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IDEAL AMENABILITY OF MODULE EXTENSIONS OF BANACH ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a Banach algebra. $\mathcal{A}$ is called ideally amenable if for every closed ideal I of $\mathcal{A}$, the first cohomology group of $\mathcal{A}$ with coefficients in I^* is zero, i.e. $H^1(\mathcal{A}, I^*) = \{0\}$. Some examples show that ideal amenability is different from weak amenability and amenability. Also for $n \in \mathbb{N}$, $\mathcal{A}$ is called n-ideally amenable if for every closed ideal I of $\mathcal{A}$, $H^1(\mathcal{A}, I^{(n)}) = \{0\}$. In this paper we find the necessary and sufficient conditions for a module extension Banach algebra to be 2-ideally amenable.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}$-bimodule. Then X^*, the dual space of X, with the following module actions is a Banach $\mathcal{A}$-bimodule:

$$\langle x, a \cdot x^* \rangle = \langle x \cdot a, x^* \rangle\,, \quad \langle x, x^* \cdot a \rangle = \langle a \cdot x, x^* \rangle\,, \quad (a \in \mathcal{A},\ x \in X,\ x^* \in X^*)\,.$$

In particular, if I is a closed ideal in $\mathcal{A}$, then I and I^* will be a Banach $\mathcal{A}$-bimodule and a dual Banach $\mathcal{A}$-bimodule respectively. A bounded linear operator $D\colon \mathcal{A} \to X$ is called a derivation if

$$D(ab) = D(a) \cdot b + a \cdot D(b) \qquad (a, b \in \mathcal{A})\,.$$

For $x \in X$, we put $\delta_x\colon \mathcal{A} \to X$ by

$$\delta_x(a) = a \cdot x - x \cdot a \qquad (a \in \mathcal{A})\,.$$

It is clear that δ_x is a derivation. Derivations of this form are called inner derivations. A Banach algebra $\mathcal{A}$ is amenable if for every Banach $\mathcal{A}$-bimodule X, every derivation from $\mathcal{A}$ into X^* is inner; i.e., $H^1(\mathcal{A}, X^*) = \{0\}$, where $H^1(\mathcal{A}, X^*)$ is the first cohomology group of $\mathcal{A}$ with coefficients in X^*. Johnson has introduced the concept of amenability of Banach algebras [12]. A Banach algebra $\mathcal{A}$ is weakly amenable if $H^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$ (see [3], [9], [10] and [13]). Bade, Curtis and Dales [1] defined the concept of weak amenability for commutative Banach algebras. Let $n \in \mathbb{N}$; a Banach algebra $\mathcal{A}$ is called n-weakly amenable if $H^1(\mathcal{A}, \mathcal{A}^{(n)}) = \{0\}$.

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Dales, Ghahramani and Grønbæk brought the concept of n-weak amenability of Banach algebras in [2].

Definition 1.1. A Banach algebra $\mathcal{A}$ is called ideally amenable if for every closed ideal I of $\mathcal{A}$; $H^1(\mathcal{A}, I^*) = \{0\}$.

Definition 1.2. A Banach algebra $\mathcal{A}$ is called n-ideally amenable if for every closed ideal I of $\mathcal{A}$; $H^1(\mathcal{A}, I^{(n)}) = \{0\}$.

2. Some Examples

Obviously, amenability implies ideal amenability and ideal amenability implies weak amenability. However, the following examples show that the converse is not valid.

Example 2.1. Consider the algebra $\mathcal{A} = B(H)$ of bounded linear operators on some infinite-dimensional separable Hilbert space H. Then $\mathcal{A}$ has exactly two nonzero closed ideals $I_0 = K(H)$, the compact operators on H, and $I_1 = B(H)$. Denoting by $N(H)$ the space of nuclear or trace-class operators on H, we have

$$H^1(\mathcal{A}, I_0^*) = H^1(B(H), N(H)) = \{0\}\,, \tag{1}$$

$$H^1(\mathcal{A}, I_1^*) = H^1(B(H), (B(H))^*) = \{0\}\,. \tag{2}$$

To prove (1), take any bounded derivation $D\colon B(H) \to N(H)$. The restriction of D to $K(H)$ is a derivation $D_0\colon K(H) \to N(H) = K(H)^*$ and hence of the form $D_0(T) = AT - TA$, $\big(T \in K(H)\big)$, for some $A \in N(H)$ [11, Corollary 4.2]. But D_0 being weakly compact, the above equation extends to all of $B(H)$, such that $D(T) = AT - TA$, $\big(T \in B(H)\big)$, showing that D is inner. The proof of (2) follows directly from the result of Haagerup just quoted. Thus $\mathcal{A} = B(H)$ is an example of an ideally amenable Banach algebra that is not amenable [15].

