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OSCILLATION THEOREMS FOR CERTAIN EVEN ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. This paper is concerned with a class of even order nonlinear differential equations of the form

$$\frac{d}{dt}\Big(\Big|(x(t)+p(t)x(\tau(t)))^{(n-1)}\Big|^{\alpha-1}(x(t)+p(t)x(\tau(t)))^{(n-1)}\Big) + F\big(t,x(g(t))\big)=0\,,$$

where n is even and $t\geq t_0$. By using the generalized Riccati transformation and the averaging technique, new oscillation criteria are obtained which are either extensions of or complementary to a number of existing results. Our results are more general and sharper than some previous results even for second order equations.

1. Introduction

Let n be an even positive integer, α a positive constant, $I=[t_0,\infty)$ and $\mathbb{R}_+=(0,\infty)$. Consider the n-th order nonlinear functional differential equation

$$\Big(\Big|\big(x(t)+p(t)x(\tau(t))\big)^{(n-1)}\Big|^{\alpha-1}(x(t)+p(t)x(\tau(t)))^{(n-1)}\Big)' + F\big(t,x(g(t))\big)=0\,,\quad t\in I\,, \tag{1}$$

where $F: I\times\mathbb{R}\to\mathbb{R}$ is a continuous function and $F(t,x)\operatorname{sgn}x = \operatorname{sgn}x$ for $(t,x)\in I\times\mathbb{R}$. In what follows, we always assume without mentioning that

(A_1) $p\colon I\to[0,\infty)$ is continuously differentiable such that p is not identically equal 1 on any interval of the form $[T,\infty)$;

(A_2) $\tau\colon I\to\mathbb{R}_+=(0,\infty)$ is continuously differentiable and strictly increasing such that $\lim\limits_{t\to\infty}\tau(t)=\infty$;

(A_3) $g\colon I\to\mathbb{R}$ is continuously differentiable with $\lim\limits_{t\to\infty}g(t)=\infty$;

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(A_4) there exists a function $q : I \to \mathbb{R}_+$ such that

$$F(t,x)\operatorname{sgn} x \ge q(t)|x|^\alpha \operatorname{sgn} x \quad \text{for} \quad x \ne 0 \quad \text{and} \quad t \ge t_0.$$

By a solution of Eq. (1) we mean a function $x \in C^{n-1}([T_x,\infty),\mathbb{R})$ for some $T_x \ge t_0$ which has the property that $\big|[x(t)+p(t)x(\tau(t))]^{(n-1)}\big|^{\alpha-1}[x(t)+p(t)x(\tau(t))]^{(n-1)} \in C^1\big([T_x,\infty),\mathbb{R}\big)$ and satisfies Eq. (1) on $[T_x,\infty)$. A nontrivial solution of Eq. (1) is called oscillatory if it has arbitrary large zeros; otherwise, it is said to be nonoscillatory. Equation (1) is oscillatory if all its solutions are oscillatory.

Qualitative properties of nonlinear special differential equations of the form (1) have been investigated by many authors (e.g. see [1-4, 6-16] and the references quoted therein). In particular, some optimal properties for oscillation of solutions of special cases such as

$$\frac{d}{dt}\Big(\big|x^{(n-1)}(t)\big|^{\alpha-1}x^{(n-1)}(t)\Big) + F\big(t, x(g(t))\big) = 0\,, \tag{2}$$

$$\frac{d}{dt}\Big(\,|x'(t)|^{\alpha-1}\,x'(t)\Big) + F\big(t, x(g(t))\big) = 0\,, \tag{3}$$

and

$$x''(t) + F\big(t, x(g(t))\big) = 0 \tag{4}$$

are contained in the papers [2, 8, 12, 13] and the references quoted therein. In particular, Agarwal et al. in [1] obtained some oscillation theorems of Eq. (1) which improve and extend several known results established in [2, 8, 9, 12, 13]. On the other hand, Yang et al. in [16] (see also Kong [7]) also obtained a number of oscillation criteria based on Wirtinger type inequalities when equation (1) becomes

$$\big(x(t) + p(t)x(t-\mu)\big)'' + q(t)f\big(x(t-\delta)\big) = 0\,, \tag{5}$$

under appropriate assumptions. For recent contributions we refer the reader to [1, 2, 6, 11–16] and the references therein.

Very extensive literature also exists (see [1–4, 11–16] and the references therein) for the oscillatory properties of equations (2) through (5), but we have found that these results are not always compatible with the results for (1) and the corresponding theory for (1) is less developed. This situation motivated us to study (1) further.

In this paper, by means of the generalized Riccati transformation and the averaging technique, we obtain new oscillation theorems for Eq. (1), thereby improving the main results in [1, 4, 7, 14]. Some results in this paper are based on the information only on a sequence of subintervals of $[t_0,\infty)$, rather than on the whole half-line. By choosing appropriate averaging functions, we can present a series of explicit oscillation criteria. Thus, results of this paper extend, improve and unify a number of existing results.

As is well known, the following lemmas are useful in working with even order nonlinear differential equations.

Lemma 1.1 ([9]). *Let $u \in C^n([t_0,\infty),\mathbb{R}_+)$. If $u^{(n)}(t)$ is eventually of constant sign for all large t, say, $t > t_0$, then there exist a $t_u \ge t_0$ and an integer l, $0 \le l \le n$, with l even for $u^{(n)}(t) \ge 0$ or l odd for $u^{(n)}(t) \le 0$ such that*

$$l > 0 \quad \textit{implies that} \quad u^{(k)}(t) > 0 \quad \textit{for} \quad t \ge t_u\,, \quad k = 0,1,\dots,l-1\,,$$

and

$$l \le n-1 \quad \textit{implies that} \quad (-1)^{l+k}u^{(k)}(t) > 0 \quad \textit{for} \quad t \ge t_u\,, \quad k = l, l+1,\dots,n-1\,.$$

Lemma 1.2 ([9]). *If the function u is as in Lemma 1.1 and*

$$u^{(n-1)}(t)u^{(n)}(t) \le 0 \quad \textit{for every} \quad t \ge t_u\,,$$

then for every λ, $0 < \lambda < 1$, we have

$$u(\lambda t) \ge \frac{2^{1-n}}{(n-1)!}\Big[\frac{1}{2} - \Big|\lambda - \frac{1}{2}\Big|\Big]^{n-1} t^{n-1}|u^{(n-1)}(t)|\,, \quad \textit{for all large} \quad t\,.$$

For the sake of convenience, we introduce some notations and state some preliminary definitions:

$$D_0 = \big\{(t,s) : t > s \ge t_0\big\}\,, \quad D = \big\{(t,s) : t \ge s \ge t_0\big\}\,;$$
$$z(t) = x(t) + p(t)x(\tau(t))\,;$$

and

$$\Theta(n,\lambda) = \frac{\lambda 2^{2-n}}{(n-2)!}\Big[\frac{1}{2} - \Big|\lambda - \frac{1}{2}\Big|\Big]^{n-2}, \quad \text{where} \quad \lambda \in (0,1)\,.$$

Definition 1.1. The triplet (H,k,ρ) is said to belong to $\mathcal{X}$ if $H \in C(D;\mathbb{R})$, k and $\rho \in C^1([t_0,\infty);\mathbb{R}_+)$ and if there exists $h \in C(D_0;\mathbb{R})$ such that the following conditions hold:

(I) $H(t,t) = 0$ for $t \ge t_0$, $H(t,s) > 0$ on D_0;
(II) $H(t,s)$ has a continuous and nonpositive partial derivatives $\partial H/\partial s$ on D_0;
(III) $\frac{\partial}{\partial s}(H(t,s)k(s)) + H(t,s)k(s)\frac{\rho'(s)}{\rho(s)} = h(t,s)$, for $(t,s) \in D_0$.

