\sum_{i\in I}c_i x(t-\sigma_i)\Big]=ax(t)+\sum_{j\in J}b_j x(t-\tau_j)\,, \tag{2.1}$$

where *I and J are initial segments of natural numbers, c_i for $i\in I$, a and $b_j\neq 0$ for $j\in J$ are real numbers, and σ_i for $i\in I$ and τ_j for $j\in J$ are positive integers such that $\sigma_{i_1}\neq\sigma_{i_2}$ for i_1, $i_2\in I$ with $i_1\neq i_2$ and $\tau_{j_1}\neq\tau_{j_2}$ for j_1, $j_2\in J$ with $j_1\neq j_2$*. Note that the *difference operator* Δ will be considered to be defined as usual, i.e.,

$$\Delta h(t)=h(t+1)-h(t)\quad\text{for}\quad t\geq t_0\,,$$

for any real-valued function h defined on an interval $[t_0,\infty)$.

Let us define the positive integers σ,τ and r by

$$\sigma=\max_{i\in I}\sigma_i\,,\quad \tau=\max_{j\in J}\tau_j\quad\text{and}\quad r=\max\{\sigma,\tau\}\,.$$

By a *solution* of the neutral delay difference equation (2.1), we mean a continuous real-valued function x defined on the interval $[-r,\infty)$ which satisfies (2.1) for all $t\geq 0$.

Along with the neutral delay difference equation (2.1), we specify an *initial condition* of the form

$$x(t)=\phi(t)\quad\text{for}\quad -r\leq t\leq 1\,, \tag{2.2}$$

where *the initial function* ϕ *is a given continuous real-valued function on the interval* $[-r, 1]$ *satisfying the "consistency condition"*

$$\phi(1) - \phi(0) + \sum_{i\in I} c_i \big[\phi(1-\sigma_i) - \phi(-\sigma_i)\big] = a\phi(0) + \sum_{j\in J} b_j \phi(-\tau_j)\,.$$

Equations (2.1) and (2.2) constitute an *initial value problem* (IVP, for short). By the use of the method of steps, one can easily see that there exists a unique solution x of the neutral delay difference equation (2.1) which satisfies the initial condition (2.2); this unique solution x will be called the *solution* of the initial value problem (2.1) and (2.2) or, more briefly, the *solution* of the IVP (2.1) and (2.2).

Together with the neutral delay difference equation (2.1), we associate the following equation

$$(\lambda - 1)\Big(1 + \sum_{i\in I} c_i \lambda^{-\sigma_i}\Big) = a + \sum_{j\in J} b_j \lambda^{-\tau_j}\,, \tag{2.3}$$

which will be called the *characteristic equation* of (2.1). Equation (2.3) is obtained from (2.1) by seeking solutions of the form $x(t) = \lambda^t$ for $t \geq -r$, where λ is a positive real number.

Our first result is the following theorem, which establishes a basic asymptotic property for the solutions of the neutral delay difference equation (2.1).

Theorem 2.1. *Let* λ_0 *be a positive root of the characteristic equation* (2.3) *such that*

$$\sum_{i\in I} |c_i| \Big(1 + \Big|1 - \frac{1}{\lambda_0}\Big|\sigma_i\Big)\lambda_0^{-\sigma_i} + \frac{1}{\lambda_0}\sum_{j\in J} |b_j|\, \tau_j \lambda_0^{-\tau_j} < 1\,. \tag{2.4}$$

Set

$$\gamma(\lambda_0) = \sum_{i\in I} c_i \Big[1 - \Big(1 - \frac{1}{\lambda_0}\Big)\sigma_i\Big]\lambda_0^{-\sigma_i} + \frac{1}{\lambda_0}\sum_{j\in J} b_j \tau_j \lambda_0^{-\tau_j}\,.$$

Then the solution x *of the IVP* (2.1) *and* (2.2) *satisfies*

$$\lim_{t\to\infty} \int_t^{t+1} \lambda_0^{-u} x(u)\, du = \frac{L(\lambda_0; \phi)}{1 + \gamma(\lambda_0)}\,,$$

where

$$\begin{aligned} L(\lambda_0; \phi) = {} & \int_0^1 \lambda_0^{-u} \phi(u)\, du \\ & + \sum_{i\in I} c_i \lambda_0^{-\sigma_i} \Big[\int_{-\sigma_i}^{-\sigma_i+1} \lambda_0^{-u}\phi(u)\, du - \Big(1 - \frac{1}{\lambda_0}\Big)\int_{-\sigma_i}^{0} \lambda_0^{-u}\phi(u)\, du\Big] \\ & + \frac{1}{\lambda_0}\sum_{j\in J} b_j \lambda_0^{-\tau_j} \Big[\int_{-\tau_j}^{0} \lambda_0^{-u}\phi(u)\, du\Big]\,. \end{aligned}$$

Note. Condition (2.4) guarantees that $1 + \gamma(\lambda_0) > 0$.

Corollary 2.2 below follows immediately from the above theorem by an application with $\lambda_0 = 1$.

Corollary 2.2. *Assume that*

$$a+\sum_{j\in J} b_j = 0 \quad \textit{and} \quad \sum_{i\in I}|c_i| + \sum_{j\in J}|b_j|\,\tau_j < 1\,. \tag{2.5}$$

Then the solution x of the IVP (2.1) *and* (2.2) *satisfies*

$$\lim_{t\to\infty}\int_t^{t+1} x(u)\,du = \frac{\int_0^1 \phi(u)\,du + \sum_{i\in I} c_i \left[\int_{-\sigma_i}^{-\sigma_i+1}\phi(u)\,du\right] + \sum_{j\in J} b_j \left[\int_{-\tau_j}^{0}\phi(u)\,du\right]}{1+\sum_{i\in I} c_i + \sum_{j\in J} b_j\tau_j}\,.$$

Note. The second condition of (2.5) guarantees that $1+\sum_{i\in I} c_i + \sum_{j\in J} b_j\tau_j > 0$.

Proof of Theorem 2.1. Let x be the solution of the IVP (2.1) and (2.2), and define

$$y(t) = \lambda_0^{-t}x(t) \quad \text{for } t \ge -r.$$

Then, for every $t\ge 0$, we obtain

$$\begin{aligned}
&\Delta\Big[x(t)+\sum_{i\in I} c_i x(t-\sigma_i) - ax(t) - \sum_{j\in J} b_j x(t-\tau_j)\Big]\\
&\quad= \Delta\Big\{\lambda_0^t\Big[y(t)+\sum_{i\in I} c_i\lambda_0^{-\sigma_i}y(t-\sigma_i)\Big]\Big\} - a\lambda_0^t y(t) - \lambda_0^t\sum_{j\in J} b_j\lambda_0^{-\tau_j}y(t-\tau_j)\\
&\quad= \lambda_0^{t+1}\Delta\Big[y(t)+\sum_{i\in I} c_i\lambda_0^{-\sigma_i}y(t-\sigma_i)\Big] + \left(\lambda_0^{t+1}-\lambda_0^t\right)\Big[y(t)+\sum_{i\in I} c_i\lambda_0^{-\sigma_i}y(t-\sigma_i)\Big]\\
&\qquad - a\lambda_0^t y(t) - \lambda_0^t\sum_{j\in J} b_j\lambda_0^{-\tau_j}y(t-\tau_j)\\
&\quad= \lambda_0^t\Big\{\lambda_0\Delta\Big[y(t)+\sum_{i\in I} c_i\lambda_0^{-\sigma_i}y(t-\sigma_i)\Big] + (\lambda_0-1-a)y(t)\\
&\qquad + (\lambda_0-1)\sum_{i\in I} c_i\lambda_0^{-\sigma_i}y(t-\sigma_i) - \sum_{j\in J} b_j\lambda_0^{-\tau_j}y(t-\tau_j)\Big\}\\
&\quad= \lambda_0^t\Big\{\lambda_0\Delta\Big[y(t)+\sum_{i\in I} c_i\lambda_0^{-\sigma_i}y(t-\sigma_i)\Big]\\
&\qquad + \Big[-(\lambda_0-1)\sum_{i\in I} c_i\lambda_0^{-\sigma_i} + \sum_{j\in J} b_j\lambda_0^{-\tau_j}\Big]y(t)\\
&\qquad + (\lambda_0-1)\sum_{i\in I} c_i\lambda_0^{-\sigma_i}y(t-\sigma_i) - \sum_{j\in J} b_j\lambda_0^{-\tau_j}y(t-\tau_j)\Big\}\\
&\quad= \lambda_0^t\Big\{\lambda_0\Delta\Big[y(t)+\sum_{i\in I} c_i\lambda_0^{-\sigma_i}y(t-\sigma_i)\Big] - (\lambda_0-1)\\
&\qquad \times\sum_{i\in I} c_i\lambda_0^{-\sigma_i}\left[y(t)-y(t-\sigma_i)\right] + \sum_{j\in J} b_j\lambda_0^{-\tau_j}\left[y(t)-y(t-\tau_j)\right]\Big\}\,.
\end{aligned}$$