We know that $B(H)$ is a C^*-algebra. So, one might wonder about the ideal amenability of C^*-algebras. Here we have:

Example 2.2. All C^*-algebras $\mathcal{A}$ are ideally amenable. Indeed let I be a closed two-sided ideal in $\mathcal{A}$ and let $D\colon \mathcal{A} \to I^*$ be a derivation. Since the restriction of D to I is again a derivation and I is a C^*-algebra in its own right, there exists by [11] an $f \in I^*$ such that $Db = bf - fb$ for all $b \in I$. We have to show that this holds true for all $a \in \mathcal{A}$. For an approximate identity (e_α) in I and $b \in I$ and $a \in \mathcal{A}$ we have

$$\begin{aligned}\big\langle e_\alpha b, D(a)\big\rangle &= \big\langle b, D(a)e_\alpha\big\rangle = \big\langle b, D(ae_\alpha) - aD(e_\alpha)\big\rangle \\ &= \big\langle b, (ae_\alpha)f - f(ae_\alpha)\big\rangle - \big\langle ba, e_\alpha f - fe_\alpha\big\rangle \\ &= \big\langle (ba)e_\alpha - a(e_\alpha b), f\big\rangle - \big\langle (ba)e_\alpha - e_\alpha(ba), f\big\rangle\,.\end{aligned}$$

Such that in the limit

$$\big\langle b, D(a)\big\rangle = \langle ba - ab, f\rangle = \langle b, af - fa\rangle\,,$$

i.e. $Da = af - fa$ for all $a \in \mathcal{A}$. This means $H^1(\mathcal{A}, I^*) = \{0\}$. Therefore $\mathcal{A}$ is ideally amenable.

Remark 2.3. A C^*-algebra is amenable if and only if it is nuclear [11]. So, a non-nuclear C^*-algebra is not amenable, but ideally amenable.

Let $n \in \mathbb{N}$, then the following assertions hold.

a) Every n-ideally amenable Banach algebra is n-weakly amenable.
b) An amenable Banach algebra is n-ideally amenable.
c) Every $(n+2)$-ideally amenable Banach algebra is n-ideally amenable.
d) Every weakly amenable commutative Banach algebra is n-ideally amenable.
e) A commutative Banach algebra $\mathcal{A}$ is weakly amenable if and only if $\mathcal{A}$ is $(2n-1)$-ideally amenable.

The assertions (a) and (b) above are obvious. Assertion (c) is Theorem 1.5 of [7]. Also (d) and (e) follow from Theorem 1.5 of [1].

Example 2.4. Let $\alpha \in (0, \frac{1}{2})$, and let (K, d) be an infinite compact metric space. Then $\mathcal{A} = \mathrm{lip}_\alpha(K)$ is weakly amenable Banach algebra that is not amenable [1]. $\mathcal{A}$ is commutative, then by assertion (d) above, $\mathcal{A}$ is ideally amenable.

There are also some examples of Banach algebras which show that ideal amenability is not equivalent to weak amenability. In the following we give one of them.

Example 2.5. Let $\mathcal{A} = L^1(G)$, where $G = SL(2, \mathbb{R})$, the set of elements in $\mathbb{M}_2(\mathbb{R})$ with determinant one. Also let $I = \{f \in L^1(G) \colon \int_G f(g)dm_G(g) = 0\}$, the augmentation ideal of $\mathcal{A}$. By Theorem 5.2 of [14]; $H^1(\mathcal{A}, I^*) \neq \{0\}$. So, $\mathcal{A}$ is not ideally amenable. On the other hand, for every locally compact group G, $L^1(G)$ is weakly amenable [13]. Thus $\mathcal{A}$ is weakly amenable.

For more examples see [5] and [8].

