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HYBRID FIXED POINT THEORY FOR RIGHT MONOTONE INCREASING MULTI-VALUED MAPPINGS AND NEUTRAL FUNCTIONAL DIFFERENTIAL INCLUSIONS

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Abstract. In this paper, some hybrid fixed point theorems for the right monotone increasing multi-valued mappings in ordered Banach spaces are proved via measure of noncompactness and they are further applied to the neutral functional nonconvex differential inclusions involving discontinuous multi-functions for proving the existence results under mixed Lipschitz, compactness and right monotonicity conditions. Our results improve the multi-valued hybrid fixed point theorems of Dhage [10] under weaker convexity conditions.

1. Introduction

Multi-valued mappings and fixed points is an important topic of multi-valued analysis and has a wide range of applications to the problems of differential and integral inclusions, control theory and optimization. Geometrical fixed point theory for multi-valued mappings initiated by Nadler (see Hu and Papageorgiou [20]) has been developed to its peak point, but the fixed point theorem of Covitz and Nadler [5] for multi-valued mappings is the only result useful for applications to differential and integral inclusions. Similarly topological fixed point theory for multi-valued mappings has also reached to its culminating point and much has been discussed in relation to differential inclusions (see Andres and Gorniewicz [3] and the references therein). But the case with the algebraic fixed point theory for multi-valued mappings is quite different. This is because of the fact that the comparison between two sets is not unique. A few results in this direction are found in Dhage [7] and Hu and Heikkilä [17]. Recently this topic is revisited by the present author(see Dhage [8, 9, 10]) and established several fixed point theorems for the multi-valued mappings in ordered spaces. In this paper, we establish some hybrid fixed point theorems for three right monotone increasing multi-valued mappings satisfying some mixed hypotheses from algebra, geometry and topology.

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Neutral functional differential equations is an important topic of functional differential equations and an exhaustive treatment may be found in Hale [15] and Ntouyas [20]. However, the study of neutral differential differential inclusions is relatively recent, but fast growing topic in the theory of differential inclusions. As already mentioned that the multi-valued hybrid fixed point theory finds several applications to differential inclusions for proving the existence theorems (see Dhage [7, 8, 9, 10, 11, 12], Dhage and Ntouyas [13] and the references therein). Almost all the results so far discussed in the literature, involve the assumption that the multi-valued functions in question satisfy certain kind of convexity condition. The order theoretic approach to the operator inclusions or differential inclusions allows us to remove this stringent condition in establishing the existence results. In this paper, we prove the existence results for certain perturbed first order neutral functional differential inclusion under the mixed Lipschitz, compactness and monotonicity conditions of multi-valued functions. We claim that our results are new to the theory of multi-valued analysis and include several existence results for operator and differential inclusions in the literature as special cases.

2. Preliminaries

Throughout this paper, unless otherwise mentioned, let X denote a Banach space with norm $\|\cdot\|$ and let $\mathcal{P}_p(X)$ denote the class of all non-empty subsets of X with property p. Here, p may be $p =$ *closed* (in short cl) or $p =$ *convex* (in short cv) or $p =$ *bounded* (in short bd) or $p =$ *compact* (in short cp). Thus $\mathcal{P}_{\mathrm{cl}}(X)$, $\mathcal{P}_{\mathrm{cv}}(X)$, $\mathcal{P}_{\mathrm{bd}}(X)$ and $\mathcal{P}_{\mathrm{cp}}(X)$ denote, respectively, the classes of all closed, convex, bounded and compact subsets of X. Similarly, $\mathcal{P}_{\mathrm{cl,bd}}(X)$ and $\mathcal{P}_{\mathrm{cp,cv}}(X)$ denote, respectively, the classes of closed-bounded and compact-convex subsets of X. For $x \in X$ and $Y, Z \in \mathcal{P}_{\mathrm{bd,cl}}(X)$ we denote by $D(x, Y) = \inf\{\|x - y\| \mid y \in Y\}$, and $\rho(Y, Z) = \sup_{a\in Y} D(a, Z)$. Define a function $d_H \colon \mathcal{P}_{\mathrm{cl}}(X) \times \mathcal{P}_{\mathrm{cl}}(X) \to \mathbb{R}^+$ by

$$d_H(Y, Z) = \max\{\rho(Y, Z)\,,\, \rho(Z, Y)\}\,. \tag{2.1}$$

The function d_H is called a Hausdorff metric on X. Note that $\|Y\|_{\mathcal{P}} = d_H(Y, \{0\})$.

A correspondence $T\colon X \to \mathcal{P}_p(X)$ is called a multi-valued mapping or operator on X. A point $x_0 \in X$ is called a fixed point of the multi-valued operator $T\colon X \to \mathcal{P}_p(X)$ if $x_0 \in T(x_0)$. The fixed points set of T in X will be denoted by $\mathcal{F}_T$.

Definition 2.1. Let $T\colon X \to \mathcal{P}_{\mathrm{cl}}(X)$ be a multi-valued operator. Then T is called $\mathcal{D}$-*Lipschitz* if there exists a continuous and nondecreasing function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$d_H(Tx, Ty) \le \psi(\|x - y\|) \tag{2.2}$$

for all $x, y \in X$, where $\psi(0) = 0$. The function ψ is called a $\mathcal{D}$-function of T on X. If $\psi(r) = k\,r$ for some $k > 0$, then T is called a multi-valued Lipschitz operator on X with the Lipschitz constant k. Further if $k < 1$, then T is called a multi-valued contraction on X with the contraction constant k. Finally, if $\psi(r) < r$ for $r > 0$, then T is called a nonlinear $\mathcal{D}$-contraction on X.

Let X be a metric space. A multi-valued mapping $T\colon X \to \mathcal{P}_{\mathrm{cl}}(X)$ is called *lower semi-continuous* (resp. *upper semi-continuous*) if G is any open subset of X then $\{x \in X \mid Tx \cap G \neq \emptyset\}$ (resp. $\{x \in X \mid Tx \subset G\}$) is an open subset of X. The multi-valued mapping T is called *totally compact* if $\overline{T(S)}$ is a compact subset of X for any $S \subset X$. T is called *compact* if $\overline{T(S)}$ is a compact subset of X for all bounded subsets S of X. Again, T is called *totally bounded* if for any bounded subset S of X, $T(S)$ is a totally bounded subset of X. A multi-valued mapping $T\colon X \to \mathcal{P}_{\mathrm{cp}}(X)$ is called *completely continuous* if it is upper semi-continuous and compact on X. Every compact multi-valued mapping is totally bounded but the converse may not be true. However, these two notions are equivalent on bounded subsets of a complete metric space X.

Let X be an ordered metric space with an order relation $\leq$. Let $a, b \in X$ be such that $a \leq b$. Then an order interval $[a, b]$ is a set in X defined by

$$[a,b] = \{x \in X \mid a \leq x \leq b\}\,.$$

When X is an ordered Banach space, the order relation " $\leq$ " in X is defined by the cone K, which is a non-empty closed set in X satisfying (i) $K + K \subset K$, (ii) $\lambda K \subset K$ for all $\lambda \in \mathbb{R}^+$, and (iii) $\{-K\} \bigcap K = 0$, where 0 is the zero element of X. A cone K in a Banach space X is called normal, if the norm $\|\cdot\|$ is semi-monotone on K. It is known that if the cone K is normal, then every order-bounded set is bounded in norm. Similarly, the cone K in X is called *regular* if every monotone increasing (resp. decreasing) order bounded sequence in X converges in norm. The details of cones and their properties appear in Guo and Lakshmikantham [14] and Heikkilä and Lakshmikantham [16]. In the following, we define an order relation in $\mathcal{P}_p(X)$ which is useful in the sequel.