Definition 1.2. The triplet (H,k,ρ) is said to belong to $\mathcal{Y}$ if $H \in C(D;\mathbb{R})$, k and $\rho \in C^1\big([t_0,\infty);\mathbb{R}_+)$ and if there exist h_1, $h_2 \in C(D_0;\mathbb{R})$ such that the following conditions hold:

(I) $H(t,t) = 0$ for $t \ge t_0, H(t,s) > 0$ on D_0;
(II) $\frac{\partial}{\partial t}(H(t,s)k(t)) + H(t,s)k(t)\frac{\rho'(t)}{\rho(t)} = h_1(t,s)$, for $(t,s) \in D_0$;
(III) $\frac{\partial}{\partial s}(H(t,s)k(s)) + H(t,s)k(s)\frac{\rho'(s)}{\rho(s)} = h_2(t,s)$, for $(t,s) \in D_0$.

2. Oscillation criteria for the case $0 \le p(t) \le 1$

In this section we always assume that the following condition holds:

(A_5) $\tau(t) < t$, $0 \le p(t) \le 1$ and there exists $\sigma\colon I \to \mathbb{R}_+$ which is continuously differentiable and satisfies

$$\sigma'(t) > 0, \quad \sigma(t) \le \inf\{t, g(t)\}\,, \quad \text{and} \quad \lim_{t\to\infty}\sigma(t) = \infty \quad \text{for} \quad t \ge t_0\,.$$

To prove the main theorems in this section, we first establish the following lemma about oscillation of solutions of the differential inequality

$$\left[\frac{d}{dt}\Big(\big|\big(x(t)+p(t)x(\tau(t))\big)^{(n-1)}\big|^{\alpha-1}\big(x(t)+p(t)x(\tau(t))\big)^{(n-1)}\Big)\right]\operatorname{sgn} x(t) + q(t)\big|x\big(g(t)\big)\big|^{\alpha} \le 0 \tag{6}$$

for $t \ge t_0$, where p, τ, g and q are defined in (A_1)-(A_4). Solutions and oscillatory solutions for (6) are defined in manners similar to those of (1).

Lemma 2.1. *Suppose $\lambda \in (0,1)$ and conditions (A_1)-(A_3) and (A_5) hold. Then the differential inequality (6) is oscillatory provided that one of the following conditions is satisfied:*

(X) *there exists $(H,k,\rho) \in \mathcal{X}$ such that either*

$$\limsup_{t\to\infty}\big[\mathcal{A}(t,t_0) - (\alpha+1)^{-(\alpha+1)}\Theta^{-\alpha}(n,\lambda)\mathcal{B}(t,t_0)\big]ds = \infty\,, \tag{7}$$

or, $n=2$ and

$$\limsup_{t\to\infty}\big[\mathcal{A}(t,t_0) - (\alpha+1)^{-(\alpha+1)}\mathcal{B}(t,t_0)\big]ds = \infty\,, \tag{8}$$

where

$$\mathcal{A}(t,t_0) = \frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)k(s)\rho(s)q(s)(1-p(g(t)))^{\alpha}ds\,,$$

$$\mathcal{B}(t,t_0) = \frac{1}{H(t,t_0)}\int_{t_0}^{t} \frac{\rho(s)|h(t,s)|^{\alpha+1}}{[H(t,s)k(s)\sigma^{n-2}(s)\sigma'(s)]^{\alpha}}ds\,.$$

(Y) *For each $T \ge t_0$, there exist $(H,k,\rho) \in \mathcal{Y}$ and $a,\ b,\ c \in \mathbb{R}$ such that $T_0 \le a < c < b$ and either*

$$\mathcal{A}_1(c,a) + \mathcal{A}(b,c) \ge (\alpha+1)^{-(\alpha+1)}\Theta^{-\alpha}(n,\lambda)\big[\mathcal{B}_1(c,a)+\mathcal{B}(b,c)\big]\,, \tag{9}$$

or, $n=2$ and

$$\mathcal{A}_1(c,a) + \mathcal{A}(b,c) \ge (\alpha+1)^{-(\alpha+1)}\big[\mathcal{B}_1(c,a)+\mathcal{B}(b,c)\big]\,, \tag{10}$$

where

$$\mathcal{A}_1(t,t_0) = \frac{1}{H(t,t_0)}\int_{t_0}^{t} H(s,t_0)k(s)\rho(s)q(s)(1-p(g(t)))^{\alpha}ds\,,$$

$$\mathcal{B}_1(t,t_0) = \frac{1}{H(t,t_0)}\int_{t_0}^{t} \frac{\rho(s)|h_1(s,t_0)|^{\alpha+1}}{[H(s,t_0)k(s)\sigma^{n-2}(s)\sigma'(s)]^{\alpha}}ds\,.$$

(Z) *For each $l \ge t_0$, there exists $(H,k,\rho) \in \mathcal{Y}$ such that either*

(i) *the following two inequalities*

$$\limsup_{t\to\infty} H(t,l)\big[\mathcal{A}_1(t,l) - (\alpha+1)^{-(\alpha+1)}\Theta^{-\alpha}(n,\lambda)\mathcal{B}_1(t,l)\big] > 0 \tag{11}$$

and

$$\limsup_{t\to\infty} H(t,l)\big[\mathcal{A}(t,l) - (\alpha+1)^{-(\alpha+1)}\Theta^{-\alpha}(n,\lambda)\mathcal{B}(t,l)\big] > 0 \tag{12}$$

hold; or,

(ii) $n = 2$ *and the following two inequalities*

$$\limsup_{t\to\infty} H(t,l)\big[\mathcal{A}_1(t,l) - (\alpha+1)^{-(\alpha+1)}\mathcal{B}_1(t,l)\big] > 0 \tag{13}$$

and

$$\limsup_{t\to\infty} H(t,l)\big[\mathcal{A}(t,l) - (\alpha+1)^{-(\alpha+1)}\mathcal{B}(t,l)\big] > 0 \tag{14}$$

hold.

Proof. Suppose (7) in (X) holds. Without loss of generality, we may assume that there exists a nonoscillatory solution x of (6), say $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \ge t_1 \ge t_0$. Then $z(t) = x(t) + p(t)x(\tau(t)) > 0$ for $t \ge t_1 \ge t_0$. By (6), we obtain

$$\big(\big|z^{(n-1)}(t)\big|^{\alpha-1} z^{(n-1)}(t)\big)' \operatorname{sgn} x + q(t)\,|x(g(t))|^{\alpha} \le 0 \tag{15}$$

which implies that $\big|z^{(n-1)}(t)\big|^{\alpha-1} z^{(n-1)}(t)$ is decreasing and $z^{(n-1)}(t)$ is eventually of one sign. If $z^{(n-1)}(t) < 0$ eventually, then

$$0 \ge \big(\big|z^{(n-1)}(t)\big|^{\alpha-1} z^{(n-1)}(t)\big)' = \alpha\big(-z^{(n-1)}(t)\big)^{\alpha-1} z^{(n)}(t)\,,$$

we find that $z^{(n)}(t) \le 0$ eventually. But then Lemma 1.1 implies that $z^{(n)}(t) > 0$ eventually. Furthermore, when $z^{(n-1)}(t) > 0$ eventually then again from Lemma 1.1 (note that n is even) we have $z'(t) > 0$ eventually. Thus there exists $t_2 \ge t_1$ such that

$$z'(t) > 0 \quad \text{and} \quad z^{(n-1)}(t) > 0 \quad \text{for} \quad t \ge t_2\,. \tag{16}$$