3. Module extension Banach algebras

Let $\mathcal{A}$ and X be a Banach algebra and a Banach $\mathcal{A}$-bimodule respectively. Consider $\mathcal{A} \oplus X$ as a Banach space with the following norm

$$\big\|(a, x)\big\| = \|a\| + \|x\| \qquad (a \in \mathcal{A}\ ,\ x \in X)\,.$$

Then $\mathcal{A} \oplus X$ is a Banach algebra with the product

$$(a_1, x_1)(a_2, x_2) = (a_1 a_2, x_1 \cdot a_2 + a_1 \cdot x_2)\,.$$

$\mathcal{A} \oplus X$ is called a module extension Banach algebra. Since $(\mathcal{A} \oplus X)^* = (0 \oplus X)^\perp \dotplus (\mathcal{A} \oplus 0)^\perp$, where $\dotplus$ denotes the direct $\mathcal{A}$-bimodule l_∞-sum, and $(0 \oplus X)^\perp$ (respectively, $(\mathcal{A} \oplus 0)^\perp$) is isometrically isomorphic to $\mathcal{A}^*$ (respectively, X^*) as $\mathcal{A}$-bimodules, for convenience, we simply identify the corresponding terms and write

$$(\mathcal{A} \oplus X)^* = \mathcal{A}^* \dotplus X^*\,.$$

Take $A^{(n)} \dotplus X^{(n)}$ as the underlying space of $(A \oplus X)^{(n)}$. The sum is an l_1-sum when n is even and is an l_∞-sum when n is odd. One can verify that the $(A \oplus X)$-bimodule actions on $(A \oplus X)^{(n)}$ for $(a, x) \in A \oplus X$ and $(a^{(n)}, x^{(n)}) \in A^{(n)} \dot{+} X^{(n)} = (A \oplus X)^{(n)}$ are formulated as follows:

$$(a, x)\big(a^{(n)}, x^{(n)}\big) = (aa^{(n)} + xx^{(n)}, ax^{(n)})$$

and

$$\big(a^{(n)}, x^{(n)}\big)(a, x) = \big(a^{(n)}a + x^{(n)}x, x^{(n)}a\big)$$

where n is odd, and

$$(a, x)\big(a^{(n)}, x^{(n)}\big) = \big(aa^{(n)}, ax^{(n)} + xa^{(n)}\big)$$

and

$$\big(a^{(n)}, x^{(n)}\big)(a, x) = \big(a^{(n)}a, a^{(n)}x + x^{(n)}a\big)$$

where n is even.

We need the following lemma for the main result of paper.

Lemma 3.1. *Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}$-bimodule. Then J is a closed two sided ideal of $\mathcal{A} \oplus X$, if and only if there exist a closed ideal I of $\mathcal{A}$ and a closed $\mathcal{A}$-submodule Y of X such that $J = I \oplus Y$ and $IX \cup XI \subseteq Y$.*

Yong Zhang in [16] found a necessary and sufficient condition for a module extension Banach algebra to be n-weakly amenable ($n = 1, 2, \dots$). Also in [6, Theorem 2.4], it has been proved that:

Theorem 3.2. *$\mathcal{A} \oplus X$ is ideally amenable if and only if for arbitrary ideal $I \oplus Y$ of $\mathcal{A} \oplus X$ the following conditions hold:*

1. *$H^1(\mathcal{A}, I^*) = \{0\}$;*
2. *$H^1(\mathcal{A}, Y^*) = \{0\}$;*
3. *For every continuous $\mathcal{A}$-bimodule morphism $\Gamma\colon X \to I^*$, there exists $F \in Y^*$ such that $aF - Fa = 0$ for $a \in \mathcal{A}$ and $\Gamma(x) = xF - Fx$ for $x \in X$;*
4. *The only continuous $\mathcal{A}$-bimodule morphism $T\colon X \to Y^*$ for which $xT(y) + T(x)y = 0$ $(x, y \in X)$ in I^* is $T = 0$.*

We prove the similar argument for n-ideal amenability when $n = 2$.

Lemma 3.3. *Suppose that $T\colon X \to Y^{**}$ is a continuous $\mathcal{A}$-bimodule morphism. Then $T\colon \mathcal{A} \oplus X \to (I \oplus Y)^{**}$, defined by $\bar{T}\big((a, x)\big) = \big(0, T(x)\big)$ is a continuous derivation. $\bar{T}$ is inner if and only if there exists $u \in I^{**}$ such that $ua = au$ for $a \in \mathcal{A}$ and $T(x) = xu - ux$ for all $x \in X$.*

Proof. Let $(a, x), (b, y) \in \mathcal{A} \oplus X$. We have

$$\bar{T}\big((a, x) \cdot (b, y)\big) = \bar{T}\big((ab, ay + xb)\big) = \big(0, T(ay + xb)\big) = \big(0, aT(y) + T(x)b\big).$$