Let $A, B \in \mathcal{P}_p(X)$. Then we define

$$\begin{aligned} A \pm B &= \{a \pm b \mid a \in A \ \text{ and } \ b \in B\}\,,\\ \lambda A &= \{\lambda a \mid a \in A \text{ and } \lambda \in \mathbb{R}\}\,,\\ \|A\| &= \{\|a\| : a \in A\}\\ \text{and} \qquad \|A\|_{\mathcal{P}} &= \sup\{\|a\| : a \in A\}\,. \end{aligned}$$

Let the Banach space X be equipped with an order relation $\leq$. Then we define the different order relations in $\mathcal{P}_p(X)$ as follows. Let $A, B \in \mathcal{P}_p(X)$. Then by $A \overset{i}{\leq} B$ we mean "for every $a \in A$ there exists a $b \in B$ such that $a \leq b$." Again, $A \overset{d}{\leq} B$ means for each $b \in B$ there exists a $a \in A$ such that $a \leq b$. Furthermore, we have $A \overset{\mathrm{id}}{\leq} B \iff A \overset{\mathrm{i}}{\leq} B$ and $A \overset{\mathrm{d}}{\leq} B$. Finally, $A \leq B$ implies that $a \leq b$ for all $a \in A$ and $b \in B$. Note that if $A \leq A$, then it follows that A is a singleton set. The details of these order relations in $\mathcal{P}_p(X)$ are given in Dhage [8] and references therein.

Definition 2.2. An operator $Q\colon X \to \mathcal{P}_p(X)$ is called right monotone increasing (resp. left monotone increasing) if $Qx \overset{\mathrm{i}}{\leq} Qy$ (resp. $Qx \overset{\mathrm{d}}{\leq} Qy$) for all $x, y \in X$ for which $x \leq y$. Similarly, Q is called monotone increasing if it is left as well as right monotone increasing on X. Finally, Q is strict monotone increasing if $Qx \leq Qy$ for all $x, y \in X$ for which $x \leq y$, $x \neq y$.

Remark 2.1. Note that every strict monotone increasing multi-valued operator is left as well as right monotone increasing, but the converse may not be true.

The Kuratowskii measure α of noncompactness in a Banach space is a nonnegative real number $\alpha(S)$ defined by

$$\alpha(S) = \inf \Big\{ r > 0 \colon S \subset \bigcup_{i=1}^{n} S_i, \text{ and } \operatorname{diam}(S_i) \leq r, \forall\, i \Big\} \tag{2.3}$$

for all bounded subsets S of X.

The Hausdorff measure of noncompactness of a bounded subset S of X is a nonnegative real number $\beta(S)$ defined by

$$\beta(S) = \inf \Big\{ r > 0 \colon S \subset \bigcup_{i=1}^{n} \mathcal{B}_i(x_i, r), \text{ for some } x_i \in X \Big\}, \tag{2.4}$$

where $\mathcal{B}_i(x_i, r) = \{x \in X \mid d(x, x_i) < r\}$.

The details of the Hausdorff measure of noncompactness and its properties appear in Deimling [6], Zeidler [22] and the references therein. The following results appear in Akhmerov et. al. [2].

Lemma 2.1 ([2, page 7])**.** *If S is a bounded set in the Banach space X, then $\alpha(S) \leq 2\,\beta(S)$.*

Lemma 2.2. *If $A \colon X \to X$ is a single-valued $\mathcal{D}$-Lipschitz mapping with the $\mathcal{D}$-function ψ, that is, $\|Ax - Ay\| \leq \psi(\|x - y\|)$ for all $x, y \in X$, then we have $\alpha(A(S)) \leq \psi(\alpha(S))$ for any bounded subset S of X.*

Definition 2.3. A multi-valued operator $T \colon X \to \mathcal{P}_{\mathrm{cp}}(X)$ is called condensing (resp. countably condensing) if for any bounded (resp. bounded and countable) subset S of X, $T(S)$ is bounded and $\beta(T(S)) < \beta(S)$ for $\beta(S) > 0$.

Note that every condensing multi-valued operator is countably condensing, but the converse may not be true. It is known that multi-valued contraction and completely continuous multi-valued operators are condensing (see Dhage [9], Petruşel [21] and the references therein). A fixed point theorem for right monotone increasing multi-valued countably condensing operators is

Theorem 2.1. *Let $[a, b]$ be a norm-bounded order interval in the ordered normed linear space X and let $T \colon [a, b] \to \mathcal{P}_{\mathrm{cl}}([a, b])$ be a upper semi-continuous and countably condensing. Furthermore, if T is right monotone increasing, then T has a fixed point in $[a, b]$.*

Proof. The proof is obtained by using essentially the same arguments that given in Dhage [9] with appropriate modifications. We omit the details. □

An improvement upon the multi-valued analogue of Tarski's fixed point theorem proved by Agarwal *et al.* [1] is embodied in the following fixed point theorem for the right monotone increasing multi-valued mappings in ordered metric spaces.

Theorem 2.2 (Dhage [11])**.** *Let $[a,b]$ be an order interval in a subset Y of an ordered Banach space X and let $Q\colon [a,b] \to \mathcal{P}_{\mathrm{cp}}([a,b])$ be a right monotone increasing (resp. left monotone increasing) multi-valued operator. If every monotone increasing (resp. decreasing) sequence $\{y_n\} \subset \bigcup Q([a,b])$ defined by $y_n \in Qx_n$, $n \in \mathbb{N}$ converges in Y, whenever $\{x_n\}$ is a monotone increasing (resp. decreasing) sequence in $[a,b]$, then Q has a fixed point.*

In the following section, we combine Theorems 2.1, and 2.2 to obtain some general hybrid fixed point theorems for multi-valued mappings on ordered Banach spaces.

3. Hybrid fixed point theory

Our main multi-valued hybrid fixed point theorem of this paper is

Theorem 3.1. *Let $[a,b]$ be a norm-bounded order interval in a subset Y of an ordered Banach space X and let $T\colon [a,b] \times [a,b] \to \mathcal{P}_{\mathrm{cp}}([a,b])$ be a multi-valued mapping satisfying the following conditions.*

(a) *The multi-valued mapping $x \mapsto T(x,y)$ is upper semi-continuous uniformly for $y \in [a,b]$.*
(b) *The multi-valued mapping $x \mapsto T(x,y)$ is countably condensing and right monotone increasing for all $y \in X$.*
(c) *$y \mapsto T(x,y)$ is right monotone increasing for all $x \in [a,b]$, and*
(d) *every monotone increasing sequence $\{z_n\} \subset \bigcup T([a,b] \times [a,b])$ defined by $z_n \in T(x,y_n)$, $n \in \mathbb{N}$ converges for each $x \in [a,b]$, whenever $\{y_n\}$ is a monotone increasing sequence in $[a,b]$.*

Then the inclusion $x \in T(x,x)$ has a solution in $[a,b]$.

Proof. Define a multi-valued operator $Q\colon [a,b] \to \mathcal{P}_{\mathrm{cp}}([a,b])$ by

$$Qy = \big\{x \in [a,b] \mid x \in T(x,y)\big\}. \tag{3.1}$$

Let $y \in [a,b]$ be fixed and define the mapping $T_y(x)\colon [a,b] \to \mathcal{P}_{\mathrm{cp}}([a,b])$ by $T_y(x) = T(x,y)$. Then T_y is a condensing, upper semi-continuous and right monotone increasing multi-valued mapping which maps the order interval $[a,b]$ of the Banach space X into itself. Therefore, an application of Theorem 2.1 yields that T_y has a fixed point in $[a,b]$, and consequently the set Qy is non-empty for each $y \in [a,b]$. Moreover, Qy is compact for each $y \in [a,b]$.