From (A_1), (A_2) and (A_5), we see that

$$\begin{aligned} x(t) &= z(t) - p(t)x(\tau(t)) = z(t) - p(t)\big[z(\tau(t)) - p(\tau(t))x(\tau\circ\tau(t))\big] \\ &\ge z(t) - p(t)z\big(\tau(t)\big) \ge \big(1 - p(t)\big)z(t) \end{aligned} \tag{17}$$

for $t \ge t_2$. By using conditions (17) in (15), we get

$$\big(\big|z^{(n-1)}(t)\big|^{\alpha-1} z^{(n-1)}(t)\big)' + q(t)\big(1 - p(g(t))\big)^{\alpha} z^{\alpha}(g(t)) \le 0 \quad \text{for} \quad t \ge t_3 \ge t_2\,.$$

Thus, it follows from (A_5) that

$$\big(\big|z^{(n-1)}(t)\big|^{\alpha-1} z^{(n-1)}(t)\big)' + q(t)(1 - p(g(t)))^{\alpha} z^{\alpha}(\sigma(t)) \le 0 \quad \text{for} \quad t \ge t_3\,. \tag{18}$$

Define

$$w(t) = \rho(t)\Big(\frac{z^{(n-1)}(t)}{z(\lambda\sigma(t))}\Big)^{\alpha}, \qquad t \ge t_3\,, \tag{19}$$

where $\lambda \in (0,1)$. Differentiating (19) and making use of (18), we may see that for $t \geq t_3$,

$$w'(t) \leq -\rho(t)q(t)(1-p(g(t)))^\alpha + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\alpha\lambda\rho(t)\sigma'(t)\left(z^{(n-1)}(t)\right)^\alpha z'(\lambda\sigma(t))}{z^{\alpha+1}(\lambda\sigma(t))}. \tag{20}$$

By Lemma 1.2 (note that since $z^{(n-1)}(t) > 0$ for $t \geq t_2$, we have

$$\left[\left(z^{(n-1)}(t)\right)^\alpha\right]' = \alpha\left(z^{(n-1)}(t)\right)^{\alpha-1} z^{(n)}(t) \leq 0 \quad \text{for} \quad t \geq t_2\,,$$

which in turn implies $z^{(n)}(t) \leq 0$ for $t \geq t_2$), there is $t_4 \geq t_3$ and a constant $\lambda \in (0,1)$ such that

$$\begin{aligned} z'(\lambda\sigma(t)) &\geq \frac{2^{2-n}}{(n-1)!}\left[\frac{1}{2} - \left|\lambda - \frac{1}{2}\right|\right]^{n-2} \sigma^{n-2}(t) z^{(n-1)}(\sigma(t)) \\ &\geq \frac{1}{\lambda}\Theta(n,\lambda)\sigma^{n-2}(t)z^{(n-1)}(t) \quad \text{for} \quad t \geq t_4\,. \end{aligned} \tag{21}$$

Using (21) in (20), we obtain

$$\begin{aligned} w'(t) \leq &- \rho(t)q(t)\big(1-p(g(t))\big)^\alpha + \frac{\rho'(t)}{\rho(t)}w(t) \\ &- \frac{\alpha\Theta(\lambda,n)\sigma^{n-2}(t)\sigma'(t)}{\rho^{1/\alpha}(t)} w^{(\alpha+1)/\alpha}(t)\,. \end{aligned} \tag{22}$$

If we replace t in (22) with s, multiply the resulting equation by $H(t,s)k(s)$ and then integrating from T to t, where $t \geq T \geq t_4$, then we have

$$\begin{aligned} \int_T^t H(t,s)k(s)\rho(s)q(s)\big(1-p(g(s))\big)^\alpha ds \leq H(t,T)k(T)w(T) + \int_T^t |h(t,s)|w(s)ds \\ - \alpha\Theta(n,\lambda)\int_T^t H(t,s)k(s)\sigma^{n-2}(s)\sigma'(s)\rho^{-1/\alpha}(s)w^{(\alpha+1)/\alpha}(s)ds\,. \end{aligned} \tag{23}$$

According to the Young inequality

$$\begin{aligned} |h(t,s)|w(s) \leq (\alpha+1)^{-(\alpha+1)}\rho(s)\big[\Theta(n,\lambda)H(t,s)k(s)\sigma^{n-2}(s)\sigma'(s)\big]^{-\alpha}|h(t,s)|^{\alpha+1} \\ + \alpha\Theta(n,\lambda)H(t,s)k(s)\sigma^{n-2}(s)\sigma'(s)\rho^{-1/\alpha}(s)w^{(\alpha+1)/\alpha}(s)\,. \end{aligned} \tag{24}$$

From (23) and (24), we get

$$\mathcal{A}(t,T) \leq w(T)k(T) + (\alpha+1)^{-(\alpha+1)}\Theta^{-\alpha}(n,\lambda)\mathcal{B}(t,T). \tag{25}$$

Let $T = t_4$ in (25), then

$$H(t,t_4)\big[\mathcal{A}(t,t_4) - (\alpha+1)^{-(\alpha+1)}\Theta^{-\alpha}(n,\lambda)\mathcal{B}(t,t_4)\big] \leq H(t,t_4)k(t_4)w(t_4) \tag{26}$$

for every $t \geq t_4 \geq t_0$. Thus we obtain

$$\begin{aligned}
&H(t,t_0)\big[\mathcal{A}(t,t_0) - (\alpha+1)^{-(\alpha+1)}\Theta^{-\alpha}(n,\lambda)\mathcal{B}(t,t_0)\big] \\
&\quad = H(t_4,t_0)\big[\mathcal{A}(t_4,t_0) - (\alpha+1)^{-(\alpha+1)}\Theta^{-\alpha}(n,\lambda)\mathcal{B}(t_0,t_4)\big] \\
&\qquad + H(t,t_4)\big[\mathcal{A}(t,t_4) - (\alpha+1)^{-(\alpha+1)}\Theta^{-\alpha}(n,\lambda)\mathcal{B}(t,t_4)\big] \\
&\quad \leq H(t,t_0)\int_{t_0}^{t_4} k(s)\rho(s)q(s)(1-p(g(s)))^{\alpha}ds + H(t,t_0)k(t_4)w(t_4) \\
&\quad = H(t,t_0)\Big[\int_{t_0}^{t_4} k(s)\rho(s)q(s)(1-p(g(s)))^{\alpha}ds + k(t_4)w(t_4)\Big]\,.
\end{aligned}$$

Dividing both sides of the above inequality by $H(t,t_0)$ and taking the superior limit as $t \to \infty$, we have

$$\begin{aligned}
&\limsup_{t\to\infty}\big[\mathcal{A}(t,t_0) - (\alpha+1)^{-(\alpha+1)}\Theta^{-\alpha}(n,\lambda)\mathcal{B}(t,t_0)\big] \\
&\qquad \leq \int_{t_0}^{t_4} k(s)\rho(s)q(s)(1-p(g(t)))^{\alpha}ds + k(t_4)w(t_4) < \infty\,,
\end{aligned}$$

which is contrary to (7).

