ARCHIVUM MATHEMATICUM (BRNO)
Tomus 43 (2007), 87 – 92

ON NEAR-RING IDEALS WITH (σ,τ)-DERIVATION

Öznur Gölbaşi and Neşet Aydin

Abstract. Let N be a 3-prime left near-ring with multiplicative center Z, a (σ,τ)-derivation D on N is defined to be an additive endomorphism satisfying the product rule $D(xy)=\tau(x)D(y)+D(x)\sigma(y)$ for all $x,y\in N$, where σ and τ are automorphisms of N. A nonempty subset U of N will be called a semigroup right ideal (resp. semigroup left ideal) if $UN\subset U$ (resp. $NU\subset U$) and if U is both a semigroup right ideal and a semigroup left ideal, it be called a semigroup ideal. We prove the following results: Let D be a (σ,τ)-derivation on N such that $\sigma D=D\sigma,\tau D=D\tau$. (i) If U is semigroup right ideal of N and $D(U)\subset Z$ then N is commutative ring. (ii) If U is a semigroup ideal of N and $D^2(U)=0$ then $D=0$. (iii) If $a\in N$ and $[D(U),a]_{\sigma,\tau}=0$ then $D(a)=0$ or $a\in Z$.

1. Introduction

H. E. Bell and G. Mason have shown several commutativity theorems for near-rings with derivation in [1]. Bell has proved in [2], that if N be a 3-prime zero symmetric left near-ring and D be a nonzero derivation on N,U is nonzero subset of N such that $UN\subset U$ or $NU\subset U$ and $D(U)\subset Z$ then N is commutative ring. The major purpose of this paper to generalize this result replacing the derivation D by (σ,τ)-derivation.

Throughout this paper, N will denote a zero-symetric left near-ring and usually will be 3-prime, that is, if $aNb=0$ then $a=0$ or $b=0$, with multiplicative center Z. A nonempty subset U of N will be called a semigroup right ideal (resp. semigroup left ideal) if $UN\subset U$ (resp. $NU\subset U$) and if U is both a semigroup right ideal and a semigroup left ideal, it be called a semigroup ideal. For subsets $X,Y\subset N$ the symbol $[X,Y]$ will denote the set $\{xy-yx\mid x\in X,y\in Y\}$. Let σ,τ be two near-ring automorphisms of N. An additive endomorphism of N with the property that $D(xy)=\tau(x)D(y)+D(x)\sigma(y)$ for all $x,y\in N$ is called $(\sigma,\tau)-$derivation of N. Given $x,y\in N$, we write $[x,y]_{\sigma,\tau}=x\sigma(y)-\tau(y)x$; in particular $[x,y]_{1,1}=[x,y]$, in the usual sense. As for terminologies used here without mention, we refer to G. Pilz [4].

2000 *Mathematics Subject Classification*: 16A72, 16A70,16Y30.
Key words and phrases: prime near-ring, derivation, (σ,τ)-derivation.
Received July 26, 2005, revised April 2004.

2. Results

Lemma 1 ([3, Lemma 2]). *Let D be a (σ,τ)-derivation on the near-ring N. Then*

$$\big(\tau(x)D(y)+D(x)\sigma(y)\big)\sigma(a)=\tau(x)D(y)\sigma(a)+D(x)\sigma(y)\sigma(a)$$

for all $x,y,a\in N$.

Lemma 2 ([2, Lemma 1.2]). *Let N be a 3-prime near-ring.*

i) *If $z\in Z\backslash\{0\}$ then z is not a zero divisor.*
ii) *If $Z\backslash\{0\}$ contains an element z for which $z+z\in Z$, then $(N,+)$ is abelian.*
iii) *If $z\in Z\backslash\{0\}$ and x is an element of N such that $xz\in Z$ or $zx\in Z$, then $x\in Z$.*