Firstly, we show that Q is a right monotone increasing multi-valued operator on $[a,b]$. Let $y_1, y_2 \in [a,b]$ be such that $y_1 \le y_2$. Then have that

$$Qy_1 = \{x \in [a,b] \mid x \in T(x,y_1) = T_{y_1}(x)\}$$

and

$$Qy_2 = \{x \in [a,b] \mid x \in T(x,y_2) = T_{y_2}(x)\}\,.$$

Let $z \in Qy_1$ be arbitrary. Take $z_0 = z$. From the right monotonicity of $T(x, y)$ in y, it follows that

$$z \in T(z, y_1) = T_{y_1}(z) \overset{i}{\leq} T_{y_2}(z) \overset{i}{\leq} T(z, y_2)\,.$$

Therefore, there is an element $z_1 \in T_{y_2}(z_0)$ such that $z_0 \leq z_1$. Again, the right monotonicity of $T(x, y)$ in y implies that $T_{y_2}(z_0) \overset{i}{\leq} T_{y_2}(z_1)$. Therefore, there is an element $z_2 \in T_{y_2}(z_1)$ such that $z_0 \leq z_1 \leq z_2$. Proceeding in the is way, by induction, we obtain a monotone increasing sequence $\{z_n\}$ in $[a, b]$ such that $z_{n+1} \in T_{y_2}(z_n)$, $n = 0, 1, \ldots$. As $T_{y_2} \colon [a, b] \to \mathcal{P}_{\mathrm{cp}}([a, b])$ is upper semi-continuous and condensing, by Theorem 2.1, $\lim\limits_{n\to\infty} z_n = z^*$ exists and $z^* \in T_{y_2}(z^*) = Qy_2$. Thus for every $z \in Qy_1$ there is a $z^* \in Qy_2$ such that $z \leq z^*$. As a result, $Qy_1 \overset{i}{\leq} Qy_2$, i.e., Q is a right monotone increasing multi-valued operator on $[a, b]$. Thus, Q defines a right monotone increasing operator $Q \colon [a, b] \to \mathcal{P}_{\mathrm{cp}}([a, b])$(see also Dhage [7, 8] and the references therein).

Next, let $\{y_n\}$be a monotone increasing sequence in $[a, b]$. We will show that the sequence $\{z_n\} \subseteq \bigcup Q([a, b])$ defined by $z_n \in Qy_n$ for each $n \in \mathbb{N}$ converges. By virtue of Q, there is a monotone increasing sequence $\{z_n\}$ in $[a, b]$ such that $z_n \in T(z_n, y_n)$, $n \in \mathbb{N}$. Let $S = \{z_n\}$. Then S is a bounded and countable subset of $[a, b]$ such that $S \subseteq \bigcup_{n\in\mathbb{N}} T(S, y_n)$. Since the multi-valued $x \mapsto T(x, y)$ is condensing for each $y \in [a, b]$, one has

$$\beta(S) \leq \beta\big(\bigcup_{n\in\mathbb{N}} T(S, y_n)\big) = \sup\,\big\{\beta(T(S, y_n)) \colon n \in \mathbb{N}\big\} < \beta(S)$$

for each $n \in \mathbb{N}$. If $\beta(S) \neq 0$, then we get a contradiction. As a result, $\beta(S) = 0$ and that $\overline{S}$ is compact. Hence the sequence $\{z_n\}$ converges to a point, say z in $[a, b]$. By upper semi-continuity of $T(x, y)$ in x uniformly for y, there exists an $n_0 \in \mathbb{N}$ such that $z_n \in T(z, y_n)$ for all $n \geq n_0$. Now, by hypothesis (d), every sequence $\{z_n\}$ in $\{T(z, y_n)\}$ converges. As a result, the sequence $\{z_n\} \subseteq \bigcup Q([a, b])$ defined by $z_n \in Qy_n$ for each $n \in \mathbb{N}$ converges, whenever $\{y_n\}$ is a monotone increasing sequence in $[a, b]$.

Thus, the multi-valued operator Q satisfies all the conditions of Theorem 2.2 on $[a, b]$ and hence an application it yields that Q has a fixed point. This further implies that the operator inclusion $x \in T(x, x)$ has a solution in $[a, b]$. This completes the proof. □

As a consequence of Theorem 3.1 we obtain

Corollary 3.2. *Let $[a, b]$ be an order interval in a subset Y of the ordered Banach space X and let $T \colon [a, b] \times [a, b] \to \mathcal{P}_{\mathrm{cp}}([a, b])$ be a mapping satisfying*

(a) *$x \mapsto T(x, y)$ is an upper semi-continuous, condensing and right monotone increasing uniformly for $y \in [a, b]$, and*
(b) *$y \mapsto T(x, y)$ is right monotone increasing for each $x \in [a, b]$.*

Then the inclusion $x \in T(x, x)$ has a solution if any one of the following conditions is satisfied.

(i) *$[a, b]$ is norm-bounded and T is compact.*
(ii) *The cone K in X is normal and $y \mapsto T(x, y)$ is compact for each $x \in [a, b]$.*
(iii) *The cone K is regular.*

The study of multi-valued hybrid fixed point theorems involving the sum of two multi-valued operators in a Banach space may be found in the works of the Adrian Petruşel [21]. See also Dhage [9] and the references therein. In this case, one operator happens to be a multi-valued contraction and another one happens to be a completely continuous on the domains of their definitions. Since every contraction is Hausdorff continuous, both operators in such theorems are upper semi-continuous continuous on the domain of their definition. Below we prove a multi-valued hybrid fixed point theorem involving the sum of three multi-valued operators in Banach spaces and relax the continuity condition of one of the operators in such hybrid fixed point theorems, instead we assume the monotonicity to yield the desired results on ordered Banach spaces.

To prove the main results in this direction, we need the following lemma in the sequel.

Lemma 3.1. *Let $A, B\colon X \to \mathcal{P}_{\mathrm{cp}}(X)$ be two multi-valued operators satisfying*

(a) *A is a multi-valued $\mathcal{D}$-contraction, and*
(b) *B is completely continuous.*

Then the multi-valued operator $T\colon X \to \mathcal{P}_{\mathrm{cp}}(X)$ defined by $Tx = Ax + Bx$ is upper semi-continuous and β-condensing on X.

Proof. The proof appears in Dhage [9]. See also Petruşel [21] for the details. □

Theorem 3.3. *Let $[a, b]$ be an order interval in the ordered Banach space X and let $A, B, C\colon [a, b] \to \mathcal{P}_{\mathrm{cp}}(X)$ be three right monotone increasing multi-valued operators satisfying*

(a) *A is a multi-valued $\mathcal{D}$-contraction,*
(b) *B is completely continuous,*
(c) *every monotone increasing sequence $\{z_n\} \subset \bigcup C([a, b])$ defined by $z_n \in C(y_n)$, $n \in \mathbb{N}$ converges, whenever $\{y_n\}$ is a monotone increasing sequence in $[a, b]$, and*
(d) *the elements a and b satisfy $a \leq Aa + Ba + Ca$ and $Ab + Bb + Cb \leq b$.*

Furthermore, if the cone K in X is normal, then the operator inclusion $x \in Ax + Bx + Cx$ has a solution in $[a, b]$.

Proof. Define a mapping T on $[a, b] \times [a, b]$ by $T(x, y) = Ax + Bx + Cy$. From hypothesis (d), it follows that T defines a multi-valued mapping $T\colon [a, b] \times [a, b] \to \mathcal{P}_{\mathrm{cp}}([a, b])$. From Lemma 3.1, it follows that the multi-valued $x \mapsto T(x, y)$ is condensing, upper semi-continuous and right monotone increasing uniformly for $y \in [a, b]$. Now the desired conclusion follows by an application of Theorem 3.1. □

When A is a single-valued operator, Theorem 3.3 reduces to

Corollary 3.4. *Let $[a,b]$ be an order interval in the ordered Banach space X. Let $B, C\colon [a,b] \to \mathcal{P}_{\mathrm{cp}}(X)$ be two right monotone increasing and $A\colon [a,b] \to X$ be a nondecreasing operator satisfying*

(a) *A is a single-valued contraction,*
(b) *B is completely continuous,*
(c) *every sequence $\{z_n\} \subset \bigcup C([a,b])$ defined by $z_n \in C(y_n), n \in \mathbb{N}$ has a cluster point, whenever $\{y_n\}$ is a monotone increasing sequence in $[a,b]$, and*
(d) *the elements a and b satisfy $a \leq Aa + Ba + Ca$ and $Ab + Bb + Cb \leq b$.*

Furthermore, if the cone K in X is normal, then the operator inclusion $x \in Ax + Bx + Cx$ has a solution in $[a,b]$.