On the other hand

$$\bar{T}\big((a, x)\big) \cdot (b, y) = \big(0, T(x)\big) \cdot (b, y) = \big(0, 0 + T(x)b\big)$$

and

$$(a,x)\cdot\bar{T}\big((b,y)\big)=(a,x)\cdot\big(0,T(y)\big)=\big(0,aT(y)+0\big)\,.$$

It is clear that $\bar{T}$ is continuous and thus $\bar{T}$ is a derivation. Let $\bar{T}$ be inner, then there exist $u\in I^{**}$ and $F\in Y^{**}$ such that

$$\bar{T}\big((a,x)\big)=(a,x)\cdot(u,F)-(u,F)\cdot(a,x)=(au-ua,aF-Fa+xu-ux)\,,$$

but

$$\big(0,T(x)\big)=\bar{T}\big((0,x)\big)=(0,xu-ux)$$

and

$$(0,0)=\bar{T}\big((a,0)\big)=(au-ua,aF-Fa)\,.$$

It shows that $au=ua$ and therefore there exists $u\in I^{**}$ such that $T(x)=xu-ux$ $(x\in X)$. For converse, let $au=ua$ and there exists $u\in I^{**}$ such that $T(x)=xu-ux$ $(x\in X)$. We have

$$\bar{T}\big((a,x)\big)=\big(0,T(x)\big)=(au-ua,xu-ux)$$

and hence $\bar{T}\big((a,x)\big)=(a,x)\cdot(u,0)-(u,0)\cdot(a,x)$, where $(u,0)\in(I\oplus Y)^{**}$. Then $\bar{T}$ is inner and proof is complete. □

If $D\colon\mathcal{A}\to Y^{**}$ is a continuous derivation, we define $\bar{D}\colon\mathcal{A}\oplus X\to(I\oplus Y)^{**}$ by $\bar{D}((a,x))=(0,D(a))$. Also, if $T\colon X\to I^{**}$ is a continuous $\mathcal{A}$-bimodule morphism such that $xT(y)+T(x)y=0$, we define $\bar{T}\colon\mathcal{A}\oplus X\to(I\oplus Y)^{**}$ by $\bar{T}((a,x))=(T(x),0)$.

Lemma 3.4. *The operators $\bar{D}$ and $\bar{T}$ defined above are continuous derivations. Furthermore, the derivation $\bar{D}$ is inner if and only if D is inner, and $\bar{T}$ is inner if and only if $T=0$.*

Proof. It is clear that $\bar{D}$ and $\bar{T}$ are continuous derivations. Let $\bar{D}$ be inner and $(a,x)\in\mathcal{A}\oplus X$ be arbitrary. There exist $u\in I^{**},F\in Y^{**}$ such that $\bar{D}\big((a,x)\big)=(a,x)\cdot(u,F)-(u,F)\cdot(a,x)=(au-ua,aF-Fa+xu-ux)$. But

$$\big(0,D(a)\big)=\bar{D}\big((a,0)\big)=(au-ua,aF-Fa)$$

and

$$(0,0)=\bar{D}\big((0,x)\big)=(0,xu-ux)\,.$$

Then $D(a)=aF-Fa$ for some $F\in Y^{**}$ and so D is inner. For converse, let D be inner. There exists $F\in Y^{**}$ such that $D(a)=aF-Fa$ $(a\in\mathcal{A})$. Then

$$\bar{D}\big((a,x)\big)=\big(0,D(a)\big)=(0,aF-Fa)=(a,x)\cdot(0,F)-(0,F)\cdot(a,x)\,.$$

This means that there exists $\xi=(0,F)\in(I\oplus Y)^{**}$ such that $\bar{D}\big((a,x)\big)=(a,x)\cdot\xi-\xi\cdot(a,x)$ $\big((a,x)\in\mathcal{A}\oplus X\big)$. Then $\bar{D}$ is inner. Now let $\bar{T}$ be inner. There exist $u\in I^{**},F\in Y^{**}$ such that for each $(a,x)\in\mathcal{A}\oplus X$,

$$\bar{T}\big((a,x)\big)=(a,x)\cdot(u,F)-(u,F)\cdot(a,x)=(au-ua,aF-Fa+xu-ux)\,.$$

But

$$\big(T(x),0\big) = \bar{T}\big((0,x)\big) = (0, xu - ux)$$

and so $T(x) = 0$, for every $x \in X$. The converse is trivial. □

Now we find a necessary and sufficient condition for a module extension Banach algebra to be 2-ideally amenable.