Lemma 3 ([3, Lemma 3]). *Let D be a nonzero (σ,τ)-derivation on prime near-ring N and $a\in N$.*

i) *If $D(N)\sigma(a)=0$ then $a=0$.*
ii) *If $aD(N)=0$ then $a=0$.*

Lemma 4. *Let N be a prime near-ring, D a (σ,τ)-derivation of N and U be nonzero semigroup right ideal (resp. semigroup left ideal). If $D(U)=0$ then $D=0$.*

Proof. U be a nonzero semigroup right ideal of N and $D(U)=0$. For any $x\in N$, $u\in U$, we get

$$0=D(ux)=\tau(u)\,D(x)+D(u)\,\sigma(x)$$

and so,

$$\tau(u)\,D(x)=0\,,\quad \text{for all}\quad u\in U,\ x\in N\,.$$

By Lemma 3(ii), we have $D=0$. If U is a nonzero semigroup left ideal of N, then the proof is similar. □

Lemma 5. *Let N be a 3-prime near-ring, D a nonzero (σ,τ)-derivation of N and U be nonzero semigroup ideal.*

i) *If $x\in N$ and $D(U)\,\sigma(x)=0$ then $x=0$.*
ii) *If $x\in N$ and $xD(U)=0$ then $x=0$.*

Proof. i) Suppose U a nonzero semigroup ideal of N and $D(U)\,\sigma(x)=0$. By Lemma 1 for all $u,v\in U$, we get

$$0=D(uv)\,\sigma(x)=\tau(u)\,D(v)\,\sigma(x)+D(u)\,\sigma(v)\,\sigma(x)\,.$$

Using the hypothesis and σ is an automorphism of N, we have

$$\sigma^{-1}\big(D(u)\big)Ux=0\,,\quad \text{for all}\quad u\in U\,.$$

Since D is a nonzero (σ,τ)-derivation of N, this relation gives us $x=0$ by [2, Lemma 1.4 (i)].

ii) A similar argument works if $xD(U)=0$. □

Theorem 1. *Let N be a 3-prime near-ring, D be a nonzero (σ,τ)-derivation of N and U a nonzero semigroup right ideal. If $D(U)\subset Z$, then N is commutative ring.*

Proof. For all $u,v\in U$, we get

$$D(uv)=\tau(u)\,D(v)+D(u)\,\sigma(v)\in Z$$

and commuting this element with $\sigma(v)$ gives us

$$\big(\tau(u)\,D(v)+D(u)\,\sigma(v)\big)\sigma(v)=\sigma(v)\big(\tau(u)\,D(v)+D(u)\,\sigma(v)\big)\,.$$

Using Lemma 1 and $D(u)\in Z$, we have

$$\tau(u)\,D(v)\,\sigma(v)+D(u)\,\sigma(v)\,\sigma(v)=\sigma(v)\,\tau(u)\,D(v)+D(u)\,\sigma(v)\,\sigma(v)$$

and so,

$$D(v)\big[\tau(u),\sigma(v)\big]=0\,,\quad \text{for all}\quad u,v\in U\,.$$

Since $D(v)\in Z$, we obtain

$$D(v)=0\quad\text{or}\quad\big[\tau(u),\sigma(v)\big]=0\,,\quad\text{for all}\quad u,v\in U\,.$$

Suppose that $D(v)=0$, then $D(uv)=\tau(u)\,D(v)+D(u)\,\sigma(v)=D(u)\,\sigma(v)\in Z$. This gives us $v\in Z$ from Lemma 2(iii). For any cases we conclude that

$$\big[\tau(u),\sigma(v)\big]=0\,,\text{for all}\quad u,v\in U\,. \tag{2.1}$$

Since τ is an automorphism of N, the relation (2.1) yields

$$\big[U,\tau^{-1}\big(\sigma(v)\big)\big]=0\,,\quad\text{for all}\quad v\in U$$

and so, $U\subset Z$ from [2, Lemma 1.3 (iii)]. Thus, we obtain that N is commutative ring by [2, Lemma 1.5]. □