Proof. Define a mapping $T\colon [a,b] \times [a,b] \to \mathcal{P}_{\mathrm{cp}}([a,b])$ by

$$T(x,y) = Ax + Bx + Cy.$$

We shall show that the mapping $T_y(\cdot) = T(\cdot, y)$ is a α-condensing on $[a,b]$. Since the order cone K in X is normal, the order interval $[a,b]$ is a norm-bounded set in X. Now for any subset S in $[a,b]$ one has

$$T_y(S) \subset A(S) + B(S) + Cy.$$

Hence, by sublinearity of α, it follows that

$$\alpha(T_y(S)) \leq \alpha(A(S)) + \alpha(B(S)) + \alpha(Cy) \leq \alpha(A(S)) \leq \psi(\alpha(S)) < \alpha(S)$$

for all $S \subset [a,b]$ with $\alpha(S) > 0$. The rest of the proof is similar to Theorem 3.1. □

The hybrid fixed point theory involving the product of two multi-valued operators in a Banach algebra is initiated by the present author in [7] and developed further in the various directions in the due course of time. Some details are given in Dhage [10] and the references therein. The main feature of these fixed point theorems in the direction of Dhage [7] is again that the operators in question satisfy certain continuity condition on their domains of definition. Below we remove the continuity of one of the operators and prove a multi-valued hybrid fixed point theorem involving the product of two operators in a Banach algebra. We need the following preliminaries in the sequel.

A cone K in a Banach algebra X is called positive, if

(iv) $K \circ K \subseteq K$, where " $\circ$ " is a multiplicative composition in X.

Let X be an ordered Banach algebra. Then for any $A, B \in \mathcal{P}_p$, we denote

$$AB = \{ab \in X \mid a \in A \text{ and } b \in B\}.$$

We need the following results in the sequel.

Lemma 3.2 (Dhage [8])**.** *Let K be a positive cone in the Banach algebra X. If $u_1, u_2, v_1, v_2 \in K$ are such that $u_1 \leq v_1$ and $u_2 \leq v_2$, then $u_1 u_2 \leq v_1 v_2$.*

Lemma 3.3 (Dhage [9])**.** *For any $A, B, C \in \mathcal{P}_p(X)$,*

$$d_H(AC, BC) \leq d_H(C, 0)\, d_H(A, B) = \|C\|_{\mathcal{P}}\, d_H(A, B).$$

Lemma 3.4 (Banas and Lecko [4]). *If $A, B \in \mathcal{P}_{\mathrm{bd}}(X)$, then*

$$\beta(AB) \leq \|A\|_{\mathcal{P}}\, \beta(B) + \|B\|_{\mathcal{P}}\, \beta(A).$$

Lemma 3.5. *Let S be a closed convex and bounded subset of a Banach algebra X and let $A, B\colon S \to \mathcal{P}_{\mathrm{cp}}(X)$ be two multi-valued operators such that*

(a) *A is a $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-function ψ,*
(b) *B is completely continuous, and*
(c) *$M\psi(r) < r$ for $r > 0$, where $M = \|B(S)\|_{\mathcal{P}} = \sup\{\|Bx\| \mid x \in S\}$.*

Then the multi-valued operator $T\colon S \to \mathcal{P}_{\mathrm{cp}}(X)$ defined by $Tx = Ax\, Bx$ is upper semi-continuous and condensing on X.

Proof. The proof appears in Dhage [7, 9]. □

Theorem 3.5. *Let $[a, b]$ be an order interval in the ordered Banach algebra X and let $A, B\colon [a, b] \to \mathcal{P}_{\mathrm{cp}}(K)$ and $C\colon [a, b] \to \mathcal{P}_{\mathrm{cp}}(X)$ be three right monotone increasing multi-valued operators satisfying*

(a) *A is $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-function ψ,*
(b) *B is completely continuous,*
(c) *every monotone increasing sequence $\{z_n\} \subset \bigcup C([a, b])$ defined by $z_n \in C(y_n)$, $n \in \mathbb{N}$ converges, whenever $\{y_n\}$ is a monotone increasing sequence in $[a, b]$, and*
(d) *the elements a and b satisfy $a \leq Aa\, Ba + Ca$ and $Ab\, Bb + Cb \leq b$.*

Furthermore, if the cone K in X is positive and normal, then the operator inclusion $x \in Ax\, Bx + Cx$ has a solution in $[a, b]$ whenever $M\psi(r) < r$ for $r > 0$, where $M = \|B([a, b])\|_{\mathcal{P}} = \sup\{\|Bx\|_{\mathcal{P}}\colon x \in [a, b]\}$.

Proof. Define the mapping T on $[a, b] \times [a, b]$ by $T(x, y) = Ax\, Bx + Cy$. From hypothesis (d), it follows that T defines a multi-valued mapping $T\colon [a, b] \times [a, b] \to \mathcal{P}_{\mathrm{cp}}([a, b])$. We show that the multi-valued $x \mapsto T_y(x) = T(x, y)$ is upper semi-continuous, condensing and right monotone increasing uniformly for $y \in [a, b]$. First we show that it is condensing on $[a, b]$. Let S be a subset of $[\overline{x}, \overline{y}]$. Since the cone K in X is normal, the order interval $[a, b]$ and consequently the set S is norm-bounded in X. Then by sublinearity of *beta*,

$$\begin{aligned}
\beta(T_y(S)) &\leq \beta(A(S)B(S)) + \beta(C(y)) \\
&\leq \|B(S)\|_{\mathcal{P}}\, \beta(A(S)) + \|B(S)\|_{\mathcal{P}}\, \beta(B(S)) + \beta(C(y)) \\
&= \|B(S)\|_{\mathcal{P}}\, \beta(A(S)) + \beta(C(S)) \\
&\leq M\, \psi(\beta(S)) < \beta(S)
\end{aligned}$$

for all sets S in $[a, b]$ for which $\beta(S) > 0$. This shows that the mapping $x \mapsto T_y(x) = T(x, y)$ is condensing uniformly for $y \in [a, b]$.

To show the mapping $x \mapsto T(x, y)$ is an upper semi-continuous uniformly for y, let $\{x_n\}$ be a sequence in $[a, b]$ converging to a point x^*. Let $\{y_n\}$ be a sequence in $Ax_nBx_n + Cy$ such that $y_n \to y^*$. It suffices to show that $y^* \in Ax^*Bx^* + Cy$. Now,

$$D\,(y^*, Ax^*Bx^* + Cy) = \lim_{n\to\infty} D(y_n, Ax^*Bx^* + Cy)$$

$$
\begin{aligned}
&\le \limsup_{n\to\infty} d_H(Ax_nBx_n + Cy, Ax^*Bx^* + Cy)\\
&\le \limsup_{n\to\infty} d_H(Ax_nBx_n, Ax^*Bx_n) + \limsup_{n\to\infty} d_H(Ax^*Bx_n, Ax^*Bx^*)\\
&\le \limsup_{n\to\infty} \big[d_H(Ax_n, Ax^*)d_H(0, Bx_n)\big] + \limsup_{n\to\infty} \big[d_H(0, Ax^*)d_H(Bx_n, Bx^*)\big]\\
&\le M\psi\big(\limsup_{n\to\infty} \|x_n - x^*\|\big) + \|Ax^*\|_{\mathcal{P}} \limsup_{n\to\infty} d_H(Bx_n, Bx^*)\\
&\to 0 \quad \text{as} \quad n \to \infty
\end{aligned}
$$

for all $y \in [a, b]$. This shows that $y^* \in Ax^*Bx^* + Cy$, and therefore, the multi-valued mapping $x \mapsto AxBx + Cy$ is an upper semi-continuous uniformly for $y \in [a, b]$. Now the desired conclusion follows by an application of Theorem 3.1.

A $\mathcal{D}$-function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is called *sumultiplicative* if $\psi(\lambda r) \le \lambda\psi(r)$ for $\lambda \in \mathbb{R}^+$. There do exist the submultiplicative $\mathcal{D}$-functions on $\mathbb{R}^+$. Indeed, the function $\psi(\lambda r) = \lambda r$, $\lambda > 0$ is a submultiplicative $\mathcal{D}$-function on $\mathbb{R}^+$.