Thus, the fact that the solution x satisfies (2.1) for $t \ge 0$ is equivalent to the fact that y satisfies

$$\Delta\Big[y(t) + \sum_{i\in I} c_i\lambda_0^{-\sigma_i} y(t-\sigma_i)\Big] = \Big(1 - \frac{1}{\lambda_0}\Big)\sum_{i\in I} c_i\lambda_0^{-\sigma_i}\,[y(t) - y(t-\sigma_i)] - \frac{1}{\lambda_0}\sum_{j\in J} b_j\lambda_0^{-\tau_j}\,[y(t) - y(t-\tau_j)] \quad \text{for} \quad t \ge 0 \tag{2.6}$$

On the other hand, the initial condition (2.2) is written in the following equivalent form

$$y(t) = \lambda_0^{-t}\phi(t) \quad \text{for} \quad -r \le t \le 1\,. \tag{2.7}$$

Next, let us introduce the function Y defined by

$$Y(t) = \int_t^{t+1} y(u)\,du \quad \text{for} \quad t \ge -r\,.$$

We observe that

$$Y'(t) = y(t+1) - y(t) = \Delta y(t) \quad \text{for} \quad t \ge -r\,.$$

So, we can immediately see that

$$\Delta\Big[y(t) + \sum_{i\in I} c_i\lambda_0^{-\sigma_i} y(t-\sigma_i)\Big] = \Big[Y(t) + \sum_{i\in I} c_i\lambda_0^{-\sigma_i} Y(t-\sigma_i)\Big]' \quad \text{for} \quad t \ge 0\,.$$

Moreover, for any $i \in I$ and every $t \ge 0$, we get

$$y(t) - y(t-\sigma_i) = \sum_{s=-\sigma_i}^{-1} [y(t+s+1) - y(t+s)] = \sum_{s=-\sigma_i}^{-1} \Delta y(t+s) = \sum_{s=-\sigma_i}^{-1} Y'(t+s)\,.$$

Consequently,

$$y(t) - y(t-\sigma_i) = \Big[\sum_{s=-\sigma_i}^{-1} Y(t+s)\Big]' \quad \text{for} \quad i \in I \quad \text{and} \quad t \ge 0\,.$$

Analogously, we have

$$y(t) - y(t-\tau_j) = \Big[\sum_{s=-\tau_j}^{-1} Y(t+s)\Big]' \quad \text{for} \quad j \in J \quad \text{and} \quad t \ge 0\,.$$

After the above observations, we see that (2.6) can equivalently be written as

$$\Big[Y(t) + \sum_{i\in I} c_i\lambda_0^{-\sigma_i} Y(t-\sigma_i)\Big]' = \Big\{\Big(1 - \frac{1}{\lambda_0}\Big)\sum_{i\in I} c_i\lambda_0^{-\sigma_i}\Big[\sum_{s=-\sigma_i}^{-1} Y(t+s)\Big] - \frac{1}{\lambda_0}\sum_{j\in J} b_j\lambda_0^{-\tau_j}\Big[\sum_{s=-\tau_j}^{-1} Y(t+s)\Big]\Big\}' \quad \text{for} \quad t \ge 0\,.$$

The last equation is equivalent to

$$Y(t)+\sum_{i\in I}c_i\lambda_0^{-\sigma_i}Y(t-\sigma_i)=\Big(1-\frac{1}{\lambda_0}\Big)\sum_{i\in I}c_i\lambda_0^{-\sigma_i}\Big[\sum_{s=-\sigma_i}^{-1}Y(t+s)\Big]$$
$$-\frac{1}{\lambda_0}\sum_{j\in J}b_j\lambda_0^{-\tau_j}\Big[\sum_{s=-\tau_j}^{-1}Y(t+s)\Big]+K\quad\text{for}\quad t\geq 0\,,$$

where the real constant K is given by

$$K=Y(0)+\sum_{i\in I}c_i\lambda_0^{-\sigma_i}Y(-\sigma_i)-\Big(1-\frac{1}{\lambda_0}\Big)\sum_{i\in I}c_i\lambda_0^{-\sigma_i}\Big[\sum_{s=-\sigma_i}^{-1}Y(s)\Big]$$
$$+\frac{1}{\lambda_0}\sum_{j\in J}b_j\lambda_0^{-\tau_j}\Big[\sum_{s=-\tau_j}^{-1}Y(s)\Big]\,.$$

By the use of the function Y, the initial condition (2.7) takes the equivalent form

$$Y(t)=\int_t^{t+1}\lambda_0^{-u}\phi(u)\,du\quad\text{for}\quad -r\leq t\leq 0\,. \tag{2.8}$$

Furthermore, by using (2.8) and taking into account the definition of $L(\lambda_0;\phi)$, we obtain

$$\begin{aligned}
K&=\int_0^1\lambda_0^{-u}\phi(u)\,du+\sum_{i\in I}c_i\lambda_0^{-\sigma_i}\Big[\int_{-\sigma_i}^{-\sigma_i+1}\lambda_0^{-u}\phi(u)\,du\Big]\\
&\quad-\Big(1-\frac{1}{\lambda_0}\Big)\sum_{i\in I}c_i\lambda_0^{-\sigma_i}\Big[\sum_{s=-\sigma_i}^{-1}\int_s^{s+1}\lambda_0^{-u}\phi(u)\,du\Big]\\
&\quad+\frac{1}{\lambda_0}\sum_{j\in J}b_j\lambda_0^{-\tau_j}\Big[\sum_{s=-\tau_j}^{-1}\int_s^{s+1}\lambda_0^{-u}\phi(u)\,du\Big]\\
&=\int_0^1\lambda_0^{-u}\phi(u)\,du+\sum_{i\in I}c_i\lambda_0^{-\sigma_i}\Big[\int_{-\sigma_i}^{-\sigma_i+1}\lambda_0^{-u}\phi(u)\,du\Big]\\
&\quad-\Big(1-\frac{1}{\lambda_0}\Big)\sum_{i\in I}c_i\lambda_0^{-\sigma_i}\Big[\int_{-\sigma_i}^{0}\lambda_0^{-u}\phi(u)\,du\Big]\\
&\quad+\frac{1}{\lambda_0}\sum_{j\in J}b_j\lambda_0^{-\tau_j}\Big[\int_{-\tau_j}^{0}\lambda_0^{-u}\phi(u)\,du\Big]\\
&=\int_0^1\lambda_0^{-u}\phi(u)\,du
\end{aligned}$$

$$+\sum_{i\in I} c_i\lambda_0^{-\sigma_i}\Big[\int_{-\sigma_i}^{-\sigma_i+1}\lambda_0^{-u}\phi(u)\,du-\Big(1-\frac{1}{\lambda_0}\Big)\int_{-\sigma_i}^{0}\lambda_0^{-u}\phi(u)\,du\Big]$$
$$+\frac{1}{\lambda_0}\sum_{j\in J} b_j\lambda_0^{-\tau_j}\Big[\int_{-\tau_j}^{0}\lambda_0^{-u}\phi(u)\,du\Big]=L(\lambda_0;\phi)\,.$$

So, we conclude that (2.6) can equivalently be written, in terms of the function Y, as follows

$$Y(t)+\sum_{i\in I} c_i\lambda_0^{-\sigma_i}Y(t-\sigma_i)=\Big(1-\frac{1}{\lambda_0}\Big)\sum_{i\in I} c_i\lambda_0^{-\sigma_i}\Big[\sum_{s=-\sigma_i}^{-1}Y(t+s)\Big]$$
$$-\frac{1}{\lambda_0}\sum_{j\in J} b_j\lambda_0^{-\tau_j}\Big[\sum_{s=-\tau_j}^{-1}Y(t+s)\Big]+L(\lambda_0;\phi)\quad\text{for}\quad t\geq 0\,. \tag{2.9}$$

Now, we define

$$z(t)=Y(t)-\frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)}\quad\text{for } t\geq -r\,.$$

By taking into account the definition of $\gamma(\lambda_0)$, we can easily verify that (2.9) reduces to the following equivalent equation

$$z(t)+\sum_{i\in I} c_i\lambda_0^{-\sigma_i}z(t-\sigma_i)=\Big(1-\frac{1}{\lambda_0}\Big)\sum_{i\in I} c_i\lambda_0^{-\sigma_i}\Big[\sum_{s=-\sigma_i}^{-1}z(t+s)\Big]$$
$$-\frac{1}{\lambda_0}\sum_{j\in J} b_j\lambda_0^{-\tau_j}\Big[\sum_{s=-\tau_j}^{-1}z(t+s)\Big]\quad\text{for}\quad t\geq 0\,. \tag{2.10}$$

On the other hand, the initial condition (2.8) becomes

$$z(t)=\int_t^{t+1}\lambda_0^{-u}\phi(u)\,du-\frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)}\quad\text{for}\quad -r\leq t\leq 0\,. \tag{2.11}$$

We use the definitions of the functions y, Y and z to conclude that all we have to prove is that

$$\lim_{t\to\infty} z(t)=0\,. \tag{2.12}$$

In the rest of the proof we will establish (2.12).