In the particular case where $n = 2$, the condition (7) can be replaced by (8). Indeed, without loss of generality, we may assume the existence of a nonoscillatory solution $x(t)$ of (6) such that $x(t) > 0$ for $t \geq t_1 \geq t_0$. Define function

$$w(t) = \rho(t)\Big(\frac{z'(t)}{z(\sigma(t))}\Big)^{\alpha}, \quad t \geq t_3\,. \tag{27}$$

Differentiating (27) and making use of (18) with $n = 2$, and (21), we may see that for $t \geq t_4$,

$$\begin{aligned}
w'(t) &\leq -\rho(t)q(t)(1-p(g(t)))^{\alpha} + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\alpha\rho(t)\sigma'(t)(z'(t))^{\alpha}z'(\sigma(t))}{z^{\alpha+1}(\sigma(t))} \\
&\leq -\rho(t)q(t)(1-p(g(t)))^{\alpha} + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\alpha\sigma'(t)}{\rho^{1/\alpha}(t)}w^{(\alpha+1)/\alpha}(t)\,. \qquad (28)
\end{aligned}$$

The rest of the proof is similar to the general case and is omitted. The proof of the implication of (X) is complete.

Next, suppose (9) of (Y) holds. As in the proof just shown, we can obtain (26). If we replace t_4 by c, then

$$H(t,c)\big[\mathcal{A}(t,c) - (\alpha+1)^{-(\alpha+1)}\Theta^{-\alpha}(n,\lambda)\mathcal{B}(t,c)\big] \leq H(t,c)k(c)w(c) \tag{29}$$

where $t \in [c,b)$. Letting $t \to b^-$ in (29) and then dividing both sides by $H(b,c)$, then we have

$$\mathcal{A}(b,c) - (\alpha+1)^{-(\alpha+1)}\Theta^{-\alpha}(n,\lambda)\mathcal{B}(b,c) \leq k(c)w(c)\,. \tag{30}$$

Next we go back to (22) and repeat the calculations by first multiplying by $H(s,t)k(s)$ instead of by $H(t,s)k(s)$ and then integrating from a to t ($t \geq a \geq t_4 \geq t_0$). Then, by symmetry considerations, we may also show that

$$\mathcal{A}_1(c,a) - (\alpha+1)^{-(\alpha+1)}\Theta^{-\alpha}(n,\lambda)\mathcal{B}_1(c,a) \leq -k(c)w(c)\,. \tag{31}$$

Now we assert that x has at least one zero in (a, b). For otherwise adding (30) and (31) would yield an inequality which contradicts our assumption (9). Finally, the proof is completed by picking $\{T_i\} \subset [t_0, \infty)$ such that $T_i \to \infty$ as $i \to \infty$, and then apply what we have just shown to conclude x has a zero in each (T_i, ∞).

The case where $n = 2$ is similarly proved.

Finally, we assert that the conditions in (Z) follow from (X) and (Y). Indeed, for any $T \geq T_0 \geq t_0$, let $a = T$. In (11) we choose $l = a$. Then there exists $c > a$ such that

$$H(c, a)\big[\mathcal{A}_1(c, a) - (\alpha + 1)^{-(\alpha+1)}\Theta^{-\alpha}(n, \lambda)\mathcal{B}_1(c, a)\big] > 0\,. \tag{32}$$

In (12) we choose $l = c$. Then there exists $b > c$ such that

$$H(b, c)\big[\mathcal{A}(b, c) - (\alpha + 1)^{-(\alpha+1)}\Theta^{-\alpha}(n, \lambda)\mathcal{B}(b, c)\big] > 0\,. \tag{33}$$

Combining (32) and (33) we obtain (9). The conclusion (i) in (Z) thus follows from (Y). The conclusion (ii) is similarly proved. The proof of Lemma 2.1 is completed.

We may now establish our main oscillation criteria in a relatively easy manner.

Theorem 2.1. *Assume that* (A_1)-(A_5)*hold and one of the conditions* (X), (Y) *or* (Z) *in Lemma 2.1 holds. Then Eq.* (1) *is oscillatory.*

Indeed, without loss of generality, we may assume that there exists a nonoscillatory solution $x(t)$ of (1) such that $x(t) > 0$ for $t \geq t_1 \geq t_0$. By (A_4), we obtain

$$\begin{aligned}
0 &= \Big\{\frac{d}{dt}\Big(\big|\big[x(t) + p(t)x(\tau(t))\big]^{(n-1)}\big|^{\alpha-1}(x(t) + p(t)x(\tau(t)))^{(n-1)}\Big)\Big\} \\
&\quad \times \operatorname{sgn} x(t) + F\big(t, x(g(t))\big)\operatorname{sgn} x(t) \\
&\geq \Big\{\frac{d}{dt}\Big(\big|\big[x(t) + p(t)x(\tau(t))\big]^{(n-1)}\big|^{\alpha-1}(x(t) + p(t)x(\tau(t)))^{(n-1)}\Big)\Big\} \\
&\quad \times \operatorname{sgn} x(t) + q(t)\big|x(g(t))\big|^{\alpha}\,,
\end{aligned}$$

which implies that $x(t)$ of (1) is a nonoscillatory solution $x(t)$ of (6). An application of Lemma 2.1 then yields our assertion.

Corollary 2.1. *Let* (A_1)-(A_5) *hold and* $\rho \in C^1(I, \mathbb{R}_+)$ *with* $\rho'(t) \geq 0$. *Assume that* (I), (II) *in Definition 1.1 and*

$$-\frac{\partial}{\partial s}(H(t, s)) + H(t, s)\frac{\rho'(s)}{\rho(s)} = h(t, s), \quad \textit{for} \quad (t, s) \in D_0 \tag{34}$$

hold. Suppose further that either (7) or (8) holds with

$$\mathcal{A}(t, t_0) = \frac{1}{H(t, t_0)}\int_{t_0}^{t} H(t, s)\rho(s)q(s)(1 - p(g(s)))^{\alpha}ds\,,$$

and

$$\mathcal{B}(t, t_0) = \frac{1}{H(t, t_0)}\int_{t_0}^{t} \frac{\rho(s)[h(t, s)]^{\alpha+1}}{[H(t, s)\sigma^{n-2}(s)\sigma'(s)]^{\alpha}}ds\,.$$

Then (1) *is oscillatory.*

Remark 2.1. If $h(t,s)$ and $h_1(t,s)$ are replaced by

$$h(t,s)\sqrt{H(t,s)k(s)} \quad \text{and} \quad h_1(t,s)\sqrt{H(t,s)k(s)}$$

in Theorem 2.1 respectively, we may show that Eq. (1) is oscillatory. The proof is similar and therefore omitted.

Next, we define

$$R(t) = \int_{t_0}^{t} \sigma^{n-2}(s)\sigma'(s)ds\,, \quad t \ge t_0 \tag{35}$$

and let

$$H(t,s) = [R(t) - R(s)]^{\mu}\,, \quad t \ge s \ge t_0 \tag{36}$$

where $\mu > \max\{1,\alpha\}$ is a constant.