Theorem 3.6. *Let $[a, b]$ be an order interval in the ordered Banach algebra X. Let $A\colon [a, b] \to K$, $C\colon [a, b] \to X$ be two nondecreasing single-valued operators and $B\colon [a, b] \to \mathcal{P}_{\mathrm{cp}}(K)$ be a right increasing multi-valued operator satisfying*

(a) *A is a $\mathcal{D}$-Lipschitz with the submultiplicative $\mathcal{D}$-function ψ,*
(b) *B is completely continuous,*
(c) *C is compact, and*
(d) *the elements a and b satisfy $a \le Aa\,Ba + Ca$ and $Ab\,Bb + Cb \le b$.*

Furthermore, if the cone K in X is positive and normal, then the operator inclusion $x \in Ax\,Bx + Cx$ has a solution in $[a, b]$ whenever $2M\psi(r) < r$, where $M = \|B([a, b])\|_{\mathcal{P}} = \sup\{\|Bx\|_{\mathcal{P}}\colon x \in [a, b]\}$.

Proof. Define a mapping $T\colon [a, b] \times [a, b] \to \mathcal{P}_{\mathrm{cp}}([a, b])$ by

$$T(x, y) = Ax\,Bx + Cy\,.$$

We shall show that the mapping $T_y(\cdot) = T(\cdot, y)$ is a β-condensing on $[a, b]$. Since the order cone K in X is normal, the order interval $[a, b]$ is a norm-bounded set in X. Now for any subset S in $[a, b]$ one has

$$T_y(S) \subset A(S)\,B(S) + Cy\,.$$

Hence from Lemmas 3.1 and 3.2, it follows that

$$
\begin{aligned}
\beta(T_y(S)) &\le \|B(S)\|_{\mathcal{P}}\beta(A(S)) + \|A(S)\|_{\mathcal{P}}\beta(B(S)) + \beta(Cy)\\
&\le \|B(S)\|_{\mathcal{P}}\,\alpha(A(S))\\
&\le 2\,M\psi(\beta(S)) < \beta(S)
\end{aligned}
$$

for all $S \subset [a, b]$ with $\beta(S) > 0$. The rest of the proof is similar to Theorem 3.3. □

Theorem 3.7. *Let $[a, b]$ be an order interval in the ordered Banach algebra X. Let $A, B\colon [a, b] \to \mathcal{P}_{\mathrm{cp}}(K)$ and $C\colon [a, b] \to \mathcal{P}_{\mathrm{cp}}(X)$ be three right monotone increasing multi-valued operators satisfying*

(a) *A is $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-function ψ,*
(b) *B is bounded and every monotone increasing sequence $\{z_n\} \subset \bigcup B([a,b])$ defined by $z_n \in B(y_n)$, $n \in \mathbb{N}$ converges, whenever $\{y_n\}$ is a monotone increasing sequence in $[a,b]$,*
(c) *C is completely continuous, and*
(d) *the elements a and b satisfy $a \leq Aa\,Ba + Ca$ and $Ab\,Bb + Cb \leq b$.*

Furthermore, if the cone K in X is positive and normal, then the operator inclusion $x \in Ax\,Bx + Cx$ has a solution in $[a,b]$ whenever $M\psi(r) < r$, where $M = \|B([a,b])\|_{\mathcal{P}} = \sup\{\|Bx\|_{\mathcal{P}} : x \in [a,b]\}$.

Proof. Define a operator T on $[a,b] \times \mathcal{P}_{\mathrm{cp}}(X)$ by $T(x,y) = Ax\,By + Cx$. From hypothesis (d), it follows that T defines a multi-valued mapping $T\colon [a,b] \times [a,b] \to \mathcal{P}_{\mathrm{cp}}([a,b])$. It can be shown as in the proof of Theorem 2.3 with appropriate modifications that the multi-valued mapping $x \mapsto T(x,y)$ is condensing and upper semi-continuous uniformly for $y \in [a,b]$. Now the desired conclusion follows by an application of Theorem 3.1. □

Theorem 3.8. *Let $[a,b]$ be an order interval in the ordered Banach algebra X with a cone K. Let $A, B\colon [a,b] \to \mathcal{P}_{\mathrm{cp}}(K)$ and $C\colon [a,b] \to \mathcal{P}_{\mathrm{cp}}(X)$ be three right monotone increasing multi-valued operators satisfying*

(a) *every monotone increasing sequence $\{z_n\} \subset \bigcup A([a,b])$ defined by $z_n \in A(y_n)$, $n \in \mathbb{N}$ converges, whenever $\{y_n\}$ is a monotone increasing sequence in $[a,b]$,*
(b) *B is completely continuous,*
(c) *C is multi-valued contraction, and*
(d) *the elements a and b satisfy $a \leq Aa\,Ba + Ca$ and $Ab\,Bb + Cb \leq b$.*

Furthermore, if the cone K in X is positive and normal, then the operator inclusion $x \in Ax\,Bx + Cx$ has a solution in $[a,b]$.

Proof. Define a mapping T on $[a,b] \times [a,b]$ by $T(x,y) = Ay\,Bx + Cx$. From hypothesis (d), it follows that T defines a multi-valued mapping $T\colon [a,b] \times [a,b] \to \mathcal{P}_{\mathrm{cp}}([a,b])$. Now the desired conclusion follows by an application of Theorem 3.1. □

Note that Theorems 3.3, 3.5, 3.6, 3.7 and 3.8 include the multi-valued hybrid fixed point theorems proved in Dhage [7, 8] for a pair of multi-valued operators in ordered Banach spaces and algebras as special cases. In the following section we prove an existence theorem for the perturbed discontinuous neutral functional differential inclusions under some mixed Lipschitz, compactness and monotonic conditions.

4. Neutral discontinuous functional differential inclusions

The method of upper and lower solutions has been successfully applied to the problems of nonlinear differential equations and inclusions. For the first direction, we refer to Heikkilä and Lakshmikantham [16] and for the second direction we refer to Dhage [9, 10, 11]. In this section, we apply the results of the previous sections to the first order initial value problems of ordinary discontinuous differential inclusions for proving the existence of solutions between the given upper and lower solutions under certain monotonicity conditions.

4.1. **Neutral functional differential inclusions.** Let $\mathbb{R}$ denote the real line. Let $I_0 = [-\delta, 0]$, $\delta > 0$ and $I = [0, T]$ be two closed and bounded intervals in $\mathbb{R}$. Let $\mathcal{C} = C(I_0, \mathbb{R})$ denote the Banach space of all continuous $\mathbb{R}$-valued functions on I_0 with the usual supremum norm $\|\cdot\|_{\mathcal{C}}$ given by

$$\|\phi\|_{\mathcal{C}} = \sup\{|\phi(\theta)| \colon -\delta \leq \theta \leq 0\}\,.$$

For any continuous $\mathbb{R}$-valued function x defined on the interval J, where $J = [-\delta, T] = I_0 \bigcup I$, and for any $t \in I$, we denote by x_t the element of $\mathcal{C}$ defined by

$$x_t(\theta) = x(t+\theta), \quad -\delta \leq \theta \leq 0\,.$$

Given a function $\phi \in \mathcal{C}$, consider the perturbed neutral functional first order differential inclusion (in short NFDI)

$$\tag{4.1} \begin{cases} \dfrac{d}{dt}[x(t) - f(t, x_t)] \in G(t, x_t) + H(t, x_t) \;\text{ a.e. }\; t \in J\,, \\ \qquad\qquad x_0 = \phi\,, \end{cases}$$

where $f \colon I \times \mathcal{C} \to \mathbb{R}$, $G, H \colon I \times \mathcal{C} \to \mathcal{P}_p(\mathbb{R})$.

By a *solution* of the NFDI (4.1) we mean a function $x \in C(J, \mathbb{R}) \cap AC(I, \mathbb{R})$ such that

(i) the mapping $t \mapsto [x(t) - f(t, x_t)]$ is absolutely continuous on I, and
(ii) there exists a $v \in L^1(I, \mathbb{R})$ such that $v(t) \in G(t, x_t) + H(t, x_t)$ a.e. $t \in I$, satisfying $\dfrac{d}{dt}[x(t) - f(t, x_t)] = v(t)$, for all $t \in I$ and $x_0 = \phi \in \mathcal{C}$,

where $AC(I, \mathbb{R})$ is the space of all absolutely continuous real-valued functions on I.