Let us consider the real constant $\mu(\lambda_0)$ defined by

$$\mu(\lambda_0)=\sum_{i\in I}|c_i|\,\Big(1+\Big|1-\frac{1}{\lambda_0}\Big|\sigma_i\Big)\lambda_0^{-\sigma_i}+\frac{1}{\lambda_0}\sum_{j\in J}|b_j|\,\tau_j\lambda_0^{-\tau_j}\,,$$

which, by condition (2.4), satisfies

$$0<\mu(\lambda_0)<1\,. \tag{2.13}$$

Moreover, we put

$$M(\lambda_0;\phi) = \max_{-r\le t\le 0}\Big|\int_t^{t+1}\lambda_0^{-u}\phi(u)\,du - \frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)}\Big|.$$

Then, because of (2.11), we have

$$|z(t)| \le M(\lambda_0;\phi) \quad \text{for} \quad -r\le t\le 0\,. \tag{2.14}$$

We will show that $M(\lambda_0;\phi)$ is a bound of the function z on the whole interval $[-r,\infty)$, i.e., that

$$|z(t)| \le M(\lambda_0;\phi) \quad \text{for all} \quad t\ge -r\,. \tag{2.15}$$

To this end, let us consider an arbitrary real number $\epsilon>0$. We claim that

$$|z(t)| < M(\lambda_0;\phi)+\epsilon \quad \text{for every} \quad t\ge -r\,. \tag{2.16}$$

Otherwise, since (2.14) guarantees that $|z(t)|<M(\lambda_0;\phi)+\epsilon$ for $-r\le t\le 0$, there exists a point $t_0>0$ so that

$$|z(t)| < M(\lambda_0;\phi)+\epsilon \quad \text{for} \quad -r\le t<t_0\,, \quad \text{and} \quad |z(t_0)| = M(\lambda_0;\phi)+\epsilon\,.$$

Then, by taking into account the definition of $\mu(\lambda_0)$ and using (2.13), from (2.10) we obtain

$$\begin{aligned}
M(\lambda_0;\phi)+\epsilon &= |z(t_0)|\\
&= \Big|\sum_{i\in I} c_i\lambda_0^{-\sigma_i}\Big[-z(t_0-\sigma_i)+\Big(1-\frac{1}{\lambda_0}\Big)\sum_{s=-\sigma_i}^{-1} z(t_0+s)\Big]\\
&\quad -\frac{1}{\lambda_0}\sum_{j\in J} b_j\lambda_0^{-\tau_j}\Big[\sum_{s=-\tau_j}^{-1} z(t_0+s)\Big]\Big|\\
&\le \sum_{i\in I}|c_i|\,\lambda_0^{-\sigma_i}\Big[|z(t_0-\sigma_i)|+\Big|1-\frac{1}{\lambda_0}\Big|\sum_{s=-\sigma_i}^{-1}|z(t_0+s)|\Big]\\
&\quad +\frac{1}{\lambda_0}\sum_{j\in J}|b_j|\,\lambda_0^{-\tau_j}\Big[\sum_{s=-\tau_j}^{-1}|z(t_0+s)|\Big]\\
&\le \Big[\sum_{i\in I}|c_i|\Big(1+\Big|1-\frac{1}{\lambda_0}\Big|\sigma_i\Big)\lambda_0^{-\sigma_i}+\frac{1}{\lambda_0}\sum_{j\in J}|b_j|\,\tau_j\lambda_0^{-\tau_j}\Big]\big[M(\lambda_0;\phi)+\epsilon\big]\\
&= \mu(\lambda_0)\big[M(\lambda_0;\phi)+\epsilon\big] < M(\lambda_0;\phi)+\epsilon\,,
\end{aligned}$$

which is a contradiction. This contradiction proves our claim, that is, (2.16) holds true. Since (2.16) is satisfied for each real number $\epsilon>0$, it follows that (2.15) is always valid. Furthermore, by using (2.15) and taking into account the definition of $\mu(\lambda_0)$, from (2.10) we obtain, for every $t \geq 0$,

$$\begin{aligned}
|z(t)| &= \Big| \sum_{i\in I} c_i \lambda_0^{-\sigma_i} \Big[-z(t-\sigma_i) + \Big(1 - \frac{1}{\lambda_0}\Big) \sum_{s=-\sigma_i}^{-1} z(t+s) \Big] \\
&\quad - \frac{1}{\lambda_0} \sum_{j\in J} b_j \lambda_0^{-\tau_j} \Big[\sum_{s=-\tau_j}^{-1} z(t+s) \Big] \Big| \\
&\leq \sum_{i\in I} |c_i| \lambda_0^{-\sigma_i} \Big[|z(t-\sigma_i)| + \Big|1 - \frac{1}{\lambda_0}\Big| \sum_{s=-\sigma_i}^{-1} |z(t+s)| \Big] \\
&\quad + \frac{1}{\lambda_0} \sum_{j\in J} |b_j| \lambda_0^{-\tau_j} \Big[\sum_{s=-\tau_j}^{-1} |z(t+s)| \Big] \\
&\leq \Big[\sum_{i\in I} |c_i| \Big(1 + \Big|1 - \frac{1}{\lambda_0}\Big| \sigma_i \Big) \lambda_0^{-\sigma_i} + \frac{1}{\lambda_0} \sum_{j\in J} |b_j|\, \tau_j \lambda_0^{-\tau_j} \Big] M(\lambda_0;\phi) \\
&= \mu(\lambda_0) M(\lambda_0;\phi)\,.
\end{aligned}$$

That is,

$$|z(t)| \leq \mu(\lambda_0) M(\lambda_0;\phi) \quad \text{for all} \quad t \geq 0\,. \tag{2.17}$$

Having in mind (2.15) and (2.17), we can use (2.10) to conclude, by the induction principle, that z satisfies

$$|z(t)| \leq [\mu(\lambda_0)]^{\nu} M(\lambda_0;\phi) \quad \text{for all} \quad t \geq (\nu-1)r \qquad (\nu = 0, 1, \dots)\,. \tag{2.18}$$

But, it follows from (2.13) that $\lim_{\nu\to\infty} [\mu(\lambda_0)]^{\nu} = 0$. Hence, it is easy to see that (2.18) implies (2.12).

The proof of the theorem is complete. □

Theorem 2.3 below gives a useful inequality for the solutions of the neutral delay difference equation (2.1).

Theorem 2.3. *Let λ_0 be a positive root of the characteristic equation* (2.3) *such that* (2.4) *holds. Let $\gamma(\lambda_0)$ be as in Theorem 2.1, and define*

$$\mu(\lambda_0) = \sum_{i\in I} |c_i| \Big(1 + \Big|1 - \frac{1}{\lambda_0}\Big| \sigma_i \Big) \lambda_0^{-\sigma_i} + \frac{1}{\lambda_0} \sum_{j\in J} |b_j| \tau_j \lambda_0^{-\tau_j}\,.$$

Then the solution x of the IVP (2.1) *and* (2.2) *satisfies*

$$\Big| \int_t^{t+1} \lambda_0^{-u} x(u)\, du \Big| \leq P(\lambda_0) \|\phi\| \quad \textit{for all} \quad t \geq 0\,,$$

where

$$P(\lambda_0) = \frac{1+\mu(\lambda_0)}{1+\gamma(\lambda_0)} \max\Big\{1, \frac{1}{\lambda_0}\Big\} + \mu(\lambda_0) \Big[\frac{1+\mu(\lambda_0)}{1+\gamma(\lambda_0)} \max\Big\{1, \lambda_0^r\Big\} + \max\Big\{\frac{1}{\lambda_0}, \lambda_0^r\Big\} \Big]$$

and

$$\|\phi\| = \sup_{-r \le t \le 1} \left|\phi(t)\right| .$$

The constant $P(\lambda_0)$ is greater than 1.

By applying Theorem 2.3 with $\lambda_0 = 1$, we can arrive at the following particular result:

Corollary 2.4. *Assume that* (2.5) *is satisfied. Then the solution x of the IVP* (2.1) *and* (2.2) *satisfies*

$$\Big| \int_t^{t+1} x(u)\,du \Big| \le p\|\phi\| \quad \textit{for all} \quad t \ge 0\,,$$

where

$$p = \frac{\left(1 + \sum_{i\in I} |c_i| + \sum_{j\in J} |b_j|\,\tau_j\right)^2}{1 + \sum_{i\in I} c_i + \sum_{j\in J} b_j\tau_j} + \sum_{i\in I} |c_i| + \sum_{j\in J} |b_j|\,\tau_j$$

and $\|\phi\|$ is defined as in Theorem 2.3. The constant p is greater than 1.

In Theorems 2.1 and 2.3, we have used a positive root λ_0 of the characteristic equation (2.3) such that (2.4) holds. The following lemma due to Kordonis and the first author [19] provides sufficient conditions (on the coefficients and the delays of the neutral delay difference equation (2.1)) for the characteristic equation (2.3) to have a positive root λ_0 satisfying (2.4).