Theorem 3.5. *$\mathcal{A} \oplus X$ is 2-ideally amenable if and only if for every arbitrary ideal $I \oplus Y$ of $\mathcal{A} \oplus X$ the following conditions hold:*

1. *the only continuous derivations $D\colon \mathcal{A} \to I^{**}$ for which there is a continuous operator $T\colon X \to Y^{**}$ such that $T(ax) = D(a)x + aT(x)$ and $T(xa) = xD(a) + T(x)a$ ($a \in \mathcal{A}, x \in X$) are the inner derivations;*
2. *$H^1(\mathcal{A}, Y^{**}) = \{0\}$;*
3. *the only continuous $\mathcal{A}$-bimodule morphism $\Gamma\colon X \to I^{**}$ for which $x\Gamma(y) + \Gamma(x)y = 0$ ($x, y \in X$) in Y^{**} is zero;*
4. *for every continuous $\mathcal{A}$-bimodule morphism $T\colon X \to Y^{**}$, there exists $u \in I^{**}$ for which $au = ua$ for $a \in \mathcal{A}$ and $T(x) = xu - ux$ for $x \in X$.*

Proof. Let $I \oplus Y$ be an arbitrary ideal of $\mathcal{A} \oplus X$. Denote by τ_1 and τ_2 the inclusion mappings from, respectively, $\mathcal{A}$ and X into $\mathcal{A} \oplus X$, and denote by Δ_1 and Δ_2 the natural projections from $(I \oplus Y)^{**}$ onto I^{**} and Y^{**}, respectively. These are $\mathcal{A}$-bimodule morphisms. To prove the sufficency we assume that Conditions 1–4 hold. Let $D\colon \mathcal{A} \oplus X \to (I \oplus Y)^{**}$ be a continuous derivation. Then $\Delta_1 \circ D \circ \tau_1\colon \mathcal{A} \to I^{**}$ and $\Delta_2 \circ D \circ \tau_1\colon \mathcal{A} \to Y^{**}$ are continuous derivations.

Claim 1: $\Delta_1 \circ D \circ \tau_2\colon X \to I^{**}$ is trivial.
Let $\Gamma = \Delta_1 \circ D \circ \tau_2$. To prove Claim 1, by Condition 3 it suffices to show that Γ is an $\mathcal{A}$-bimodule morphism satisfying $x\Gamma(y) + \Gamma(x)y = 0$ $(x, y \in X)$.

$$\begin{aligned} 0 &= D\big((0,0)\big) = D\big((0,x)\cdot(0,y)\big) \\ &= D\big((0,x)\big)\cdot(0,y) + (0,x)\cdot D\big((0,y)\big) = \big(0,\Gamma(x)y\big) + \big(0,x\Gamma(y)\big) \end{aligned}$$

Thus $x\Gamma(y) + \Gamma(x)y = 0$. On the other hand,

$$\begin{aligned} \Gamma(ax) &= \Delta_1 \circ D\big((0,ax)\big) = \Delta_1 \circ D\big((a,0)\cdot(0,x)\big) \\ &= \Delta_1\big(D\big((a,0)\big)\cdot(0,x) + (a,0)\cdot D\big((0,x)\big)\big) \\ &= \Delta_1\big((a,0)\cdot D\big((0,x)\big)\big) = \Delta_1\big(aD\circ\tau_2(x)\big) = a\Gamma(x)\,. \end{aligned}$$

Similarly, $\Gamma(xa) = \Gamma(x)a$ and so Γ is an $\mathcal{A}$-bimodule morphism. Therefore claim 1 is true. Now let $T = \Delta_2 \circ D \circ \tau_2\colon X \to Y^{**}$ and $D_1 = \Delta_1 \circ D \circ \tau_1\colon \mathcal{A} \to I^{**}$.