The special cases of NFDI (4.1) have been discussed in the literature very extensively for different aspects of the solutions under different continuity conditions. See Dhage and Ntouyas [13], Deimling [6], Hale [15], Ntouyas [20] and the references therein. But the study of NFDI (4.1) or its special cases with discontinuous multi-valued mappings have not been made so far in the literature for the existence results. In this section, we will prove the existence theorems for NFDI (4.1) via functional theoretic approach embodied in Corollary 3.4 under the mixed Lipschitz, compactness and right monotonic conditions.

We shall seek the solution of NFDI (4.1) in the space $C(J, \mathbb{R})$ of continuous and real-valued functions on J. Define a norm $\|\cdot\|$ and an order relation " $\leq$ " in $C(J, \mathbb{R})$ by

$$\tag{4.2} \|x\| = \sup_{t \in J} |x(t)|$$

and

$$\tag{4.3} x \leq y \iff x(t) \leq y(t) \;\text{ for all }\; t \in J\,.$$

Here, the cone K in $C(J,\mathbb{R})$ is defined by

$$K = \{x \in C(J,\mathbb{R}) \mid x(t) \geq 0 \text{ for all } t \in J\},$$

which is obviously positive and normal. See Guo and Lakshmikantham [14] and Heikkilä and Lakshmikantham [16].

For any multi-valued mapping $\beta\colon I \times \mathcal{C} \to \mathcal{P}_{\mathrm{cp}}(\mathbb{R})$, we denote

$$S^1_\beta(x) = \{v \in L^1(I,\mathbb{R}) \mid v(t) \in F(t,x_t) \quad \text{a.e.} \quad t \in I\}$$

for some $x \in C(J,\mathbb{R})$. The integral of the multi-valued mapping β is defined as

$$\int_0^t \beta(s,x_s)\,ds = \Big\{ \int_0^t v(s)\,ds \colon v \in S^1_\beta(x) \Big\}.$$

Definition 4.1. A multi-valued function $\beta\colon I \to \mathcal{P}_{\mathrm{cp}}(\mathbb{R})$ is said to be measurable if for every $y \in X$, the function $t \to d(y,\beta(t)) = \inf\{|y-x| \colon x \in \beta(t)\}$ is measurable.

Definition 4.2. A measurable multi-valued function $\beta\colon I \to \mathcal{P}_{\mathrm{cp}}(\mathbb{R})$ is said to be integrably bounded if there exists a function $h \in L^1(I,\mathbb{R})$ such that $|v| \leq h(t)$ a.e. $t \in I$ for all $v \in \beta(t)$.

Remark 4.1. It is known that if $\beta\colon I \to \mathcal{P}_{\mathrm{cp}}(\mathbb{R})$ is an integrably bounded multi-valued function , then the set S^1_β of all Lebesgue integrable selections of β is closed and non-empty. See Hu and Papageorgiou [18].

Definition 4.3. A multi-valued mapping $\beta\colon I \times \mathcal{C} \to \mathcal{P}_{\mathrm{cp}}(\mathbb{R})$ is said to be L^1-Carathéodory if

(i) $t \mapsto \beta(t,x)$ is measurable for each $x \in \mathbb{C}$,
(ii) $x \mapsto \beta(t,x)$ is upper semi-continuous almost everywhere for $t \in I$, and
(iii) for each real number $k > 0$, there exists a function $h_k \in L^1(I,\mathbb{R})$ such that

$$\|\beta(t,x)\|_{\mathcal{P}} = \sup\{|u| \colon u \in \beta(t,x)\} \leq h_k(t), \quad a.e. \quad t \in I$$

for all $x \in \mathcal{C}$ with $\|x\|_{\mathcal{C}} \leq k$.

Then, we have the following lemmas due to Lasota and Opial [19].

Lemma 4.1. *Let E be a Banach space. If $\dim(E) < \infty$ and $\beta\colon J \times E \to \mathcal{P}_{\mathrm{cp}}(E)$ is L^1-Carathéodory, then $S^1_\beta(x) \neq \emptyset$ for each $x \in E$.*

Lemma 4.2. *Let E be a Banach space, $\beta\colon J \times E \to \mathcal{P}_{\mathrm{cp}}(E)$ an L^1-Carathéodory multi-valued mapping with $S^1_\beta \neq \emptyset$ and let $\mathcal{K}\colon L^1(I,\mathbb{R}) \to C(I,E)$ be a linear continuous mapping. Then the composition operator $\mathcal{K} \circ S^1_\beta\colon C(I,E) \longrightarrow \mathcal{P}_{\mathrm{cp}}(C(I,E))$ is a closed graph operator in $C(I,E) \times C(I,E)$.*

Remark 4.2. It is known that a *compact* multi-valued mapping $T\colon E \to \mathcal{P}_{\mathrm{cp}}(E)$ is upper semi-continuous if and only if it has a *closed graph* in E, that is, if $\{x_n\}$ and $\{y_n\}$ are sequences in E such that $y_n \in Tx_n$ for $n = 0,1,\ldots$; and $x_n \to x^*$, $y_n \to y^*$, then $y^* \in Tx^*$.

We need the following definitions in the sequel.

Definition 4.4. A multi-valued mapping $\beta(t,x)$ is called right monotone increasing in x almost everywhere for $t \in I$ if $\beta(t,x) \overset{i}{\leq} \beta(t,y)$ a.e. $t \in I$, for all $x, y \in \mathcal{C}$, for which $x \leq y$.

Definition 4.5. A multi-valued mapping $\beta\colon I \times \mathcal{C} \to \mathcal{P}_{\mathrm{cp}}(\mathbb{R})$ is called L^1-Chandrabhan if

(i) $t \mapsto \beta(t, x_t)$ is Lebesgue measurable for each $x \in C(J,\mathbb{R})$,
(ii) $x \mapsto \beta(t,x)$ is right monotone increasing almost everywhere for $t \in I$, and
(iii) for each real number $r > 0$ there exists a function $h_r \in L^1(I,\mathbb{R})$ such that

$$\|\beta(t,x)\|_{\mathcal{P}} = \sup\{|u| \colon u \in \beta(t,x)\} \leq h_r(t) \text{ a.e. } t \in I$$

for all $x \in \mathcal{C}$ with $\|x\|_{\mathcal{C}} \leq r$.

Definition 4.6. A function $a \in C(J,\mathbb{R}) \cap AC(I,\mathbb{R})$ is called a strict lower solution of NFDI (4.1) if $t \mapsto [a(t) - f(t,a_t)]$ is absolutely continuous on I and for all $v_1 \in S^1_G(a)$ and $v_2 \in S^1_H(a)$ we have that $\frac{d}{dt}[a(t) - f(t,a_t)] \leq v_1(t) + v_2(t)$ for all $t \in I$ and $a_0 \leq \phi$. Similarly, a function $b \in C(J,\mathbb{R}) \cap AC(I,\mathbb{R})$ is called a strict upper solution of NFDI (4.1) if $t \mapsto [b(t) - f(t,b_t)]$ is absolutely continuous on I and for all $v_1 \in S^1_G(b)$ and $v_2 \in S^1_H(b)$ we have that $\frac{d}{dt}[b(t) - f(t,b_t)] \geq v_1(t) + v_2(t)$ for all $t \in I$ and $b_0 \geq \phi$.

We now introduce the following hypotheses in the sequel.

(f_0) $f(0,x) = 0$ for each $x \in \mathcal{C}$.
(f_1) The mapping f is continuous on $I \times \mathcal{C}$ and there exists a real-valued bounded function ℓ on I such that

$$|f(t,x) - f(t,y)| \leq \ell(t)\|x - y\|_{\mathcal{C}}\,,$$

for all $(t,x), (t,y) \in I \times \mathcal{C}$.
(f_2) The mapping $f(t,x)$ is nondecreasing in x for almost everywhere $t \in I$.
(G_1) $G(t,x)$ is compact subset of $\mathbb{R}$ for each $t \in I$ and $x \in \mathcal{C}$.
(G_2) G is L^1-Carathéodory.
(G_3) The multi-valued mapping $G(t,x)$ is right monotone increasing in x for almost everywhere $t \in I$.
(G_4) The multi-valued $x \mapsto S^1_G(x)$ is right monotone increasing in $C(J,\mathbb{R})$.
(H_1) $H(t,x)$ is compact subset of $\mathbb{R}$ for each $t \in I$ and $x \in \mathcal{C}$.
(H_2) H is L^1-Chandrabhan.
(H_3) The multi-valued $x \mapsto S^1_H(x)$ is right monotone increasing in $C(J,\mathbb{R})$.
(H_4) NFDI (4.1) has a strict lower solution a and a strict upper solution b with $a \leq b$.