Lemma 2.5 ([19])**.** *Assume that*

$$\frac{1}{r+1}\Big[1 + \sum_{i\in I} c_i \Big(1 + \frac{1}{r}\Big)^{\sigma_i}\Big] + a + \sum_{j\in J} b_j \Big(1 + \frac{1}{r}\Big)^{\tau_j} > 0$$

and

$$\sum_{i\in I} |c_i| \Big[1 + \Big(2 + \frac{1}{r}\Big)\sigma_i\Big]\Big(1 + \frac{1}{r}\Big)^{\sigma_i} + \Big(1 + \frac{1}{r}\Big)\sum_{j\in J} |b_j|\tau_j \Big(1 + \frac{1}{r}\Big)^{\tau_j} \le 1\,.$$

Then, in the interval $(\frac{r}{r+1}, \infty)$, the characteristic equation (2.3) *has a unique* (*positive*) *root λ_0; this root is such that* (2.4) *holds.*

Proof of Theorem 2.3. Consider the constant $L(\lambda_0;\phi)$ defined as in Theorem 2.1. Then we get

$$\begin{aligned} |L(\lambda_0;\phi)| \le & \int_0^1 \lambda_0^{-u}\,|\phi(u)|\,du \\ & + \sum_{i\in I} |c_i|\lambda_0^{-\sigma_i}\Big[\int_{-\sigma_i}^{-\sigma_i+1} \lambda_0^{-u}\,|\phi(u)|\,du + \Big|1 - \frac{1}{\lambda_0}\Big| \int_{-\sigma_i}^{0} \lambda_0^{-u}\big|\phi(u)\big|\,du\Big] \end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{\lambda_0} \sum_{j\in J} |b_j| \lambda_0^{-\tau_j} \Big[\int_{-\tau_j}^{0} \lambda_0^{-u} |\phi(u)|\, du \Big] \\
&\leq \Big[\int_0^1 \lambda_0^{-u}\, du + \sum_{i\in I} |c_i|\, \lambda_0^{-\sigma_i} \Big(\int_{-\sigma_i}^{-\sigma_i+1} \lambda_0^{-u}\, du + \Big| 1 - \frac{1}{\lambda_0} \Big| \int_{-\sigma_i}^{0} \lambda_0^{-u}\, du \Big) \\
&+ \frac{1}{\lambda_0} \sum_{j\in J} |b_j| \lambda_0^{-\tau_j} \Big(\int_{-\tau_j}^{0} \lambda_0^{-u}\, du \Big) \Big] \|\phi\| \,.
\end{aligned}
$$

We observe that

$$\max_{u\in[0,1]} \lambda_0^{-u} = \max\Big\{1, \frac{1}{\lambda_0}\Big\}$$

and consequently

$$\int_0^1 \lambda_0^{-u}\, du \leq \max\Big\{1, \frac{1}{\lambda_0}\Big\} \,.$$

Moreover, we see that

$$\max_{u\in[-r,0]} \lambda_0^{-u} = \max\{1, \lambda_0^r\}$$

and so we have

$$
\begin{aligned}
\int_{-\sigma_i}^{-\sigma_i+1} \lambda_0^{-u}\, du &\leq \max\{1, \lambda_0^r\} \quad \text{for} \quad i \in I \,, \\
\int_{-\sigma_i}^{0} \lambda_0^{-u}\, du &\leq \sigma_i \max\{1, \lambda_0^r\} \quad \text{for} \quad i \in I
\end{aligned}
$$

and

$$\int_{-\tau_j}^{0} \lambda_0^{-u}\, du \leq \tau_j \max\{1, \lambda_0^r\} \quad \text{for} \quad j \in J \,.$$

After the above observations, we find

$$
\begin{aligned}
\big| L(\lambda_0; \phi) \big| \leq \Big\{ \max\Big\{1, \frac{1}{\lambda_0}\Big\} &+ \Big[\sum_{i\in I} |c_i| \Big(1 + \Big| 1 - \frac{1}{\lambda_0} \Big| \sigma_i \Big) \lambda_0^{-\sigma_i} \\
&+ \frac{1}{\lambda_0} \sum_{j\in J} |b_j| \tau_j \lambda_0^{-\tau_j} \Big] \max\{1, \lambda_0^r\} \Big\} \|\phi\| \,.
\end{aligned}
$$

Hence, by the definition of $\mu(\lambda_0)$, it holds

$$\big| L(\lambda_0; \phi) \big| \leq \Big[\max\Big\{1, \frac{1}{\lambda_0}\Big\} + \mu(\lambda_0) \max\{1, \lambda_0^r\} \Big] \|\phi\| \,. \tag{2.19}$$

Now, let x be the solution of the IVP (2.1) and (2.2), and define the functions y, Y and z as in the proof of Theorem 2.1. Moreover, consider the constant $M(\lambda_0; \phi)$ defined as in the proof of Theorem 2.1. As it has been shown in the proof of

Theorem 2.1, the function z satisfies (2.17). By the definition of the function z, it follows from (2.17) that

$$|Y(t)| \le \frac{|L(\lambda_0;\phi)|}{1+\gamma(\lambda_0)} + \mu(\lambda_0) M(\lambda_0;\phi) \quad \text{for every} \quad t \ge 0\,. \tag{2.20}$$

Since

$$\max_{u\in[-r,1]} \lambda_0^{-u} = \max\left\{\frac{1}{\lambda_0}, \lambda_0^r\right\},$$

we have

$$\int_t^{t+1} \lambda_0^{-u}\,du \le \max\left\{\frac{1}{\lambda_0}, \lambda_0^r\right\} \quad \text{for} \quad -r \le t \le 0\,.$$

Thus, by taking into account the definition of $M(\lambda_0;\phi)$, we obtain

$$\begin{aligned}
M(\lambda_0;\phi) &\le \max_{-r\le t\le 0}\Big[\int_t^{t+1} \lambda_0^{-u}\big|\phi(u)\big|\,du\Big] + \frac{|L(\lambda_0;\phi)|}{1+\gamma(\lambda_0)} \\
&\le \Big[\max_{-r\le t\le 0}\Big(\int_t^{t+1} \lambda_0^{-u}\,du\Big)\Big]\|\phi\| + \frac{|L(\lambda_0;\phi)|}{1+\gamma(\lambda_0)} \\
&\le \Big(\max\Big\{\frac{1}{\lambda_0}, \lambda_0^r\Big\}\Big)\|\phi\| + \frac{|L(\lambda_0;\phi)|}{1+\gamma(\lambda_0)}\,.
\end{aligned}$$

So, (2.20) gives

$$|Y(t)| \le \frac{1+\mu(\lambda_0)}{1+\gamma(\lambda_0)}\,|L(\lambda_0;\phi)| + \mu(\lambda_0)\Big(\max\Big\{\frac{1}{\lambda_0}, \lambda_0^r\Big\}\Big)\|\phi\| \quad \text{for} \quad t \ge 0\,. \tag{2.21}$$

By combining (2.19) and (2.21), we obtain, for each $t \ge 0$,

$$\begin{aligned}
\big|Y(t)\big| &\le \Big\{\frac{1+\mu(\lambda_0)}{1+\gamma(\lambda_0)}\Big[\max\Big\{1, \frac{1}{\lambda_0}\Big\} + \mu(\lambda_0)\max\big\{1, \lambda_0^r\big\}\Big] + \mu(\lambda_0)\max\Big\{\frac{1}{\lambda_0}, \lambda_0^r\Big\}\Big\}\,\|\phi\| \\
&= \Big\{\frac{1+\mu(\lambda_0)}{1+\gamma(\lambda_0)}\max\Big\{1, \frac{1}{\lambda_0}\Big\} + \mu(\lambda_0)\Big[\frac{1+\mu(\lambda_0)}{1+\gamma(\lambda_0)}\max\{1, \lambda_0^r\} \\
&\quad + \max\Big\{\frac{1}{\lambda_0}, \lambda_0^r\Big\}\Big]\Big\}\,\|\phi\|\,.
\end{aligned}$$

Hence, because of the definition of $P(\lambda_0)$, we have

$$|Y(t)| \le P(\lambda_0)\,\|\phi\| \quad \text{for all} \quad t \ge 0\,.$$

By taking into account the definitions of the functions y and Y, we immediately see that the last inequality coincides with the inequality in the conclusion of the theorem. Finally, as $|\gamma(\lambda_0)| \le \mu(\lambda_0)$, we have

$$\frac{1+\mu(\lambda_0)}{1+\gamma(\lambda_0)} \ge 1\,.$$

Also, it holds

$$\max\Big\{1, \frac{1}{\lambda_0}\Big\} \ge 1\,.$$

So, it is easy to conclude that $P(\lambda_0) > 1$.

The proof of the theorem is now complete. $\square$

We proceed with a result (Theorem 2.7 below) concerning the behavior of the solutions of the neutral delay difference equation (2.1); this result will be established via two distinct positive roots of the characteristic equation (2.3). Before stating and proving Theorem 2.7, we give a lemma obtained by the authors in [33], which provides some useful information about the positive roots of the characteristic equation (2.3).

Lemma 2.6 ([33]). *Suppose that*

$$c_i \leq 0 \quad \text{for} \quad i \in I\,, \quad \text{and} \quad b_j < 0 \quad \text{for} \quad j \in J\,.$$

(I) *Let λ_0 be a positive root of the characteristic equation* (2.3) *with $\lambda_0 \leq 1$, and let $\gamma(\lambda_0)$ be defined as in Theorem 2.1. Then*

$$1 + \gamma(\lambda_0) > 0$$

if (2.3) *has another positive root less than λ_0, and*

$$1 + \gamma(\lambda_0) < 0$$

if (2.3) *has another positive root greater than λ_0 and less than or equal to* 1.