Theorem 2.2. *Suppose (A_1)-(A_5) hold. Then Eq. (1) is oscillatory provided that there is some $\mu > \max\{1,\alpha\}$ such that one of the following conditions is satisfied:*

(I) *for any $l \ge t_0$,*

$$\limsup_{t\to\infty} \frac{1}{R^{\mu-\alpha}(t)} \int_l^t [R(s) - R(l)]^{\mu} q(s)(1 - p(g(s)))^{\alpha} ds > \frac{1}{\Theta^{\alpha}(n,\lambda)} \frac{\mu^{\alpha+1}}{(\alpha+1)^{(\alpha+1)}(\mu-\alpha)} \quad \textit{and} \tag{37}$$

$$\limsup_{t\to\infty} \frac{1}{R^{\mu-\alpha}(t)} \int_l^t [R(t) - R(s)]^{\mu} q(s)(1 - p(g(s)))^{\alpha} ds > \frac{1}{\Theta^{\alpha}(n,\lambda)} \frac{\mu^{\alpha+1}}{(\alpha+1)^{(\alpha+1)}(\mu-\alpha)}; \tag{38}$$

(II) $n = 2$ *and for any $l \ge t_0$, one of the following conditions is satisfied:*

$$\limsup_{t\to\infty} \frac{1}{R^{\mu-\alpha}(t)} \int_l^t [R(s) - R(l)]^{\mu} q(s)(1 - p(g(s)))^{\alpha} ds > \frac{\mu^{\alpha+1}}{(\alpha+1)^{(\alpha+1)}(\mu-\alpha)} \quad \textit{or} \tag{39}$$

$$\limsup_{t\to\infty} \frac{1}{R^{\mu-\alpha}(t)} \int_l^t [R(t) - R(s)]^{\mu} q(s)(1 - p(g(s)))^{\alpha} ds > \frac{\mu^{\alpha+1}}{(\alpha+1)^{(\alpha+1)}(\mu-\alpha)}\,. \tag{40}$$

Proof. (I) Pick $H(t,s)$ as in (36) and $k(t) \equiv \rho(t) \equiv 1$ for $t > t_0$. By Definition 1.2, it is easy to see that

$$|h_1(t,s)| = \mu[R(t) - R(s)]^{\mu-1}\sigma^{n-2}(t)\sigma'(t) \qquad \text{and}$$

$$|h_2(t,s)| = \mu[R(t) - R(s)]^{\mu-1}\sigma^{n-2}(s)\sigma'(s)\,.$$

Note further that

$$H(t,l)\mathcal{B}_1(t,l) = \int_l^t \mu^{\alpha+1}[R(s)-R(l)]^{\mu-(\alpha+1)}\sigma^{n-2}(s)\sigma'(s)\,ds$$

$$= \frac{\mu^{\alpha+1}}{\mu-\alpha}[R(t)-R(l)]^{\mu-\alpha} \qquad \text{and}$$

$$H(t,l)\mathcal{B}(t,l) = \int_l^t \mu^{\alpha+1}[R(t)-R(s)]^{\mu-(\alpha+1)}\sigma^{n-2}(s)\sigma'(s)\,ds$$

$$= \frac{\mu^{\alpha+1}}{\mu-\alpha}[R(t)-R(l)]^{\mu-\alpha}\,. \tag{41}$$

In view of the fact that $\limsup_{t\to\infty} R(t) = \infty$, we see that

$$\limsup_{t\to\infty} \frac{1}{R^{\mu-\alpha}(t)} \int_l^t \frac{\rho(s)|h_1(s,l)|^{\alpha+1}}{[H(s,l)k(s)\sigma^{n-2}(s)\sigma'(s)]^\alpha} ds = \frac{\mu^{\alpha+1}}{(\alpha+1)^{(\alpha+1)}(\mu-\alpha)} \tag{42}$$

and

$$\limsup_{t\to\infty} \frac{1}{R^{\mu-\alpha}(t)} \int_l^t \frac{\rho(s)|h(t,s)|^{\alpha+1}}{[H(t,s)k(s)\sigma^{n-2}(s)\sigma'(s)]^\alpha} ds = \frac{\mu^{\alpha+1}}{(\alpha+1)^{(\alpha+1)}(\mu-\alpha)}\,. \tag{43}$$

From (37) and (42), we have

$$0 < \limsup_{t\to\infty} \frac{1}{R^{\mu-\alpha}(t)} H(t,l)\big[\mathcal{A}_1(t,l) - (\alpha+1)^{-(\alpha+1)}\Theta^{-\alpha}(n,\lambda)\mathcal{B}_1(t,l)\big]$$
$$- \Theta^{-\alpha}(n,\lambda)\frac{\mu^{\alpha+1}}{(\alpha+1)^{(\alpha+1)}(\mu-\alpha)}\,,$$

i.e. (11) holds. Similarly, (38) and (41) imply that (12) hold. By the case (Z) (i) of Theorem 2.1, Eq. (1) is oscillatory.

(II) The proof is similar to the previous case by means of the condition (Z) (ii) of Theorem 2.1.

Remark 2.2. Corollary 2.1 is an improvement or an extension of the results by Agarwal et al. [1, Theorem 2.1], Grammatikopoulos et al. [4], Xu and Xia [14, Theorem 2.1]. Moreover, Theorem 2.2 is an improvement of Kong [7, Theorem 2.3].

3. Oscillation results for the case $p(t) \geq 1$

In this section we consider the oscillation of Eq. (1) when the function $p(t) \geq 1$. In this section we always assume the following condition.

(A_5^*) $\tau(t) > t$, $p(t) \geq 1$ and there exists $\sigma^* : I \to \mathbb{R}_+$ which is continuously differentiable and satisfies

$$(\sigma^*(t))' > 0, \quad \sigma^*(t) \leq \inf\{t, \tau^{-1}\circ g(t)\}, \quad \text{and} \quad \lim_{t\to\infty}\sigma^*(t) = \infty \quad \text{for} \quad t \geq t_0\,,$$

where τ^{-1} is the inverse function of τ.

We also let

$$P(t) = \frac{1}{p(\tau^{-1}(t))}\left(1 - \frac{1}{p(\tau^{-1}\circ\tau^{-1}(t))}\right) \quad \text{for all large } t\,.$$

Lemma 3.1. *Let conditions (A_1)–(A_3) and (A_5^*) hold. Then the differential inequality* (6) *is oscillatory provided one of the following conditions is satisfied:*

(X*) *there exists $(H;k,\rho)\in\mathcal{X}$ such that either*

$$\limsup_{t\to\infty}\big[\mathcal{A}^*(t,t_0)-(\alpha+1)^{-(\alpha+1)}\Theta^{-\alpha}(n,\lambda)\mathcal{B}^*(t,t_0)\big]\,ds=\infty\,; \tag{44}$$

or, $n=2$ and

$$\limsup_{t\to\infty}\big[\mathcal{A}^*(t,t_0)-(\alpha+1)^{-(\alpha+1)}\mathcal{B}^*(t,t_0)\big]\,ds=\infty\,.$$

where

$$\mathcal{A}^*(t,t_0)=\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)k(s)\rho(s)q(s)(P(g(t)))^{\alpha}ds\,,$$
$$\mathcal{B}^*(t,t_0)=\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{\rho(s)|h(t,s)|^{\alpha+1}}{[H(t,s)k(s)(\sigma^*(s))^{n-2}(\sigma^*(s))']^{\alpha}}\,ds\,.$$

(Y*) *For each $T\ge t_0$, there exist $(H;k,\rho)\in\mathcal{Y}$ and a, b, $c\in\mathbb{R}$ such that $T\le a<c<b$ and either*

$$\mathcal{A}_1^*(c,a)+\mathcal{A}^*(b,c)>(\alpha+1)^{-(\alpha+1)}\Theta^{-\alpha}(n,\lambda)\big[\mathcal{B}_1^*(c,a)+\mathcal{B}^*(b,c)\big]\,,$$

or, $n=2$ and

$$\mathcal{A}_1^*(c,a)+\mathcal{A}^*(b,c)>(\alpha+1)^{-(\alpha+1)}\big[\mathcal{B}_1^*(c,a)+\mathcal{B}^*(b,c)\big]\,,$$

where

$$\mathcal{A}_1^*(t,t_0)=\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(s,t_0)k(s)\rho(s)q(s)(P(g(t)))^{\alpha}\,ds\,,$$
$$\mathcal{B}_1^*(t,t_0)=\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{\rho(s)|h_1(s,t_0)|^{\alpha+1}}{[H(s,t_0)k(s)(\sigma^*(s))^{n-2}(\sigma^*(s))']^{\alpha}}\,ds\,;$$