Theorem 2. *Let N be a 3-prime near-ring, D a nonzero $(\sigma,\tau)-$derivation of N and U be nonzero semigroup left ideal. If $D(U)\subset Z$, then N is commutative ring.*

Proof. If we use the same argument in the proof of Theorem 1, we conclude that

$$D(v)=0\quad\text{or}\quad\big[\tau(u),\sigma(v)\big]=0\,,\quad\text{for all}\quad u,v\in U\,. \tag{2.2}$$

Suppose that $D(v)=0$, then $D(uv)=\tau(u)\,D(v)+D(u)\,\sigma(v)=D(u)\,\sigma(v)\in Z$, for all $u\in U$, so that

$$D(u)\,\sigma(v)x=xD(u)\,\sigma(v)=D(u)\,x\sigma(v)\,,\quad\text{for all}\quad u\in U,\ x\in N\,.$$

Thus, $D(u)\big[\sigma(v),x\big]=0$, for all $u\in U,x\in N$. By Lemma 2 (i) and Lemma 4, we have show that

$$\text{If}\quad v\in U\quad\text{and}\quad D(v)=0\,,\quad\text{then}\quad v\in Z\,. \tag{2.3}$$

Thus, the relation (2.2) yields

$$\big[\tau(u),\sigma(v)\big]=0\,,\quad\text{for all}\quad u,v\in U\,.$$

By the hypothesis, we get $D(wu)\in Z$, for all $u,w\in U$. That is,

$$D(wu)=\tau(w)\,D(u)+D(w)\,\sigma(u)\in Z\,.$$

Commuting this element with $\sigma(v)$, one can obtain

$$\big(\tau(w)\,D(u)+D(w)\,\sigma(u)\big)\sigma(v)=\sigma(v)\big(\tau(w)\,D(u)+D(w)\,\sigma(u)\big)\,.$$

Applying Lemma 1 and using $\sigma(v)\,\tau(w)=\tau(w)\,\sigma(v)$, $D(u)\in Z$, we have

$$D(u)\,\tau(w)\,\sigma(v)+D(w)\,\sigma(u)\,\sigma(v)=D(u)\,\tau(w)\,\sigma(v)+D(w)\,\sigma(v)\,\sigma(u)$$

and so,

$$D(w)\,\sigma\big([u,v]\big)=0\,,\quad \text{for all}\quad u,v,w\in U\,.$$

Since $D(w)\in Z$, we have

$$D(w)=0\quad\text{or}\quad [u,v]=0\,,\quad \text{for all}\quad u,v,w\in U\,.$$

In the first case, we find that $D=0$, a contradiction. So, we must have U is commutative.

Now, we assume that $U\cap Z\neq\{0\}$. Then U contains a nonzero central element w, we have

$$(wx)u=(xw)u=u(xw)=u(wx)=(uw)x=(wu)x$$

that is,

$$w[x,u]=0\,,\quad \text{for all}\quad u\in U\,.$$

Since w is nonzero central element, we have $U\subset Z$. Thus N is commutative ring by [2, Lemma 1.5].

If $U\cap Z=\{0\}$, in which case (2.3) shows that $D(u)\neq 0$ for all $u\in U\backslash\{0\}$. For each such u, $D(u^2)=\tau(u)\,D(u)+D(u)\,\sigma(u)=D(u)\big(\tau(u)+\sigma(u)\big)\in Z$. Since $D(u)$ is noncentral element of N, we get

$$\tau(u)+\sigma(u)\in Z\,,\quad \text{for all}\quad u\in U\,.$$

Suppose that $\tau(u)+\sigma(u)=0$, for all $u\in U\backslash\{0\}$. By the hypothesis, $D(u^3)\in Z$, we get

$$\begin{aligned}D(u^3)&=\tau(u\,)D(u^2)+D(u)\,\sigma(u^2)\\&=\tau(u)\,\tau(u)\,D(u)+\tau(u)\,D(u)\,\sigma(u)+D(u)\,\sigma(u^2)\in Z\,.\end{aligned}$$

Using $\tau(u)=-\sigma(u)$ and $D(u)\in Z$, we get

$$\tau(u)\,\tau(u)\,D(u)-\tau(u)\,\tau(u)\,D(u)+D(u)\,\sigma(u^2)\in Z$$

which implies $u^2\in Z$. Hence we get $u^2\in U\cap Z=\{0\}$, and so $u^2=0$.