Remark 4.3. Note that if the multi-function $H(t,x)$ is L^1-Chandrabhan and (H_4) holds, then it is measurable in t and integrably bounded on $I \times [a,b]$. It follows from a selection theorem (see Deimling [6]) that S^1_H is non-empty and has closed values on $[a,b]$, i.e.,

$$S^1_H(x) = \big\{v \in L^1(I,\mathbb{R}) \mid v(t) \in H(t,x_t) \text{ a.e. } \ t \in I\big\} \neq \emptyset$$

for all $x \in [a,b] \subset C(J,\mathbb{R})$.

Theorem 4.1. *Assume that the hypotheses* $(f_0)-(f_2)$, $(G_1)-(G_4)$ *and* $(H_1)-(H_4)$ *hold. Furthermore, if* $\|\ell\| < 1$, *then the NFDI (4.1) has a solution in* $[a,b]$ *defined on* J.

Proof. Let $X = C(J,\mathbb{R})$ and define an order interval $[a,b]$ in $C(J,\mathbb{R})$ which does exist in view of hypothesis (H_4). Note that the cone K is normal in X, and therefore, the order interval $[a,b]$ is norm bounded in X. As a result, there is a constant $r > 0$ such that $\|x\| \le r$ for all $x \in [a,b]$.

Now NFDI (4.1) is equivalent to the integral inclusion

$$x(t) \in \phi(0) - f(0,\phi) + f(t,x_t) + \int_0^t G(t,x_s)\,ds + \int_0^t H(t,x_s)\,ds\,, \quad \text{if} \quad t \in I\,, \tag{4.4}$$

satisfying

$$x(t) = \phi(t), \quad ,\text{if} \quad t \in I_0\,. \tag{4.5}$$

Define three multi-valued operators $A,B,C\colon [a,b] \to \mathcal{P}_p(X)$ by

$$Ax(t) = \begin{cases} -f(0,\phi) + f(t,x_t)\,, & \text{if } t \in I, \\ 0\,, & \text{if } t \in I_0\,, \end{cases} \tag{4.6}$$

$$Bx(t) = \begin{cases} \phi(0) + \displaystyle\int_0^t G(s,x_s)\,ds\,, & \text{if } t \in I\,, \\ \phi(t)\,, & \text{if } t \in I_0\,, \end{cases} \tag{4.7}$$

and

$$Cx(t) = \begin{cases} \displaystyle\int_0^t H(s,x_s)\,ds\,, & \text{if } t \in I\,, \\ 0\,, & \text{if } t \in I_0\,. \end{cases} \tag{4.8}$$

Clearly, the multi-valued operators A, B and C are well defined in view of hypotheses (G_2) and (H_2) and map $[a,b]$ into X. Now the NFDI (4.1) is transformed into an operator inclusion as

$$x(t) \in Ax(t) + Bx(t) + Cx(t), \quad t \in J.$$

We shall show that A, B and C satisfy all the conditions of Corollary 3.4 on $[a,b]$.

Step I. Firstly, we show that A is monotone increasing and B and C are right monotone increasing on $[a,b]$. Let $x,y \in [a,b]$ be such that $x \le y$. Then, by (f_2),

$$\begin{aligned} Ax(t) &= \begin{cases} -f(0,\phi) + f(t,x_t)\,, & \text{if } t \in I\,, \\ 0, & \text{if } t \in I_0, \end{cases} \\ &\le \begin{cases} -f(0,\phi) + f(t,y_t)\,, & \text{if } t \in I\,, \\ 0\,, & \text{if } t \in I_0\,, \end{cases} \\ &= Ay(t) \end{aligned}$$

for all $t \in J$. Hence, $Ax \leq Ay$, and so, the operator A is monotone increasing on $[a,b]$. Since (G_4) and (H_3), we have that $S^1_G(x) \stackrel{i}{\leq} S^1_G(y)$ and $S^1_H(x) \stackrel{i}{\leq} S^1_H(y)$. As a result, we obtain $Bx \stackrel{i}{\leq} By$ and $Cx \stackrel{i}{\leq} Cy$. Thus B and C are right monotone increasing on $[a,b]$. By (H_4), $a \leq Aa + Ba + Ca$ and $Ab + Bb + Cb \leq b$

Step II. Next, we show that A is a contraction operator on $[a,b]$. Let $x, y \in [a,b]$ be arbitrary. Then by hypothesis (f_1),

$$\|Ax - Ay\| \leq \sup_{t\in J} |f(t,x_t) - f(t,y_t)| \leq \sup_{t\in J} \ell(t)\|x_t - y_t\|_{\mathcal{C}} \leq \|\ell\| \, \|x - y\| \,.$$

This shows that A is contraction on $[a,b]$ with the contraction constant $\|\ell\| < 1$.

Step III. Secondly, we show that the multi-valued operator B satisfies all the conditions of Theorem 2.2. It can be proved as in the Step I that B is a right monotone increasing mapping on $[a,b]$. We only prove that it is completely continuous on $[a,b]$. First we show B maps bounded sets into bounded sets in X. If S is a bounded set in X, then there exists $r > 0$ such that $\|x\| \leq r$ for all $x \in S$. Now for each $u \in Bx$, there exists a $v \in S^1_G(x)$ such that

$$u(t) = \begin{cases} \phi(0) + \displaystyle\int_0^t v(s)\,ds\,, & \text{if } \ t \in I\,, \\ \phi(t)\,, & \text{if } \ t \in I_0\,. \end{cases}$$

Then, for each $t \in J$,

$$|u(t)| \leq \|\phi\|_{\mathcal{C}} + \int_0^t |v(s)|\,ds \leq \|\phi\|_{\mathcal{C}} + \int_0^t h_r(s)\,ds \leq \|\phi\|_{\mathcal{C}} + \|h_r\|_{L^1}\,.$$

This further implies that $\|u\| \leq \|\phi\|_{\mathcal{C}} + \|h_r\|_{L^1}$ for all $u \in Bx \subset \bigcup B(S)$. Hence, $\bigcup B(S)$ is bounded.

Next we show that B maps bounded sets into equicontinuous sets. Let S be, as above, a bounded set and $u \in Bx$ for some $x \in S$. Then there exists $v \in S^1_G(x)$ such that

$$u(t) = \begin{cases} \phi(0) + \displaystyle\int_0^t v(s)\,ds\,, & \text{if } \ t \in I\,, \\ \phi(t)\,, & \text{if } \ t \in I_0\,. \end{cases}$$

Then for any $t_1, t_2 \in I$ with $t_1 \leq t_2$, we have

$$|u(t_1) - u(t_2)| \leq \left| \int_0^{t_1} v(s)\,ds - \int_0^{t_2} v(s)\,ds \right| = \int_{t_1}^{t_2} |v(s)|\,ds \leq \int_{t_1}^{t_2} h_r(s)\,ds\,.$$

If $t_1, t_2 \in I_0$, then $|u(t_1) - u(t_2)| = |\phi(t_1) - \phi(t_2)|$. For the case when $t_1 \leq 0 \leq t_2$, we have that

$$|u(t_1) - u(t_2)| \leq |\phi(t_1) - \phi(0)| + \int_0^{t_2} |v(s)|\,ds \leq |\phi(t_1) - \phi(0)| + \int_0^{t_2} h_r(s)\,ds\,.$$

Hence, in all three cases, we have

$$|u(t_1) - u(t_2)| \to 0 \ \text{ as } \ t_1 \to t_2\,.$$

As a result, $\bigcup B(Q)$ is an equicontinuous set in X. Now an application of Arzelá-Ascoli theorem yields that the multi B is totally bounded on X. Consequently, $B\colon [a,b] \to \mathcal{P}_{\mathrm{cp}}(X)$ is a compact multi-valued operator.