(Z*) *For each $l\ge t_0$, there exists $(H;k,\rho)\in\mathcal{Y}$ such that either*

(i) *the following two inequalities*

$$\limsup_{t\to\infty}H(t,l)\big[\mathcal{A}_1^*(t,l)-(\alpha+1)^{-(\alpha+1)}\Theta^{-\alpha}(n,\lambda)\mathcal{B}_1^*(t,l)\big]>0 \tag{45}$$

and

$$\limsup_{t\to\infty}H(t,l)\big[\mathcal{A}^*(t,l)-(\alpha+1)^{-(\alpha+1)}\Theta^{-\alpha}(n,\lambda)\mathcal{B}^*(t,l)\big]>0 \tag{46}$$

hold; or

(ii) *$n=2$ and the following two inequalities*

$$\limsup_{t\to\infty}H(t,l)\big[\mathcal{A}_1^*(t,l)-(\alpha+1)^{-(\alpha+1)}\mathcal{B}_1^*(t,l)\big]>0$$

$$\limsup_{t\to\infty} H(t,l)\big[\mathcal{A}^*(t,l) - (\alpha+1)^{-(\alpha+1)}\mathcal{B}^*(t,l)\big] > 0$$

hold.

Proof. Assume (44) holds. Without loss of generality, we may assume that there exists a nonoscillatory solution $x(t)$ of (6), say $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \ge t_1 \ge t_0$. Then $z(t) = x(t) + p(t)x(\tau(t)) > 0$ for $t \ge t_1 \ge t_0$. Proceeding as in the proof Lemma 2.1, we see that (15) and (16) hold for $t \ge t_2$. From $(A_1) - (A_2)$ and (A_5^*), it follows that

$$\begin{aligned} x(t) &= \frac{1}{p(\tau^{-1}(t))}\big(z(\tau^{-1}(t)) - x(\tau(t))\big) \\ &= \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(t))}\Big(\frac{\tau(\tau^{-1}\circ\tau^{-1}(t))}{p(\tau^{-1}\circ\tau^{-1}(t))} - \frac{x(\tau^{-1}\circ\tau^{-1}(t))}{p(\tau^{-1}\circ\tau^{-1}(t))}\Big) \\ &\ge \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{z(\tau^{-1}\circ\tau^{-1}(t))}{p(\tau^{-1}(t))p(\tau^{-1}\circ\tau^{-1}(t))} \\ &\ge \frac{1}{p(\tau^{-1}(t))}\Big[1 - \frac{1}{p(\tau^{-1}\circ\tau^{-1}(t))}\Big]z\big(\tau^{-1}(t)\big) \\ &= P(t)z(\tau^{-1}(t)) \end{aligned} \tag{47}$$

for $t \ge t_2$. By using conditions (47) and (A_5^*) in (15), we obtain

$$\begin{aligned} 0 &\ge \Big(\big|z^{(n-1)}(t)\big|^{\alpha-1}z^{(n-1)}(t)\Big)' + q(t)(P(g(t)))^\alpha z^\alpha\big(\tau^{-1}\circ g(t)\big) \\ &\ge \Big(\big|z^{(n-1)}(t)\big|^{\alpha-1}z^{(n-1)}(t)\Big)' + q(t)\big(P(g(t))\big)^\alpha z^\alpha\big(\sigma^*(t)\big)\,, \end{aligned} \tag{48}$$

for $t \ge t_3 \ge t_2$. Define

$$w(t) = \rho(t)\Big(\frac{z^{(n-1)}(t)}{z(\lambda\sigma^*(t))}\Big)^\alpha, \quad t \ge t_3\,. \tag{49}$$

Thus, for $t \ge t_3$, in view of (49) and (48), we have

$$w'(t) \le -\rho(t)q(t)(P(g(t)))^\alpha + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\alpha\lambda\rho(t)[\sigma^*(t)]'\left(z^{(n-1)}(t)\right)^\alpha z'(\lambda\sigma^*(t))}{z^{\alpha+1}(\lambda\sigma^*(t))}\,. \tag{50}$$

By Lemma 1.2, there is $t_4 \ge t_3$ and a constant λ, $\lambda \in (0,1)$ such that

$$z'\big(\lambda\sigma^*(t)\big) \ge \frac{1}{\lambda}\Theta(n,\lambda)\big[\sigma^*(t)\big]^{n-2}z^{(n-1)}(t) \quad \text{for} \quad t \ge t_4\,. \tag{51}$$

Using (51) in (50), we obtain

$$w'(t) \le -\rho(t)\,q(t)\,\big(P(g(t))\big)^\alpha + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\alpha\Theta(\lambda,n)[\sigma^*(t)]^{n-2}[\sigma^*(t)]'}{\rho^{1/\alpha}(t)}w^{(\alpha+1)/\alpha}(t)\,. \tag{52}$$

The remainder of the proof is similar to that of Lemma 2.1. So we omit the details. □

The following theorem is an immediate result of Lemma 3.1.

Theorem 3.1. *Suppose conditions* (A_1)*–*(A_4) *and* (A_5^*) *hold. Suppose further that one of the conditions* (X^*)*,* (Y^*) *and* (Z^*)*in Lemma 3.1 holds. Then Eq.* (1) *is oscillatory.*

Corollary 3.1. *Suppose* (A_1)*–*(A_4) *and* (A_5^*) *hold and* $\rho \in C^1(I, \mathbb{R}_+)$ *with* $\rho'(t) \geq 0$*. Assume that* (I)*,* (II) *in Definition 1.1 and* (34) *hold. Suppose further that the condition* (X^*) *in Lemma 3.1 holds with* $\mathcal{A}^*(t, t_0)$ *and* $\mathcal{B}^*(t, t_0)$ *replaced by*

$$\mathcal{A}^*(t, t_0) = \frac{1}{H(t, t_0)} \int_{t_0}^{t} H(t, s)\rho(s)q(s)(P(g(t)))^{\alpha}\, ds\,,$$

and

$$\mathcal{B}^*(t, t_0) = \frac{1}{H(t, t_0)} \int_{t_0}^{t} \frac{\rho(s)[h(t, s)]^{\alpha+1}}{[H(t, s)(\sigma^*(s))^{n-2}(\sigma^*(s))']^{\alpha}}\, ds$$

respectively. Then (1)*) is oscillatory.*

Now, we define

$$\mathcal{R}(t) = \int_{t_0}^{t} [\sigma^*(s)]^{n-2}[\sigma^*(s)]'\, ds\,, \quad t \geq t_0\,, \tag{53}$$

and let

$$H(t, s) = [\mathcal{R}(t) - \mathcal{R}(s)]^{\mu}\,, \quad t \geq s \geq t_0$$

where $\mu > \max\{1, \alpha\}$ is a constant.