(II) *If $a = 0$, then $\lambda = 1$ is not a root of the characteristic equation* (2.3).

(III) *Assume that $a = 0$ and that*

$$\sum_{i \in I} (-c_i) \leq 1\,.$$

Then, in the interval $(1, \infty)$, the characteristic equation (2.3) *has no roots.*

(IV) *Assume that*

$$\sum_{j \in J} (-b_j) \geq a \tag{2.22}$$

and

$$\sum_{i \in I} (-c_i) + \sum_{j \in J} (-b_j)\tau_j \leq 1\,.$$

Then, in the interval $(1, \infty)$, the characteristic equation (2.3) *has no roots.*

(V) *Assume that* (2.22) *holds, and that*

$$\sum_{i \in I} (-c_i) \frac{(r+1)^{\sigma_i}}{r^{\sigma_i}} + \sum_{j \in J} (-b_j) \frac{(r+1)^{\tau_j + 1}}{r^{\tau_j}} < 1 + a(r+1)\,. \tag{2.23}$$

Then: (i) *$\lambda = \frac{r}{r+1}$ is not a root of the characteristic equation* (2.3). (ii) *In the interval $(\frac{r}{r+1}, 1]$,* (2.3) *has a unique root.* (iii) *In the interval $(0, \frac{r}{r+1})$,* (2.3) *has a unique root.* (*Note: Assumption* (2.23) *guarantees that $1 + a(r+1) > 0$ and so $a > -\frac{1}{r+1}$.*)

Theorem 2.7. *Suppose that*

$$c_i \leq 0 \quad \text{for} \quad i \in I\,, \quad \text{and} \quad b_j < 0 \quad \text{for} \quad j \in J\,.$$

Let λ_0 be a positive root of the characteristic equation (2.3) *with $\lambda_0 \leq 1$ and such that*

$$1 + \gamma(\lambda_0) \neq 0\,,$$

where $\gamma(\lambda_0)$ is defined as in Theorem 2.1. Let also λ_1 be a positive root of (2.3) *with $\lambda_1 \neq \lambda_0$.*

Then the solution x of the IVP (2.1) *and* (2.2) *satisfies*

$$U(\lambda_0, \lambda_1; \phi) \leq \Big(\frac{\lambda_0}{\lambda_1}\Big)^t \Big[\int_t^{t+1} \lambda_0^{-u} x(u)\, du - \frac{L(\lambda_0; \phi)}{1+\gamma(\lambda_0)}\Big] \leq V(\lambda_0, \lambda_1; \phi)$$

for all $t \geq 0$, where $L(\lambda_0; \phi)$ is defined as in Theorem 2.1 and:

$$U(\lambda_0, \lambda_1; \phi) = \min_{-r \leq t \leq 0} \Big\{ \Big(\frac{\lambda_0}{\lambda_1}\Big)^t \Big[\int_t^{t+1} \lambda_0^{-u} \phi(u)\, du - \frac{L(\lambda_0; \phi)}{1+\gamma(\lambda_0)}\Big]\Big\},$$

$$V(\lambda_0, \lambda_1; \phi) = \max_{-r \leq t \leq 0} \Big\{ \Big(\frac{\lambda_0}{\lambda_1}\Big)^t \Big[\int_t^{t+1} \lambda_0^{-u} \phi(u)\, du - \frac{L(\lambda_0; \phi)}{1+\gamma(\lambda_0)}\Big]\Big\}.$$

Note. By Lemma 2.6 (Part (I)), we always have $1 + \gamma(\lambda_0) \neq 0$ if $\lambda_1 \leq 1$.

We immediately observe that the double inequality in the conclusion of Theorem 2.7 can equivalently be written in the following form

$$\begin{aligned} U(\lambda_0, \lambda_1; \phi)\Big(\frac{\lambda_1}{\lambda_0}\Big)^t &\leq \int_t^{t+1} \lambda_0^{-u} x(u)\, du - \frac{L(\lambda_0; \phi)}{1+\gamma(\lambda_0)} \\ &\leq V(\lambda_0, \lambda_1; \phi)\Big(\frac{\lambda_1}{\lambda_0}\Big)^t \quad \text{for} \quad t \geq 0\,. \end{aligned}$$

Consequently, we have

$$\lim_{t\to\infty} \int_t^{t+1} \lambda_0^{-u} x(u)\, du = \frac{L(\lambda_0; \phi)}{1+\gamma(\lambda_0)}\,,$$

provided that $\lambda_1 < \lambda_0$.

Proof of Theorem 2.7. Consider the solution x of the IVP (2.1) and (2.2), and let y, Y and z be defined as in the proof of Theorem 2.1. As it has been proved in the proof of Theorem 2.1, the fact that x satisfies (2.1) for $t \geq 0$ is equivalent to the fact that z satisfies (2.10). Also, the initial condition (2.2) can equivalently be written in the form (2.11). Furthermore, let us define

$$w(t) = \Big(\frac{\lambda_0}{\lambda_1}\Big)^t z(t) \quad \text{for} \quad t \geq -r\,.$$

Then we can see that (2.10) reduces to the following equivalent equation

$$\begin{aligned} w(t) + \sum_{i\in I} c_i \lambda_1^{-\sigma_i} w(t-\sigma_i) &= \Big(1 - \frac{1}{\lambda_0}\Big) \sum_{i\in I} c_i \lambda_0^{-\sigma_i} \Big[\sum_{s=-\sigma_i}^{-1} \Big(\frac{\lambda_0}{\lambda_1}\Big)^{-s} w(t+s)\Big] \\ &\quad - \frac{1}{\lambda_0} \sum_{j\in J} b_j \lambda_0^{-\tau_j} \Big[\sum_{s=-\tau_j}^{-1} \Big(\frac{\lambda_0}{\lambda_1}\Big)^{-s} w(t+s)\Big] \quad \text{for} \quad t \geq 0\,. \end{aligned} \tag{2.24}$$

On the other hand, the initial condition (2.11) becomes

$$w(t) = \Big(\frac{\lambda_0}{\lambda_1}\Big)^t \Big[\int_t^{t+1} \lambda_0^{-u} \phi(u)\, du - \frac{L(\lambda_0; \phi)}{1+\gamma(\lambda_0)}\Big] \quad \text{for} \quad -r \leq t \leq 0\,. \tag{2.25}$$

By taking into account the definitions of y, Y, z and w, we have

$$w(t) = \Big(\frac{\lambda_0}{\lambda_1}\Big)^t \Big[\int_t^{t+1} \lambda_0^{-u} x(u)\,du - \frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)}\Big] \quad \text{for} \quad t \geq -r\,.$$

Moreover, it follows from (2.25) that

$$U(\lambda_0,\lambda_1;\phi) = \min_{-r\leq t\leq 0} w(t) \quad \text{and} \quad V(\lambda_0,\lambda_1;\phi) = \max_{-r\leq t\leq 0} w(t)\,.$$

So, what we have to prove is that w satisfies

$$\min_{-r\leq s\leq 0} w(s) \leq w(t) \leq \max_{-r\leq s\leq 0} w(s) \quad \text{for all} \quad t \geq 0\,.$$

We will confine our discussion only to proving the inequality

$$w(t) \geq \min_{-r\leq s\leq 0} w(s) \quad \text{for every} \quad t \geq 0\,. \tag{2.26}$$

The inequality

$$w(t) \leq \max_{-r\leq s\leq 0} w(s) \quad \text{for every} \quad t \geq 0$$

can be shown by an analogous procedure. In the rest of the proof, we will establish (2.26).

The proof that (2.26) holds can be accomplished, by showing that, for any real number D with $D < \min\limits_{-r\leq s\leq 0} w(s)$, it holds

$$w(t) > D \quad \text{for all} \quad t \geq 0\,. \tag{2.27}$$

For this purpose, let us consider an arbitrary real number D with $D < \min\limits_{-r\leq s\leq 0} w(s)$. Then we obviously have

$$w(t) > D \quad \text{for} \quad -r \leq t \leq 0\,. \tag{2.28}$$

Assume, for the sake of contradiction, that (2.27) fails to hold. Then, because of (2.28), there exists a point $t_0 > 0$ so that

$$w(t) > D \quad \text{for} \quad -r \leq t < t_0\,, \quad \text{and} \quad w(t_0) = D\,.$$