Step IV. Next, we prove that B has a *closed graph* in X. Let $\{x_n\} \subset X$ be a sequence such that $x_n \to x_*$ and let $\{y_n\}$ be a sequence defined by $y_n \in Bx_n$ for each $n \in \mathbb{N}$ such that $y_n \to y_*$. We will show that $y_* \in Bx_*$. Since $y_n \in Bx_n$, there exists a $v_n \in S^1_G(x_n)$ such that

$$y_n(t) = \begin{cases} \phi(0) + \displaystyle\int_0^t v_n(s)\,ds\,, & \text{if } \ t \in I\,, \\ \phi(t)\,, & \text{if } \ t \in I\,. \end{cases}$$

Consider the linear and continuous operator $\mathcal{K}\colon L^1(I,\mathbb{R}) \to C(I,\mathbb{R})$ defined by

$$\mathcal{K}v(t) = \int_0^t v(s)\,ds\,.$$

Now, when $n \to \infty$,we obtain

$$|y_n(t) - \phi(0) - (y_*(t) - \phi(0))| \leq |y_n(t) - y_*(t)| \leq \|y_n - y_*\| \to 0\,.$$

Therefore, from Lemma 4.2 it follows that $(\mathcal{K} \circ S^1_G)$ is a closed graph operator and from the definition of $\mathcal{K}$ one has

$$y_n(t) - \phi(0) \in (\mathcal{K} \circ S^1_F(x_n)).$$

As $x_n \to x_*$ and $y_n \to y_*$, there is a $v \in S^1_G(x_*)$ such that

$$y_*(t) = \begin{cases} \phi(0) + \displaystyle\int_0^t v_*(s)\,ds\,, & \text{if } \ t \in I\,, \\ \phi(t)\,, & \text{if } \ t \in I_0\,. \end{cases}$$

Hence, B is an upper semi-continuous multi-valued operator on $[a,b]$.

Step V. Finally, we show that the multi-valued operator C satisfies all the conditions of Theorem 2.2. First, we show that C has compact values on $[a,b]$. Observe first that the operator C is equivalent to

$$Cx(t) = \begin{cases} (\mathcal{L} \circ S^1_H)(x)(t)\,, & \text{if } \ t \in I\,, \\ 0\,, & \text{if } \ t \in I_0\,, \end{cases} \tag{4.9}$$

where $\mathcal{L}\colon L^1(I,\mathbb{R}) \to X$ is the continuous operator defined by

$$\mathcal{L}v(t) = \int_0^t v(s)\,ds\,, \ \text{ if } \ t \in I\,.$$

To show C has compact values, it then suffices to prove that the composition operator $\mathcal{L} \circ S^1_H$ has compact values on $[a,b]$. Let $x \in [a,b]$ be arbitrary and let $\{v_n\}$ be a sequence in $S^1_H(x)$. Then, by the definition of S^1_H, $v_n(t) \in H(t,x_t)$ a.e. for $t \in I$. Since $H(t,x_t)$ is compact, there is a convergent subsequence of $v_n(t)$ (for simplicity call it $v_n(t)$ itself) that converges in measure to some $v(t)$, where $v(t) \in H(t,x_t)$ a.e. for $t \in I$. From the continuity of $\mathcal{L}$, it follows that

$\mathcal{L}v_n(t) \to \mathcal{L}v(t)$ pointwise on I as $n \to \infty$. In order to show that the convergence is uniform, we first show that $\{\mathcal{L}v_n\}$ is an equi-continuous sequence. Let t, $\tau \in I$; then

$$|\mathcal{L}v_n(t) - \mathcal{L}v_n(\tau)| \leq \Big| \int_0^t v_n(s)\, ds - \int_0^\tau v_n(s)\, ds \Big| \leq \Big| \int_\tau^t |v_n(s)|\, ds \Big| . \tag{4.10}$$

Now, $v_n \in L^1(I, \mathbb{R})$, so the right hand side of (4.10) tends to 0 as $t \to \tau$. Hence, $\{\mathcal{L}v_n\}$ is equi-continuous, and an application of the Ascoli theorem implies that it has a uniformly convergent subsequence. We then have $\mathcal{L}v_{n_j} \to \mathcal{L}v \in (\mathcal{L} \circ S^1_H)(x)$ as $j \to \infty$, and so $(\mathcal{L} \circ S^1_H)(x)$ is compact. Therefore, C is a compact-valued multi-valued operator on $[a, b]$.

Let $\{y_n\}$ be a sequence in $\bigcup C([a, b])$ defined by $y_n \in Cx_n$, $n \in \mathbb{N}$, where $\{x_n\}$ is a monotone increasing sequence in $[a, b]$. Then there is a sequence $v_n \in S^1_H(x_n)$ such that

$$y_n(t) = \begin{cases} \displaystyle\int_0^t v_n(s)\, ds\,, & \text{if } \ t \in I\,, \\ 0\,, & \text{if } \ t \in I_0\,. \end{cases}$$

We show that $\{y_n\}$ has a cluster point. Since (H_3) holds, we have

$$|y_n(t)| \leq \int_0^t |v(s)|\, ds \leq \int_0^t h_r(s)\, ds \leq \|h_r\|_{L^1}$$

for all $t \in J$. This implies that $\|y_n\| \leq \|h_r\|_{L^1}$ and so, $\{y_n\}$ is uniformly bounded.

Next we show that $\{y_n\}$ equicontinuous. Now for any $t_1, t_2 \in I$ with $t_1 \leq t_2$ we have

$$|y_n(t_1) - y_n(t_2)| \leq \Big| \int_0^{t_1} v_n(s)\, ds - \int_0^{t_2} v_n(s)\, ds \Big| \leq \int_{t_1}^{t_2} h_r(s)\, ds\,.$$

If $t_1, t_2 \in I_0$ then $|y_n(t_1) - y_n(t_2)| = 0$. For the case, where $t_1 \leq 0 \leq t_2$ we have that

$$|y_n(t_1) - y_n(t_2)| \leq \Big| \int_0^{t_2} v_n(s)\, ds \Big| \leq |p(t_2) - p(0)|\,,$$

where $p(t) = \displaystyle\int_0^t h_r(s) ds$. Hence, in all three cases, we have

$$|u(t_1) - u(t_2)| \to 0 \ \text{ as } \ t_1 \to t_2\,.$$

As a result $\{y_n\}$ is an equicontinuous set in X. Now an application of Arzelá-Ascoli theorem yields that the sequence $\{y_n\}$ has a cluster point. Thus all the conditions of Corollary 3.4 are satisfied and hence the operator inclusion $x \in Ax + Bx + Cx$ has a solution in $[a, b]$. This further implies that the NFDI (4.1) has a solution in $[a, b]$ defined on J. $\square$

5. Remarks and conclusion

In this paper, we have established the multi-valued hybrid fixed point theorems only for right monotone increasing operators, however, similar results can also be obtained for left monotone increasing multi-valued operators with appropriate modifications. As mentioned earlier, we do not need the multi-valued operators to be continuous and to have convex values in any of the hybrid fixed point theorems of section 3. Therefore, the results of this paper are the improvement upon the hybrid fixed point theorems for multi-valued operators obtained in Dhage [10] under weaker conditions. Thus, our hybrid fixed point theorems of this paper are useful in the study of nonconvex differential inclusions involving the discontinuous multi-valued functions for existence of the solutions. In this paper, we have dealt with some quite general forms of the neutral functional differential inclusions and so the results of section 4 include some known results in the literature as special cases under weaker continuity and convexness conditions. Again, we remark that the hypotheses (G_4) and (H_3) are somewhat new to the literature in the existence theory for differential inclusions and the sufficient condition guaranteeing these conditions, so far we know, are that the multi-valued functions G and H should be strictly monotone increasing in the state variable (see Agarwal et al. [1]). Finally, we mention that the study of such sufficient conditions for the validity of the assumptions (G_4) and (H_3) is again a problem and the further study in this direction forms a scope for the future research work and while concluding, we conjecture that the possible answers to this question are the hypotheses (G_3) and (H_2) under suitable conditions.

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