Theorem 3.2. *Suppose* (A_1)*–*(A_4) *and* (A_5^*) *hold. Then Eq.* (1) *is oscillatory provided that there is some* $\mu > \max\{1, \alpha\}$ *such that one of the following conditions is satisfied:*

(I) *for any* $l \geq t_0$*,*

$$\limsup_{t\to\infty} \frac{1}{\mathcal{R}^{\mu-\alpha}(t)} \int_{l}^{t} [\mathcal{R}(s) - \mathcal{R}(l)]^{\mu} q(s)(P(g(s)))^{\alpha}\, ds > \frac{1}{\Theta^{\alpha}(n, \lambda)} \frac{\mu^{\alpha+1}}{(\alpha+1)^{(\alpha+1)}(\mu-\alpha)} \tag{54}$$

and

$$\limsup_{t\to\infty} \frac{1}{\mathcal{R}^{\mu-\alpha}(t)} \int_{l}^{t} [\mathcal{R}(t) - \mathcal{R}(s)]^{\mu} q(s)(P(g(s)))^{\alpha} ds > \frac{1}{\Theta^{\alpha}(n, \lambda)} \frac{\mu^{\alpha+1}}{(\alpha+1)^{(\alpha+1)}(\mu-\alpha)}\,; \tag{55}$$

(II) $n = 2$ *and for any* $l \geq t_0$*, one of the following conditions is satisfied:*

$$\limsup_{t\to\infty} \frac{1}{\mathcal{R}^{\mu-\alpha}(t)} \int_{l}^{t} [\mathcal{R}(s) - \mathcal{R}(l)]^{\mu} q(s)(P(g(s)))^{\alpha}\, ds > \frac{\mu^{\alpha+1}}{(\alpha+1)^{(\alpha+1)}(\mu-\alpha)} \tag{56}$$

or

$$\limsup_{t\to\infty} \frac{1}{\mathcal{R}^{\mu-\alpha}(t)} \int_l^t [\mathcal{R}(t)-\mathcal{R}(s)]^\mu q(s)(P(g(s)))^\alpha\, ds$$

$$> \frac{\mu^{\alpha+1}}{(\alpha+1)^{(\alpha+1)}(\mu-\alpha)}\,. \tag{57}$$

This theorem can be proved in a manner quite similar to the proof of Theorem 2.2. The details are omitted here.

We remark that different choices of $k(s)$, $\rho(s)$ include 1, s, etc.; while choices of H include $H(t,s) = [R(t) - R(s)]^\beta$, $H(t,s) = [\log Q(t)/Q(s)]^\beta$, or $H(t,s) = \big[\int_s^t \frac{1}{w(z)}\, dz\big]^\beta$, etc., for $t \geq s \geq t_0$, where $\beta > \max\{1,\alpha\}$ is a constant, $R(t) = \int_{t_0}^t ds/u(s)$, $Q(t) = \int_t^\infty ds/u(s) < \infty$, for $t \geq t_0$, and $w \in C([t_0,\infty),\mathbb{R}_+)$ satisfying $\int_{t_0}^\infty ds/w(s) = \infty$.

Remark 3.1. Our results are general since the function $g(t)$ in (1) is only required to satisfy $\lim_{t\to\infty} g(t) = \infty$. Therefore Theorems 2.1–2.2, Theorems 3.1–3.2, Corollaries 2.1 and 3.1 may hold for ordinary, retarded or advanced type equations.

4. Examples

In the following, we will give some applications of our oscillation criteria. We will see that there are equations that cannot be handled by results in [1-4, 6–16], but we may show that they are oscillatory based on our results.

Example 4.1. Let $(n-3)\alpha > 2$, consider even order nonlinear equation

$$\frac{d}{dt}\Big(\big|[x(t)+px(\gamma t)]^{(n-1)}\big|^{\alpha-1}(x(t)+px(\gamma t))^{(n-1)}\Big) + q(t)|x(\nu t)|^{\alpha-1}x(\nu t) = 0\,, \quad n \ \text{ even}, \tag{58}$$

where $p \geq 0$, α, γ, ν are positive constants and $q \in C([1,\infty),\mathbb{R}_+)$. By Corollaries 2.1 and 3.1, we can show that Eq. (58) is oscillatory under some appropriate assumptions.

In fact, choose $\rho(t) = t^\mu$, $k(t) = 1$ and $H(t,s) = (t-s)^\mu$ for $t \geq s \geq 1$ such that

$$\alpha + 1 < \mu < (n-2)\alpha - 1\,, \quad \rho(t)q(t) \geq \frac{c}{t} \quad \text{for} \quad c > 0\,.$$

Then, by simple computations, we may check that (A_1)-(A_4) and

$$h(t,s) = -\frac{\partial}{\partial s}\big(H(t,s)\big) + H(t,s)\frac{\rho'(s)}{\rho(s)} = \mu(t-s)^\mu\Big(1 + \frac{s}{t-s}\Big)$$

hold. From Theorem 4.1 in [5], we have the inequality

$$(t-s)^\mu \geq t^\mu - \mu s t^{\mu-1}\,, \quad \text{for} \quad t \geq s \geq 1\,.$$

(i) Consider the case $0 \le p < 1$. It is easy to see that $\sigma(t) = \nu t$ for $0 < \nu \le 1$. Further (A_5) holds if $0 < \gamma < 1$. Thus

$$\begin{aligned}\limsup_{t\to\infty} \mathcal{A}(t,1) &= \limsup_{t\to\infty} \frac{1}{t^\mu}\int_l^t (t-s)^\mu \rho(s) q(s)(1-p(s))^\alpha \, ds \\ &\ge c(1-p)^\alpha \limsup_{t\to\infty} \frac{1}{t^\mu}\int_1^t \frac{t^\mu - \mu s t^{\mu-1}}{s}\, ds = \infty\end{aligned}$$

and

$$\begin{aligned}\limsup_{t\to\infty} \mathcal{B}(t,1) &= \limsup_{t\to\infty} \frac{1}{H(t,1)}\int_1^t \frac{\rho(s) h^{\alpha+1}(t,s)}{[H(t,s)\sigma^{n-2}(s)\sigma(s)]^\alpha}\, ds \\ &= \limsup_{t\to\infty} \frac{\mu^{\alpha+1}}{\nu^{(n-1)\alpha}} \frac{1}{t^{\mu-\alpha-1}} \int_1^t s^{\mu-(n-2)\alpha}(t-s)^{\mu-\alpha-1} ds \\ &\le \limsup_{t\to\infty} \frac{\mu^{\alpha+1}}{\nu^{(n-1)\alpha}}\Big(1-\frac{1}{t}\Big)^{\mu-\alpha-1}\int_1^t s^{\mu-(n-2)\alpha}\, ds < \infty\,,\end{aligned}$$

i.e., (7) holds for $\lambda \in (0.1)$. Applying Corollary 2.1, Eq. (58) is oscillatory if $0 < p < 1$, $0 < \gamma < 1$ and $0 < \nu \le 1$.

(ii) Consider the case $p > 1$. It is easy to check that

$$\tau^{-1}(t) = \frac{1}{\gamma} t\,, \quad \tau^{-1} o g(t) = \frac{\nu}{\gamma} t\,, \quad \sigma^*(t) = \frac{\nu}{\gamma} t\,, \quad P(t) = \frac{1}{p}\Big(1 - \frac{1}{p}\Big)\,, \quad \text{for} \quad 0 < \nu \le \gamma$$

and (A_5^*) hold if $\gamma > 1$. Similar to the case (i), we get that

$$\limsup_{t\to\infty} \mathcal{A}^*(t,1) = \limsup_{t\to\infty} \frac{1}{t^\mu}\int_l^t (t-s)^\mu \rho(s) q(s) P^\alpha(s)\, ds = \infty$$

and

$$\limsup_{t\to\infty} \mathcal{B}^*(t,1) = \limsup_{t\to\infty} \mathcal{B}(t,1) < \infty\,,$$

imply (44) holds for $\lambda \in (0,1)$. If $p > 1$, $\gamma > 1$ and $0 < \nu \le \gamma$, then (58) is oscillatory by Corollary 3.1.