Hence, by using the hypothesis that $c_i \leq 0$ for $i \in I$ and $b_j < 0$ for $j \in J$ and taking into account the assumption that $\lambda_0 \leq 1$, i.e., that $1 - \frac{1}{\lambda_0} \leq 0$, from (2.24) we obtain

$$
\begin{aligned}
D = w(t_0) &= -\sum_{i\in I} c_i\lambda_1^{-\sigma_i} w(t_0-\sigma_i) + \Big(1-\frac{1}{\lambda_0}\Big)\sum_{i\in I} c_i\lambda_0^{-\sigma_i}\Big[\sum_{s=-\sigma_i}^{-1}\Big(\frac{\lambda_0}{\lambda_1}\Big)^{-s} w(t+s)\Big] \\
&\quad - \frac{1}{\lambda_0}\sum_{j\in J} b_j\lambda_0^{-\tau_j}\Big[\sum_{s=-\tau_j}^{-1}\Big(\frac{\lambda_0}{\lambda_1}\Big)^{-s} w(t+s)\Big] \\
&> D\Big\{-\sum_{i\in I} c_i\lambda_1^{-\sigma_i} + \Big(1-\frac{1}{\lambda_0}\Big)\sum_{i\in I} c_i\lambda_0^{-\sigma_i}\Big[\sum_{s=-\sigma_i}^{-1}\Big(\frac{\lambda_0}{\lambda_1}\Big)^{-s}\Big] \\
&\quad - \frac{1}{\lambda_0}\sum_{j\in J} b_j\lambda_0^{-\tau_j}\Big[\sum_{s=-\tau_j}^{-1}\Big(\frac{\lambda_0}{\lambda_1}\Big)^{-s}\Big]\Big\} \\
&= D\Big\{-\sum_{i\in I} c_i\lambda_1^{-\sigma_i} + \Big(1-\frac{1}{\lambda_0}\Big)\sum_{i\in I} c_i\lambda_0^{-\sigma_i}\Big[\sum_{\nu=1}^{\sigma_i}\Big(\frac{\lambda_0}{\lambda_1}\Big)^{\nu}\Big] \\
&\quad - \frac{1}{\lambda_0}\sum_{j\in J} b_j\lambda_0^{-\tau_j}\Big[\sum_{\nu=1}^{\tau_j}\Big(\frac{\lambda_0}{\lambda_1}\Big)^{\nu}\Big\} \\
&= \frac{D}{\lambda_0-\lambda_1}\Big\{-(\lambda_0-\lambda_1)\sum_{i\in I} c_i\lambda_1^{-\sigma_i} + (\lambda_0-1)\sum_{i\in I} c_i\lambda_0^{-\sigma_i}\Big[\Big(\frac{\lambda_0}{\lambda_1}\Big)^{\sigma_i}-1\Big] \\
&\quad - \sum_{j\in J} b_j\lambda_0^{-\tau_j}\Big[\Big(\frac{\lambda_0}{\lambda_1}\Big)^{\tau_j}-1\Big]\Big\} \\
&= \frac{D}{\lambda_0-\lambda_1}\Big\{-\big[(\lambda_0-1)-(\lambda_1-1)\big]\sum_{i\in I} c_i\lambda_1^{-\sigma_i} + (\lambda_0-1)\sum_{i\in I} c_i\left(\lambda_1^{-\sigma_i}-\lambda_0^{-\sigma_i}\right) \\
&\quad - \sum_{j\in J} b_j\big(\lambda_1^{-\tau_j}-\lambda_0^{-\tau_j}\big)\Big\} \\
&= \frac{D}{\lambda_0-\lambda_1}\Big\{\Big[-(\lambda_0-1)\sum_{i\in I} c_i\lambda_0^{-\sigma_i} + \sum_{j\in J} b_j\lambda_0^{-\tau_j}\Big] \\
&\quad - \Big[-(\lambda_1-1)\sum_{i\in I} c_i\lambda_1^{-\sigma_i} + \sum_{j\in J} b_j\lambda_1^{-\tau_j}\Big]\Big\} \\
&= \frac{D}{\lambda_0-\lambda_1}\big[(\lambda_0-1-a)-(\lambda_1-1-a)\big] = D\,.
\end{aligned}
$$

We have thus arrived at a contradiction, which shows that (2.27) is always satisfied.

The proof of the theorem has been completed. □

3. Autonomous linear delay difference equations with continuous variable

In this section, we will concentrate on the special case of the difference equation (2.1), where *the coefficients* c_i *for* $i \in I$ *are equal to zero*, and the initial segment of natural numbers I and the delays σ_i for $i \in I$ are chosen arbitrarily so that $\max_{i\in I} \sigma_i \equiv \sigma \leq \tau \equiv \max_{j\in J} \tau_j$. (For example, it can be considered that $I = J$, and $\sigma_i = \tau_i$ for $i \in I$.) In this particular case, the difference equation (2.1) reduces to the (*non-neutral*) delay difference equation with continuous variable

$$\Delta x(t) = ax(t) + \sum_{j\in J} b_j x(t - \tau_j)\,. \tag{3.1}$$

As it concerns the delay difference equation (3.1), we have the integer $\tau \equiv \max_{j\in J} \tau_j$ in place of the integer r (which is used in the general case of the neutral delay difference equation (2.1)). A *solution* of the delay difference equation (3.1) is a continuous real-valued function x defined on the interval $[-\tau, \infty)$, which satisfies (3.1) for all $t \geq 0$.

We will consider the *initial value problem* (IVP, for short) consisting of the delay difference equation (3.1) and an *initial condition* of the form

$$x(t) = \psi(t) \quad \text{for} \quad -\tau \leq t \leq 1\,, \tag{3.2}$$

where *the initial function* ψ *is a given continuous real-valued function on the interval* $[-\tau, 1]$ *satisfying the "consistency condition"*

$$\psi(1) - \psi(0) = a\psi(0) + \sum_{j\in J} b_j \psi(-\tau_j)\,.$$

The initial value problem (3.1) and (3.2) (more briefly, the IVP (3.1) and (3.2)) has a unique *solution* x; that is, there exists a unique solution x of the delay difference equation (3.1) which satisfies the initial condition (3.2).

The *characteristic equation* of the delay difference equation (3.1) is

$$\lambda - 1 = a + \sum_{j\in J} b_j \lambda^{-\tau_j}\,. \tag{3.3}$$

In the special case of the (non-neutral) delay difference equation (3.1), Theorem 2.1, Corollary 2.2, Theorem 2.3 and Corollary 2.4 are formulated as follows:

Theorem 3.1. *Let* λ_0 *be a positive root of the characteristic equation* (3.3) *such that*

$$\frac{1}{\lambda_0} \sum_{j\in J} |b_j|\, \tau_j \lambda_0^{-\tau_j} < 1\,. \tag{3.4}$$

Then the solution x *of the IVP* (3.1) *and* (3.2) *satisfies*

$$\lim_{t\to\infty} \int_t^{t+1} \lambda_0^{-u} x(u)\, du = \frac{L_0(\lambda_0; \psi)}{1 + \frac{1}{\lambda_0} \sum_{j\in J} b_j \tau_j \lambda_0^{-\tau_j}}\,,$$

where

$$L_0(\lambda_0;\psi) = \int_0^1 \lambda_0^{-u}\psi(u)\,du + \frac{1}{\lambda_0}\sum_{j\in J} b_j\lambda_0^{-\tau_j}\Big[\int_{-\tau_j}^0 \lambda_0^{-u}\psi(u)\,du\Big].$$

Note. Condition (3.4) guarantees that $1+\frac{1}{\lambda_0}\sum_{j\in J} b_j\tau_j\lambda_0^{-\tau_j} > 0$.

Corollary 3.2. *Assume that*

$$a+\sum_{j\in J} b_j = 0 \quad \textit{and} \quad \sum_{j\in J}|b_j|\,\tau_j < 1\,. \tag{3.5}$$

Then the solution x of the IVP (3.1) *and* (3.2) *satisfies*

$$\lim_{t\to\infty}\int_t^{t+1} x(u)\,du = \frac{\int_0^1 \psi(u)\,du + \sum_{j\in J} b_j\left[\int_{-\tau_j}^0 \psi(u)\,du\right]}{1+\sum_{j\in J} b_j\tau_j}\,.$$

Note. The second condition of (3.5) guarantees that $1+\sum_{j\in J} b_j\tau_j > 0$.

Theorem 3.3. *Let λ_0 be a positive root of the characteristic equation* (3.3) *such that* (3.4) *holds. Then the solution x of the IVP* (3.1) *and* (3.2) *satisfies*

$$\Big|\int_t^{t+1} \lambda_0^{-u}x(u)\,du\Big| \le P_0(\lambda_0)\,\|\psi\| \quad \textit{for all} \quad t\ge 0\,,$$

where

$$P_0(\lambda_0) = \frac{1+\frac{1}{\lambda_0}\sum_{j\in J}|b_j|\,\tau_j\lambda_0^{-\tau_j}}{1+\frac{1}{\lambda_0}\sum_{j\in J} b_j\tau_j\lambda_0^{-\tau_j}}\max\Big\{1,\frac{1}{\lambda_0}\Big\} + \Big(\frac{1}{\lambda_0}\sum_{j\in J}|b_j|\,\tau_j\lambda_0^{-\tau_j}\Big)$$
$$\times\Big(\frac{1+\frac{1}{\lambda_0}\sum_{j\in J}|b_j|\,\tau_j\lambda_0^{-\tau_j}}{1+\frac{1}{\lambda_0}\sum_{j\in J} b_j\tau_j\lambda_0^{-\tau_j}}\max\{1,\lambda_0^r\} + \max\Big\{\frac{1}{\lambda_0},\lambda_0^r\Big\}\Big)$$

and

$$\|\psi\| = \sup_{-\tau\le t\le 1}|\psi(t)|\,.$$

The constant $P_0(\lambda_0)$ is greater than 1.

