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ON THE FINITE DIMENSION OF ATTRACTORS OF DOUBLY NONLINEAR PARABOLIC SYSTEMS WITH L-TRAJECTORIES

Hamid El Ouardi

Abstract. This paper is concerned with the asymptotic behaviour of a class of doubly nonlinear parabolic systems. In particular, we prove the existence of the global attractor which has, in one and two space dimensions, finite fractal dimension.

Introduction

We consider the following doubly nonlinear system $(\mathcal{P})$ of the form

$$\begin{aligned}
&\frac{\partial b_1(u_1)}{\partial t} - \operatorname{div}\big(a_1(\nabla u_1)\big) + f_1(u_1) = \delta_1 \frac{\partial H}{\partial u_1}(u_1,u_2) && \text{in } \ \Omega\times\mathbb{R}^+\,,\\
&\frac{\partial b_2(u_2)}{\partial t} - \operatorname{div}\big(a_2(\nabla u_2)\big) + f_2(u_2) = \delta_2 \frac{\partial H}{\partial u_2}(u_1,u_2) && \text{in } \ \Omega\times\mathbb{R}^+\,,\\
&\qquad u_1 = u_2 = 0 && \text{on } \ \partial\Omega\times\mathbb{R}^+\,,\\
&\big(b_1(u_1(x,0)),b_2(u_2(x,0))\big) = \big(b_1(\varphi_0(x)),b_2(\psi_0(x))\big) && \text{in } \ \Omega\,,
\end{aligned}$$

where Ω is a bounded and open subset in $\mathbb{R}^d$, with a smooth boundary $\partial\Omega$.

Problems of form $(\mathcal{P})$ arise in flow in porous media, nonlinear heat equation with absorption, chemical isothermic reactions and unidirectional Non-Newtonian fluids, see [4] . Therefore, it is important to obtain information about the existence of the global attractor and its regularity.

In recent years, numbers of works have contributed to the study of the single equation of the type $(\mathcal{P})$. Where $a_i = Id$, the elegant work [6, 7] plays a critical role in this area. The method is extensively used in many other works; we refer the readers to [1, 2, 3, 4, 5, 8, 9, 19] and the references therein. The basic tools that have been used are a priori estimates, degree theory and super-subsolution method.

In [21], Miranville has considered the following single equation

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$$\frac{\partial \alpha(u)}{\partial t} - \operatorname{div}\big(\beta(u)\big) + f(u) = g\,,$$

with Dirichlet boundary condition and has obtained the existence of the global attractor which has, in one and two space dimensions, finite fractal dimension.

Finally, we remind here that doubly nonlinear parabolic systems of the type

$$\frac{\partial \beta_i(u_i)}{\partial t} - \Delta u_i = f_i(,x,t,u_1,u_2)\,, \quad (i=1,2)$$

have been studied by El Ouardi and El Hachimi [12] who obtained the global attractor, the regularity and the estimates of Hausdorf and fractal dimensions. Our aim in this paper is to extend the results of [12] and [21] to the more general systems $(\mathcal{P})$.

The outline of the paper is as follows: in section 1 we make some assumptions and we prove the existence of the global attractor. In section 2 we show the results on the regularity of the global attractor. Section 3 is devoted to the fractal dimension using the method of l-trajectories.

1. Existence and uniqueness

1.1. **Notations and assumptions.**

Let Ω be a smooth and bounded domain in $\mathbb{R}^d, d \in \{1,2,3\}$. Set for $t>0, Q_t := \Omega \times (0,t), S_t := \partial\Omega \times (0,t)$ and also $((.,.))\,, \|.\|,\ (.,.)\,, |.|$ be respectively the scalar product and the norms in $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$.

Let b_i, $(i=1,2)$ be continuous functions with $b_i(0)=0$.

We define for $s \in \mathbb{R}$

$$\Psi_i(s) = \int_0^s \tau b_i'(\tau)d\tau\,,$$

$$d_s A_i(s) \cdot w = a_i(s) \cdot w\,, \quad \text{for all} \quad s, w \in \mathbb{R}^d$$

(d_s denoting the differential).

In the sequel, the same symbol c will be used to indicate some positive constants, possibly different from each other, appearing in the various hypotheses and computations and depending only on data. When we need to fix the precise value of one constant, we shall use a notation like $M_i, i=1,2,\dots$, instead.

In the sequel, we shall present the following assumptions:

$$(A1) \qquad \begin{cases} b_i \in C^3(\mathbb{R}), b_i(0)=0, & (i=1,2)\,, \\ \gamma_i \le b_i'(s) \le \alpha_i, & \text{for all} \ \ s \in \mathbb{R}, (i=1,2). \end{cases}$$

$$(A2) \qquad \begin{cases} a_i \in C^1(\mathbb{R}^d)^d, a_i(0)=0, d_s a_i\,, & \text{is bounded} \ \ (i=1,2), \\ c_{i1}\,|s|^2 - \beta_i \le A_i(s) \le d_i\,|s|^2 + e_i, & \text{for all} \ \ s \in \mathbb{R}^d (i=1,2), \\ a_i(s)\cdot s \ge c_{i2}\,|s|^2 & (i=1,2), \\ d_s a_i(s)\cdot w \cdot w \ge c_{i3}\,|w|^2\,, & \text{for all} \ \ s,w \in \mathbb{R}^d \ (i=1,2)\,. \end{cases}$$