Claim 2: $T(ax) = D_1(a)x + aT(x)$ and $T(xa) = xD_1(a) + T(x)a$ for $a \in \mathcal{A}$ and $x \in X$.

$$\begin{aligned} \big(0,T(ax)\big) &= \big(0,\Delta_2\circ D\big((0,ax)\big)\big) = D\big((0,ax)\big) = D\big((a,0)\cdot(0,x)\big) \\ &= D\big((a,0)\big)\cdot(0,x) + (a,o)\cdot D\big((0,x)\big) = \big(0,D_1(a)x\big) + a\big(0,T(x)\big) \\ &= \big(0,D_1(a)x + aT(x)\big)\,. \end{aligned}$$

Similarly, for every $a \in \mathcal{A}$ and $x \in X$, we have $\big(0, T(ax)\big) = \big(0, xD_1(a) + T(x)a\big)$. Thus Claim 2 holds. Therefore by Condition 1, $D_1 = \Delta_1 \circ D\tau_1$ is inner.

Now suppose that $u \in I^{**}$ satisfies $D_1(a) = au - ua$ for $a \in \mathcal{A}$. Let $T_1\colon X \to Y^{**}$ be defined by $T_1(x) = xu - ux$ for $x \in X$. Then $T - T_1\colon X \to Y^{**}$ is a continuous $\mathcal{A}$-bimodule morphisms. In fact, from Claim 2, for every $a \in \mathcal{A}$ and $x \in X$, we have

$$\begin{aligned}(T - T_1)(ax) &= T(ax) - T_1(ax) = \big(D_1(a)x + aT(x)\big) - (axu - uax)\\ &= (au - ua)x + aT(x) - (axu - uax) = a(ux - xu) + aT(x)\\ &= a(T - T_1)(x)\,.\end{aligned}$$

Similarly, $T - T_1$ is a right $\mathcal{A}$-bimodule morphism. From Condition 4, there is a $v \in I^{**}$ such that $av = va$ for $a \in \mathcal{A}$ and $(T - T_1)(x) = xv - vx$ for $x \in X$. By Lemma 3.2, we know that

$$\overline{T - T_1}\colon \mathcal{A} \oplus X \to (I \oplus Y)^{**}, \quad (a, x) \mapsto \big(0, (T - T_1)(x)\big)$$

is an inner derivation. Since $\Delta_2 \circ D \circ \tau_1\colon A \to Y^{**}$ is a continuous derivation, it is inner by Condition 2. By Lemma 2.4, the mapping

$$\overline{\Delta_2 \circ D \circ \tau_1}\colon \mathcal{A} \oplus X \to (I \oplus Y)^{**}, \quad (a, x) \mapsto \big(0, \Delta_2 \circ D \circ \tau_1(a)\big)$$

is also inner derivation. Using Claim 1, we now have

$$\begin{aligned}D\big((a, x)\big) &= \big(D_1(a), \Delta_2 \circ D \circ \tau_1(a) + T(x)\big)\\ &= \overline{\Delta_2 \circ D \circ \tau_1}\big((a, x)\big) + \overline{(T - T_1)}\big((a, x)\big) + \big(D_1(a), T(x)\big)\,.\end{aligned}$$

Since

$$\big(D_1(a), T_1(x)\big) = (au - ua, xu - ux) = (a, x) \cdot (u, 0) - (u, 0) \cdot (a, x)$$

for $a \in \mathcal{A}$ and $x \in X$, it gives an inner derivation from $\mathcal{A} \oplus X$ into $(I \oplus Y)^{**}$. Hence as a sum of three inner derivations, D is inner. Thus under Conditions 1-4, $\mathcal{A} \oplus X$ is 2-ideally amenable.

Now we prove the necessity. Suppose that $\mathcal{A} \oplus X$ is 2-ideally amenable. Let $D\colon \mathcal{A} \to I^{**}$ be a continuous derivation with the property given in Condition 1. We define $\bar{D}\colon \mathcal{A} \oplus X \to (I \oplus Y)^{**}$ by

$$\bar{D}\big((a, x)\big) = \big(D(a), T(x)\big) \quad \big((a, x) \in \mathcal{A} \oplus X\big)\,.$$

$\bar{D}$ is a continuous derivation. $\bar{D}$ is inner, so there exists $(u, F) \in (I \oplus Y)^{**}$ such that

$$\bar{D}\big((a, x)\big) = (a, x) \cdot (u \cdot F) - (u, F) \cdot (a, x)\,,$$

and then for some $u \in I^{**}$, we have $(D(a), T(x)) = (au - ua, xF - Fx)$, thus $D(a) = au - ua$, this means that D is inner, and Condition 1 holds.
Conditions 2 and 3 hold by Lemma 3.4. Also Condition 4 holds by Lemma 3.3. □

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