Now, for any $x\in N$, $D(xu)=\tau(x)\,D(u)+D(x)\,\sigma(u)\in Z$. Hence,

$$\sigma(u)\,\big(\tau(x)\,D(u)+D(x)\,\sigma(u)\big)=\big(\tau(x)\,D(u)+D(x)\,\sigma(u)\big)\,\sigma(u)\,.$$

Using Lemma 1 and $u^2=0$, we have

$$\sigma(u)\,\big(\tau(x)\,D(u)+D(x)\,\sigma(u)\big)=\tau(x)\,D(u)\,\sigma(u)\,.$$

Left-multiplying this relation by $\sigma(u)$ and using $u^2=0$, we obtain

$$\sigma(u)\,\tau(x)\,D(u)\sigma(u)=0\,,\quad \text{for all}\quad u\in U,\ x\in N\,.$$

The primenessly of the near-ring N, we obtain that $D(u)\sigma(u)=0$. Since $0\neq D(u)\in Z$, we have $\sigma(u)=0$, a contradiction. Thus, we must have $z=\sigma(u_0)+\tau(u_0)\in Z\backslash\{0\}$ for an $u_0\in U$. Since $D(zu)=\tau(z)\,D(u)+D(z)\,\sigma(u)\in Z$, for any $u\in U$, we have

$$\big(\tau(z)\,D(u)+D(z)\,\sigma(u)\big)\,\sigma(y)=\sigma(y)\big(\tau(z)\,D(u)+D(z)\,\sigma(u)\big)\,,\quad \text{for all}\quad y\in N$$

and so,

$$\tau(z)\,D(u)\,\sigma(y)+D(z)\,\sigma(u)\,\sigma(y)=\sigma(y)\,\tau(z)\,D(u)+\sigma(y)\,D(z)\,\sigma(u)\,.$$

Using $z, D(u) \in Z$, we get

$$D(u)\,\tau(z)\,\sigma(y) + D(z)\,\sigma(u)\,\sigma(y) = D(u)\,\tau(z)\,\sigma(y) + \sigma(y)\,\sigma(u)\,D(z)\,.$$

That is,

$$D(z)\,\sigma\big([u,y]\big) = 0\,, \quad \text{for all} \quad u \in U\,,\ y \in N\,.$$

Since $D(z)$ is a nonzero central element of N, we have $U \subset Z$. Thus, we conclude that N is commutative ring from [2, Lemma 1.5]. ☐

Theorem 3. *Let N be a 3-prime near-ring, D be a (σ,τ)-derivation of N such that $\sigma D = D\sigma, \tau D = D\tau$ and U a nonzero semigroup ideal of N. If $D^2(U) = 0$ then $D = 0$.*

Proof. For arbitrary $u, v \in U$, we have $0 = D^2(uv) = D\big(D(uv)\big) = D\big(\tau(u)\,D(v) + D(u)\,\sigma(v)\big) = \tau^2(u)\,D^2(v) + D\big(\tau(u)\big)\,\sigma\big(D(v)\big) + \tau\big(D(u)\big)\,D\big(\sigma(v)\big) + D^2(u)\,\sigma^2(v)$. Using the hypothesis and $\sigma D = D\sigma, \tau D = D\tau$, we get

$$2D\big(\tau(u)\big)\,D\big(\sigma(v)\big) = 0\,, \quad \text{for all} \quad u, v \in U$$

and so,

$$2\sigma^{-1}\big(\tau(D(u))\big)\,D(U) = 0\,, \quad \text{for all} \quad u \in U\,.$$