Therefore, under the following condition

$$q(t) \ge \frac{c}{t^{\mu+1}} \quad \text{for} \quad c > 0\,,$$

where $\alpha + 1 < \mu < (n-2)\alpha - 1$, we conclude the following:

(i) If $0 < p < 1$, $0 < \gamma < 1$ and $0 < \nu \le 1$, then (58) is oscillatory by Corollary 2.1.

(ii) If $p > 1$, $\gamma > 1$ and $0 < \nu \le \gamma$, then (58) is oscillatory by Corollary 3.1.

Next, we shall construct an example including the following Euler equation as a special case:

$$x'' + \frac{c}{t^2} x = 0\,. \tag{59}$$

The following example also illustrates Theorem 2.2.

Example 4.2. Let $0 < c$, $0 \le p < 1$, $0 < \alpha$ and $0 < \gamma < 1$. Consider the even order nonlinear equation

$$\frac{d}{dt}\Big(\big|[x(t)+px(\gamma t)]^{(n-1)}\big|^{\alpha-1}(x(t)+px(\gamma t))^{(n-1)}\Big) + \frac{c\sigma^{n-2}(t)\sigma'(t)}{R^{\alpha+1}(t)}\big|x(g(t))\big|^{\alpha-1}x\big(g(t)\big) = 0\,, \quad n \text{ even}, \quad t \ge t_0\,, \tag{60}$$

where g satisfies (A_3), σ satisfies (A_5) and R is defined as in (35). Let $\alpha_0 := \max\{1,\alpha\}$. Then we can verify that (60) is oscillatory for

$$c > c_0 := \Big(\frac{(n-2)!2^{2n-4}}{1-p}\Big)^{\alpha}\frac{\alpha_0^{\alpha+1}}{(\alpha+1)^{\alpha+1}}$$

by the case (I) of Theorem 2.2.

Choose $H(t,s) = [R(t)-R(s)]^{\mu}$ for $\mu > \alpha_0$. Note that $\mu > \alpha_0 \ge 1$ and

$$[R(s)-R(l)]^{\mu} \ge R^{\mu}(s) - \mu R(l)R^{\mu-1}(s) \quad \text{for} \quad s \ge l \ge t_0$$

and

$$[R(t)-R(s)]^{\mu} \ge R^{\mu}(t) - \mu R(s)R^{\mu-1}(t) \quad \text{for} \quad t \ge s \ge t_0\,.$$

It follows from $R'(t) = \sigma^{n-2}(t)\sigma'(t)$ that for each $l \ge t_0$

$$d\,R(s) = \sigma^{n-2}(s)\sigma'(s)ds$$

and thus

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{R^{\mu-\alpha}(t)}\int_l^t [R(s)-R(l)]^{\mu}\frac{c\sigma^{n-2}(s)\sigma'(s)}{R^{\alpha+1}(s)}(1-p(s))^{\alpha}\,ds\\ &\ge c(1-p)^{\alpha}\limsup_{t\to\infty}\frac{1}{R^{\mu-\alpha}(t)}\int_1^t \frac{R^{\mu}(s)-\mu R(l)R^{\mu-1}(s)}{R^{\alpha+1}(s)}dR(s) = \frac{c(1-p)^{\alpha}}{\mu-\alpha}\,.\end{aligned}$$

For any $c > c_0$, there exists $\mu > \alpha_0$ such that

$$\frac{c(1-p)^{\alpha}}{\mu-\alpha} > \big[(n-2)!2^{2n-4}\big]^{\alpha}\,\frac{\mu^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\mu-\alpha)}\,. \tag{61}$$

Moreover, it is easy to see that

$$\big[(n-2)!2^{2n-4}\big]^{\alpha}\,\frac{\mu^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\mu-\alpha)} \ge \frac{1}{\Theta^{\alpha}(n,\lambda)}\,\frac{\mu^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\mu-\alpha)}\,, \tag{62}$$

for $\lambda \in (0,1)$. From (61) and (62), we see that (37) holds.

On the other hand, we get

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{R^{\mu-\alpha}(t)}\int_l^t [R(t)-R(s)]^{\mu}\frac{c\sigma^{n-2}(s)\sigma'(s)}{R^{\alpha+1}(s)}\,(1-p(s))^{\alpha}\,ds\\ &\quad\ge c(1-p)^{\alpha}\limsup_{t\to\infty}\frac{1}{R^{\mu-\alpha}(t)}\int_1^t \big[R^{\mu}(t)-\mu R(s)R^{\mu-1}(t)\big]\,\frac{1}{R^{\alpha+1}(s)}\,dR(s)\,.\end{aligned}$$

Noting that when $\alpha = 1$,

$$\limsup_{t\to\infty} \frac{1}{R^{\mu-\alpha}(t)} \int_1^t \left[R^{\mu}(t) - \mu R(s) R^{\mu-1}(t)\right] \frac{1}{R^{\alpha+1}(s)}\, dR(s)$$
$$= \lim_{t\to\infty} \left(\frac{1}{R(l)} R(t) - \mu \ln R(t) + \mu \ln R(l) - 1 \right) = \infty$$

when $\alpha \neq 1$,

$$\limsup_{t\to\infty} \frac{1}{R^{\mu-\alpha}(t)} \int_1^t \left[R^{\mu}(t) - \mu R(s) R^{\mu-1}(t)\right] \frac{1}{R^{\alpha+1}(s)}\, dR(s)$$
$$= \lim_{t\to\infty} \left(\frac{R^{-\alpha}(l)}{\alpha} R^{\alpha}(t) + \frac{\mu}{1-\alpha} R^{1-\alpha}(l) R^{1-\alpha}(t) - \frac{1}{\alpha} - \frac{\mu}{1-\alpha} \right) = \infty\,.$$

Then for any $c > 0$, $1 > p \geq 0$, $\alpha > 0$ and $\mu > \alpha_0$, we obtain that

$$\limsup_{t\to\infty} \frac{1}{R^{\mu-\alpha}(t)} \int_l^t [R(t) - R(s)]^{\mu} \frac{c\sigma^{n-2}(s)\sigma'(s)}{R^{\alpha+1}(s)} (1 - p(s))^{\alpha}\, ds = \infty\,.$$

In view of Theorem 2.2 (I), we see that (60) is oscillatory for $\mu > c_0$.

Remark 4.1. The results in [1-16] fail to apply to Eq. (60). However, there are many equations which satisfy the hypotheses of Example 4.2. For example, we may choose $0 < g(t) \leq \nu t^{\beta}$ with $0 < \nu$, $\beta \leq 1$ for $t \geq 0$; here we omit the details. In particular, noting that Eq. (60) with $p = 0$, $n = 2$ and $g(t) = t - \delta$ $(\delta \geq 0)$ for $t \geq t_0\colon\ = 0$ becomes

$$\text{(63)} \qquad \left(|x'(t)|^{\alpha-1} x'(t)\right)' + \frac{c}{t^{\alpha+1}} |x(t-\delta)|^{\alpha-1} x(t-\delta) = 0\,, \qquad t \geq 0\,.$$

Then, by Example 4.2, Eq. (63) is oscillatory for $c > c_0 = \alpha_0^{\alpha+1}/(\alpha+1)^{\alpha+1}$. We note that this conclusion does not appear to follow from the known oscillation criteria in the literature. Moreover, when $\alpha = 1$ and $\delta = 0$, Eq. (63) reduces to Eq. (59). In this case, $c_0 = 1/4$, then Example 4.2 is consistent with the well-known result of (59) that Eq. (59) is oscillatory if $c > 1/4$ and to a certain extent it also reveals some of the peculiar nature of the Euler equation (59).

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