Corollary 3.4. *Assume that* (3.5) *is satisfied. Then the solution x of the IVP* (3.1) *and* (3.2) *satisfies*

$$\Big|\int_t^{t+1} x(u)\,du\Big| \le p_0\,\|\psi\| \quad \textit{for all} \quad t\ge 0\,,$$

where

$$p_0 = \frac{\left(1+\sum_{j\in J}|b_j|\,\tau_j\right)^2}{1+\sum_{j\in J} b_j\tau_j} + \sum_{j\in J}|b_j|\,\tau_j$$

and $\|\psi\|$ is defined as in Theorem 3.3. The constant p_0 is greater than 1.

Lemma 3.5 below gives sufficient conditions (on the coefficients and the delays of the delay difference equation (3.1)) for the characteristic equation (3.3) to have a positive root λ_0 such that (3.4) holds. This lemma has been established by Kordonis and the authors [20]; it is also a consequence of Lemma 2.5.

Lemma 3.5 ([20]). *Assume that*

$$\sum_{j\in J} b_j \frac{(\tau+1)^{\tau_j+1}}{\tau^{\tau_j}} > -1 - a(\tau+1)$$

and

$$\sum_{j\in J} |b_j| \frac{\tau_j}{\tau} \cdot \frac{(\tau+1)^{\tau_j+1}}{\tau^{\tau_j}} \le 1\,.$$

Then, in the interval $(\frac{\tau}{\tau+1}, \infty)$, *the characteristic equation* (3.3) *has a unique* (*positive*) *root* λ_0; *this root is such that* (3.4) *holds.*

The following lemma due to the authors [33] is concerned with the positive roots of the characteristic equation (3.3).

Lemma 3.6 ([33]). *Suppose that*

$$b_j < 0 \quad for \quad j \in J\,.$$

(I) *Let* λ_0 *be a positive root of the characteristic equation* (3.3). *Then*

$$1 + \frac{1}{\lambda_0} \sum_{j\in J} b_j \tau_j \lambda_0^{-\tau_j} > 0$$

if (3.3) *has another positive root less that* λ_0, *and*

$$1 + \frac{1}{\lambda_0} \sum_{j\in J} b_j \tau_j \lambda_0^{-\tau_j} < 0$$

if (3.3) *has another positive root greater than* λ_0.

(II) $a > -1$ *is a necessary condition for the characteristic equation* (3.3) *to have at least one positive root.*

(III) *The characteristic equation* (3.3) *has no positive roots greater than or equal to* $a+1$.

(IV) *Assume that*

$$\sum_{j\in J} (-b_j) \frac{(\tau+1)^{\tau_j+1}}{\tau^{\tau_j}} < 1 + a(\tau+1)\,.$$

[*This condition implies that* $1+a(\tau+1) > 0$ *and so* $a+1 > \frac{\tau}{\tau+1}$.] *Then:* (i) $\lambda = \frac{\tau}{\tau+1}$ *is not a root of the characteristic equation* (3.3). (ii) *In the interval* $(\frac{\tau}{\tau+1}, a+1)$, (3.3) *has a unique root.* (iii) *In the interval* $(0, \frac{\tau}{\tau+1})$, (3.3) *has a unique root.*

The need in assuming, in Theorem 2.7, that the root λ_0 of the characteristic equation (2.3) is such that $\lambda_0 \le 1$ is due only to the existence of the term

$$\Big(1 - \frac{1}{\lambda_0}\Big) \sum_{i\in I} c_i \lambda_0^{-\sigma_i} \Big[\sum_{s=-\sigma_i}^{-1} \Big(\frac{\lambda_0}{\lambda_1}\Big)^{-s} w(t+s) \Big]$$

in (2.24). This term does not exist in the special case of the (non-neutral) delay difference equation (3.1). More precisely, in this particular case, (2.24) becomes

$$w(t) = -\frac{1}{\lambda_0}\sum_{j\in J} b_j\lambda_0^{-\tau_j}\Big[\sum_{s=-\tau_j}^{-1}\Big(\frac{\lambda_0}{\lambda_1}\Big)^{-s} w(t+s)\Big] \quad \text{for} \quad t \geq 0\,.$$

So, following the lines of the proof of Theorem 2.7 and taking into consideration the above observation, we can prove the following theorem.

Theorem 3.7. *Suppose that*

$$b_j < 0 \quad \textit{for} \quad j \in J\,,$$

and let λ_0 *and* λ_1, $\lambda_0 \neq \lambda_1$, *be two positive roots of the characteristic equation* (3.3). *Then the solution* x *of the IVP* (3.1) *and* (3.2) *satisfies*

$$U_0(\lambda_0,\lambda_1;\psi) \leq \Big(\frac{\lambda_0}{\lambda_1}\Big)^t\Big[\int_t^{t+1}\lambda_0^{-u}x(u)\,du - \frac{L_0(\lambda_0;\psi)}{1+\frac{1}{\lambda_0}\sum_{j\in J} b_j\tau_j\lambda_0^{-\tau_j}}\Big]$$
$$\leq V_0(\lambda_0,\lambda_1;\psi) \quad \textit{for all} \quad t \geq 0\,,$$

where $L_0(\lambda_0;\psi)$ *is defined as in Theorem 3.1 and:*

$$U_0(\lambda_0,\lambda_1;\psi) = \min_{-\tau\leq t\leq 0}\Big\{\Big(\frac{\lambda_0}{\lambda_1}\Big)^t\Big[\int_t^{t+1}\lambda_0^{-u}\psi(u)\,du - \frac{L_0(\lambda_0;\psi)}{1+\frac{1}{\lambda_0}\sum_{j\in J} b_j\tau_j\lambda_0^{-\tau_j}}\Big]\Big\}\,,$$

$$V_0(\lambda_0,\lambda_1;\psi) = \max_{-\tau\leq t\leq 0}\Big\{\Big(\frac{\lambda_0}{\lambda_1}\Big)^t\Big[\int_t^{t+1}\lambda_0^{-u}\psi(u)\,du - \frac{L_0(\lambda_0;\psi)}{1+\frac{1}{\lambda_0}\sum_{j\in J} b_j\tau_j\lambda_0^{-\tau_j}}\Big]\Big\}\,.$$

Note. By Lemma 3.6 (Part (I)), we always have $1+\frac{1}{\lambda_0}\sum_{j\in J} b_j\tau_j\lambda_0^{-\tau_j} \neq 0$.

Now, let us consider the delay difference equation with continuous variable

$$w(t) - w(t-\theta) = aw(t-\theta) + \sum_{j\in J} b_j w(t-\eta_j), \tag{3.6}$$

where θ is a positive real number, and η_j for $j \in J$ are real numbers such that: $\eta_j > \theta$ for $j \in J$, and $\eta_{j_1} \neq \eta_{j_2}$ for $j_1, j_2 \in J$ with $j_1 \neq j_2$. Let us assume that there exist integers $m_j > 1$ for $j \in J$ so that $\eta_j = m_j\theta$ for $j \in J$. Consider the positive real number η defined by $\eta = \max_{j\in J}\eta_j$. By a solution of the delay difference equation (3.6), we mean a continuous real-valued function w defined on the interval $[-\eta,\infty)$ which satisfies (3.6) for all $t \geq 0$.

Set $\tau_j = m_j - 1$ for $j \in J$. Clearly, τ_j for $j \in J$ are positive integers such that $\tau_{j_1} \neq \tau_{j_2}$ for $j_1, j_2 \in J$ with $j_1 \neq j_2$. Moreover, we put $\tau = \max_{j\in J}\tau_j$. We immediately see that $\eta = (\tau+1)\theta$.

Let w be a solution of the delay difference equation (3.6), and define

$$x(t) = w\big(\theta(t-1)\big) \quad \text{for} \quad t \geq -\tau\,.$$

Then, for every $t \geq 0$, we obtain

$$\begin{aligned}
\Delta x(t) - ax(t) - \sum_{j\in J} b_j x(t-\tau_j) \\
&= w(\theta t) - w(\theta t - \theta) - aw(\theta t - \theta) - \sum_{j\in J} b_j w(\theta t - (\tau_j + 1)\theta) \\
&= w(\theta t) - w(\theta t - \theta) - aw(\theta t - \theta) - \sum_{j\in J} b_j w(\theta t - m_j\theta) \\
&= w(\theta t) - w(\theta t - \theta) - aw(\theta t - \theta) - \sum_{j\in J} b_j w(\theta t - \eta_j) = 0\,.
\end{aligned}$$

Consequently, x is a solution of the delay difference equation (3.1). Conversely, if x is a solution of (3.1), then the function w defined by

$$w(t) = x\Big(\frac{t}{\theta} + 1\Big) \quad \text{for} \quad t \geq -\eta$$

is a solution of (3.6).

Delay difference equations with continuous variable of the form (3.6) have been studied by the authors in [31] as well as in the last section of [33]. Note that, in [31], more general forced delay difference equations with continuous variable of the form

$$w(t) - w(t-\theta) = aw(t-\theta) + \sum_{j\in J} b_j w(t-\eta_j) + f(t)$$

are considered, where f is a continuous real-valued function on the interval $[0,\infty)$.