From Lemma 5(ii), we obtain that

$$2D(u) = 0\,, \quad \text{for all} \quad u \in U\,.$$

Now for any $y \in N$ and $u \in U$, $0 = D^2(yu) = D\big(\tau(y)\,D(u) + D(y)\,\sigma(u)\big) = \tau^2(y)\,D^2(v) + 2\tau\big(D(y)\big)\,\sigma\big(D(u)\big) + D^2(y)\,\sigma^2(u)$. Hence,

$$D^2(y)\,\sigma^2(u) = 0\,, \quad \text{for all} \quad y \in N,\ u \in U\,.$$

Using σ is an automorphism on N and [2, Lemma 1.4 (i)], we can take $D^2(N) = 0$. That is $D = 0$ by [3, Lemma 4]. ☐

Theorem 4. *Let N be a 3-prime near-ring, D be a (σ,τ)-derivation of N such that $\sigma D = D\sigma, \tau D = D\tau$ and U a nonzero semigroup ideal of N. If $a \in N$ and $\big[D(U), a\big]_{\sigma,\tau} = 0$ then $D(a) = 0$ or $a \in Z$.*

Proof. For $u \in U$, we get $D(au)\,\sigma(a) = \tau(a)\,D(au)\,\tau(a)$. Expanding this equation, one can obtain,

$$\tau(a)\,D(u)\,\sigma(a) + D(a)\,\sigma(u)\,\sigma(a) = \tau(a)\,\tau(a)\,D(u) + \tau(a)\,D(a)\,\sigma(u)\,.$$

By the hypothesis, we get $D(u)\,\sigma(a) = \tau(a)\,D(u)$. Hence,

$$\tau(a)\,\tau(a)\,D(u) + D(a)\,\sigma(u)\,\sigma(a) = \tau(a)\,\tau(a)\,D(u) + \tau(a)\,D(a)\,\sigma(u)\,.$$

That is,

$$D(a)\,\sigma(u)\,\sigma(a) = \tau(a)\,D(a)\,\sigma(u)\,, \quad \text{for all} \quad u \in U\,. \tag{2.4}$$

Replacing $ux, x \in N$ by u in (2.4) and using (2.4), we have

$$D(a)\,\sigma(u)\,\sigma(x)\,\sigma(a) = \tau(a)\,D(a)\,\sigma(u)\,\sigma(x) = D(a)\,\sigma(u)\,\sigma(a)\,\sigma(x)$$

and so,

$$D(a)\,\sigma(u)\,\sigma\big([x,a]\big) = 0\,, \quad \text{for all} \quad x \in N,\ u \in U\,.$$

Since σ is an automorphism of N, this relation can be written,

$$\sigma^{-1}\big(D(a)\,U[x,a]\big) = 0\,, \quad \text{for all} \quad x \in N\,.$$

Thus we conclude that $a \in Z$ or $D(a) = 0$ by [2, Lemma 1.4 (i)]. $\square$

References

[1] Bell, H. E. and Mason G., *On Derivations in near-rings, Near-rings and Near-fields*, North-Holland Math. Stud. **137** (1987).

[2] Bell, H. E., *On Derivations in Near-Rings II*, Kluwer Acad. Publ., Dordrecht **426** (1997), 191–197.

[3] Gölbaşı, Ö., Aydın, N., *Results on Prime Near-Rings with (σ,τ)-Derivation*, Math. J. Okayama Univ. **46** (2004), 1–7.

[4] Pilz, G., *Near-rings*, 2nd Ed., North-Holland Math. Stud. **23** (1983).

Cumhuriyet University, Faculty of Arts and Science
Department of Mathematics, Sivas - TURKEY
E-mail: ogolbasi@cumhuriyet.edu.tr
URL: http://www.cumhuriyet.edu.tr

Çanakkale 18 Mart University, Faculty of Arts and Science
Department of Mathematics, Çanakkale - TURKEY
E-mail: neseta@comu.edu.tr
URL: http://www.comu.edu.tr