After the above analysis on the connection between the delay difference equations (3.1) and (3.6), it is not difficult to see that the main result in [31], applied to the unforced case, coincides with Theorem 3.1, while Theorem III in [33] is essentially the same result with Theorem 3.7. We notice here that Pituk [44] established an important result on Cesàro summability to a linear autonomous difference equation with continuous variable. A detailed comparison between Pituk's result and the authors' main result in [31] is contained in [44].

References

[1] Agarwal, R. P., *Difference Equations and Inequalities*, Marcel Dekker, New York, 1992.

[2] Arino, O. and Pituk, M., *More on linear differential systems with small delays*, J. Differential Equations **170** (2001), 381–407.

[3] de Bruijn, N. G., *On some linear functional equations*, Publ. Math. Debrecen **1** (1950), 129–134.

[4] Dix, J. G., Philos, Ch. G. and Purnaras, I. K., *An asymptotic property of solutions to linear nonautonomous delay differential equations*, Electron. J. Differential Equations **2005** (2005), No. 10, 1–9.

[5] Dix, J. G., Philos, Ch. G. and Purnaras, I. K., *Asymptotic properties of solutions to linear non-autonomous neutral differential equations*, J. Math. Anal. Appl. **318** (2006), 296–304.

[6] Domshlak, Y., *Oscillatory properties of linear difference equations with continuous time*, Differential Equations Dynam. Systems **1** (1993), 311–324.

[7] Driver, R. D., *Some harmless delays*, Delay and Functional Differential Equations and Their Applications, Academic Press, New York, 1972, 103–119.

[8] Driver, R. D., *Linear differential systems with small delays*, J. Differential Equations **21** (1976), 148–166.

[9] Driver, R. D., Ladas, G. and Vlahos, P. N., *Asymptotic behavior of a linear delay difference equation*, Proc. Amer. Math. Soc. **115** (1992), 105–112.

[10] Driver, R. D., Sasser, D. W. and Slater, M. L., *The equation $x'(t) = ax(t) + bx(t - \tau)$ with "small" delay*, Amer. Math. Monthly **80** (1973), 990–995.

[11] Elaydi, S. N., *An Introduction to Difference Equations*, Springer, New York, 1996.

[12] Frasson, M. V. S. and Verduyn Lunel, S. M., *Large time behaviour of linear functional differential equations*, Integral Equations Operator Theory **47** (2003), 91–121.

[13] Graef, J. R. and Qian, C., *Asymptotic behavior of forced delay equations with periodic coefficients*, Commun. Appl. Anal. **2** (1998), 551–564.

[14] Győri, I., *Invariant cones of positive initial functions for delay differential equations*, Appl. Anal. **35** (1990), 21–41.

[15] Győri, I., *Sharp conditions for existence of nontrivial invariant cones of nonnegative initial values of difference equations*, Appl. Math. Comput. **36** (1990), 89–111.

[16] Kelley, W. G. and Peterson, A. C., *Difference Equations: An Introduction with Applications*, Academic Press, Inc., New York, 1991.

[17] Kordonis, I.-G. E., Niyianni, N. T. and Philos, Ch. G., *On the behavior of the solutions of scalar first order linear autonomous neutral delay differential equations*, Arch. Math. (Basel) **71** (1998), 454–464.

[18] Kordonis, I.-G. E. and Philos, Ch. G., *The behavior of solutions of linear integro-differential equations with unbounded delay*, Comput. Math. Appl. **38** (1999), 45–50.

[19] Kordonis, I.-G. E. and Philos, Ch. G., *On the behavior of the solutions for linear autonomous neutral delay difference equations*, J. Differ. Equations Appl. **5** (1999), 219–233.

[20] Kordonis, I.-G. E., Philos, Ch. G. and Purnaras, I. K., *Some results on the behavior of the solutions of a linear delay difference equation with periodic coefficients*, Appl. Anal. **69** (1998), 83–104.

[21] Kordonis, I.-G. E., Philos, Ch. G. and Purnaras, I. K., *On the behavior of solutions of linear neutral integrodifferential equations with unbounded delay*, Georgian Math. J. **11** (2004), 337–348.

[22] Ladas, G., *Recent developments in the oscillation of delay difference equations*, Differential Equations (Colorado Springs, CO, 1989), Lecture Notes in Pure and Appl. Math. **127**, Dekker, New York, 1991, 321–332.

[23] Ladas, G., Pakula, L. and Wang, Z., *Necessary and sufficient conditions for the oscillation of difference equations*, Panamer. Math. J. **2** (1992), 17–26.

[24] Lakshmikantham, V. and Trigiante, D., *Theory of Difference Equations: Numerical Methods and Applications*, Academic Press, Inc., New York, 1988.

[25] Mickens, R., *Difference Equations*, Van Nostrand Reinhold, New York, 1990.

[26] Norris, M. J., *Unpublished notes on the delay differential equation $x'(t) = bx(t - 1)$ where* $-1/e \leq b < 0$, October 1967.

[27] Philos, Ch. G., *Asymptotic behaviour, nonoscillation and stability in periodic first-order linear delay differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **128** (1998), 1371–1387.

[28] Philos, Ch. G. and Purnaras, I. K., *Periodic first order linear neutral delay differential equations*, Appl. Math. Comput. **117** (2001), 203–222.

[29] Philos, Ch. G. and Purnaras, I. K., *The behavior of solutions of linear Volterra difference equations with infinite delay*, Comput. Math. Appl. **47** (2004), 1555–1563.

[30] Philos, Ch. G. and Purnaras, I. K., *Asymptotic properties, nonoscillation, and stability for scalar first order linear autonomous neutral delay differential equations*, Electron. J. Differential Equations **2004** (2004), No. 03, 1–17.

[31] Philos, Ch. G. and Purnaras, I. K., *An asymptotic result for some delay difference equations with continuous variable*, Advances in Difference Equations **2004:1** (2004), 1–10.

[32] Philos, Ch. G. and Purnaras, I. K., *An asymptotic result and a stability criterion for linear nonautonomous delay difference equations*, Arch. Math. (Basel) **83** (2004), 243–255.

[33] Philos, Ch. G. and Purnaras, I. K., *On the behavior of the solutions for certain linear autonomous difference equations*, J. Differ. Equations Appl. **10** (2004), 1049–1067.

[34] Philos, Ch. G. and Purnaras, I. K., *The behavior of the solutions of periodic linear neutral delay difference equations*, J. Comput. Appl. Math. **175** (2005), 209–230.

[35] Philos, Ch. G. and Purnaras, I. K., *On the behavior of the solutions for certain first order linear autonomous functional differential equations*, Rocky Mountain J. Math. **36** (2006), 1999–2019.

[36] Philos, Ch. G. and Purnaras, I. K., *A result on the behavior of the solutions for scalar first order linear autonomous neutral delay differential equations*, Math. Proc. Cambridge Philos. Soc. **140** (2006), 349–358.

[37] Philos, Ch. G. and Purnaras, I. K., *Asymptotic behaviour and stability to linear nonautonomous neutral delay difference equations*, J. Differ. Equations Appl. **11** (2005), 503–513.

[38] Philos, Ch. G. and Purnaras, I. K., *More on the behavior of solutions to linear integrodifferential equations with unbounded delay*, Funkcial. Ekvac. **48** (2005), 393–414.

[39] Philos, Ch. G. and Purnaras, I. K., *On linear Volterra difference equations with infinite delay*, Advances in Difference Equations **2006** (2006), Art. 78470, pp. 1–28.

[40] Philos, Ch. G. and Purnaras, I. K., *On the behavior of the solutions to periodic linear delay differential and difference equations*, J. Math. Anal. Appl. **322** (2006), 847–863.

[41] Philos, Ch. G. and Purnaras, I. K., *On periodic linear neutral delay differential and difference equations*, Electron. J. Differential Equations **2006** (2006), No. 110, 1–25.

[42] Pituk, M., *The limits of the solutions of a nonautonomous linear delay difference equation*, Comput. Math. Appl. **42** (2001), 543–550.

[43] Pituk, M., *Asymptotic behavior of a nonhomogeneous linear recurrence system*, J. Math. Anal. Appl. **267** (2002), 626–642.

[44] Pituk, M., *Cesàro summability in a linear autonomous difference equation*, Proc. Amer. Math. Soc. **133** (2005), 3333–3339.

[45] Sharkovsky, A. N., Maistrenko, Yu. L. and Romanenko, E. Yu., *Difference Equations and Their Applications*, Kluwer Academic Publishers, Dordrecht, 1993.

[46] Shen, J. H., *Comparison and oscillation results for difference equations with continuous variable*, Indian J. Pure Appl. Math. **31** (2000), 1633–1642.

[47] Yan, J. and Zhang, F., *Oscillation for system of delay difference equations*, J. Math. Anal. Appl. **230** (1999), 223–231.

[48] Zhang, Y., Yan, J. and Zhao, A., *Oscillation criteria for a difference equation*, Indian J. Pure Appl. Math. **28** (1997), 1241–1249.

Department of Mathematics, University of Ioannina
P. O. Box 1186, 451 10 Ioannina, Greece
E-mail: cphilos@cc.uoi.gr

Department of Mathematics, University of Ioannina
P. O. Box 1186, 451 10 Ioannina, Greece
E-mail: ipurnara@cc.uoi.gr