$$(A3)\qquad \begin{cases} f_i \in C^2(\mathbb{R})\,, & \\ \operatorname{sign}(s) f_i(s) \geq c_{i4}\,|s|^{p_i+1}\,, & \text{for all}\;\; s\in\mathbb{R},\ p_i>0,\ (i=1,2)\,,\\ |f_i(s)| \leq c_{i5}\,|s|^{p_i+1}\,, & \text{for all}\;\; s\in\mathbb{R},\ (i=1,2)\,,\\ f_i'(s) \geq -K_i\,, & \text{for all}\;\; s\in\mathbb{R}\,,\ (i=1,2)\,. \end{cases}$$

$$(A4)\qquad \begin{cases} H \in \mathcal{C}^2(\mathbb{R}\times\mathbb{R}\,, & \\ \frac{\partial H}{\partial u_i}(0,0)=0\,, & (i=1,2)\,,\\ \text{there is positive constant}\;\; M_i>0 & \text{such that:}\\ \max\limits_{s\in\mathbb{R}^2}\left|\frac{\partial H}{\partial u_i}(s)\right| \leq M_i\,, & (i=1,2)\,. \end{cases}$$

(A5) $d = 1, 2$ or 3 when $b_i = c_i Id, c_i > 0$, and that $d = 1$, or 2 otherwise.

(A6) $(\varphi_0, \psi_0) \in \left(L^{+\infty}(\Omega)\right)^2$.

The following lemma are useful and used.

Lemma 1.1 (Ghidaghia lemma, cf.[19]). *Let y be a positive absolutely continuous function on $(0,\infty)$ which satisfies*

$$y' + \mu y^{q+1} \leq \lambda\,,$$

with $q > 0$, $\mu > 0$, $\lambda \geq 0$. Then for $t > 0$

$$y(t) \leq \left(\frac{\lambda}{\mu}\right)^{\frac{1}{q+1}} + [\mu(q)t]^{\frac{-1}{q}}\,.$$

Lemma 1.2 (Uniform Gronwall's lemma, cf. [27]). *Let y and h be locally integrable functions such that: $\exists r > 0$, $a_1 > 0$, $a_2 > 0$, $\tau > 0$, $\forall t \geq \tau$*

$$\int_t^{t+r} y(s)\,ds \leq a_1\,, \quad \int_t^{t+r} |h(s)|\,ds \leq a_2\,, \quad y' \leq h\,.$$

Then

$$y(t+r) \leq \frac{a_1}{r} + a_2\,, \quad \forall t \geq \tau\,.$$

Lemma 1.3 (Theorem 2.4., cf. [21]). *For every $\varphi \in L^2(\Omega)$, the problem*

$$-\operatorname{div}(a(\nabla u)) = \varphi\,, \quad u = 0 \quad on \quad \partial\Omega\,,$$

possesses a unique solution u such that $u \in H_0^1(\Omega) \cap H^2(\Omega)$.

1.2. **Existence theorem.**

Theorem 1. *Let (A1) to (A6) be satisfied. Then there exists a solution (u_1, u_2) of problem $(\mathcal{P})$ such that for $i = (1,2)$, we have*

$$u_i \in L^{p_i+1}\big(0,T;L^{p_i+1}(\Omega)\big) \cap L^2\big(0,T;H_0^1(\Omega)\big) \cap L^\infty\big(t_0,T;L^\infty(\Omega)\big), \quad \forall t_0 > 0\,.$$

Proof. By Theorem 2 in [6] , we can choose $u_i^0 \in L^{p_i+1}(Q_T) \cap L^2\big(0,T;H_0^1(\Omega)\big) \cap L^\infty\big(\tau,T;L^\infty(\Omega)\big)$ for any $\tau > 0$ such that:

$$
\begin{aligned}
&\frac{\partial b_1(u_1^0)}{\partial t} - \operatorname{div}\big(a_1(\nabla u_1^0)\big) + f_1(u_1^0) = 0 && \text{in} \quad Q_T\,,\\
&u_1^0 = 0 && \text{on} \quad S_T\,,\\
&b_1(u_1^0)_{t=0} = b_1(\varphi_0) && \text{in} \quad \Omega\,,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial b_2(u_2^0)}{\partial t} - \operatorname{div}\big(a_2(\nabla u_2^0)\big) + f_2(u_2^0) = 0 && \text{in} \quad Q_T\,,\\
&u_2^0 = 0 && \text{on} \quad S_T\,,\\
&b_1(u_2^0)_{t=0} = b_1(\psi_0) && \text{in} \quad \Omega\,.
\end{aligned}
$$

We construct two sequences of functions (u_1^n) and (u_2^n), such that:

$$\frac{\partial b_1(u_1^n)}{\partial t} - \operatorname{div}\big(a_1(\nabla u_1^n)\big) + f_1(u_1^n) = \delta_1 \frac{\partial H}{\partial u_1}(u_1^{n-1}, u_2^{n-1}) \qquad \text{in} \quad Q_T\,, \tag{1.1}$$

$$u_1^n = 0 \qquad \text{on} \quad S_T\,, \tag{1.2}$$

$$b_1(u_1^n)_{t=0} = b_1(\psi_0) \qquad \text{in} \quad \Omega\,, \tag{1.3}$$

$$\frac{\partial b_2(u_2^n)}{\partial t} - \operatorname{div}\big(a_2(\nabla u_2^n)\big) + f_2(u_2^n) = \delta_2 \frac{\partial H}{\partial u_2}(u_1^{n-1}, u_2^{n-1}) \qquad \text{in} \quad Q_T\,, \tag{1.4}$$

$$u_2^n = 0 \qquad \text{on} \quad S_T\,, \tag{1.5}$$

$$b_2(u_2^n)_{t=0} = b_1(\varphi_0 \qquad \text{in} \quad \Omega\,. \tag{1.6}$$

We need Lemma 1.4 and Lemma 1.5 below to complete the proof of Theorem 1. □

Lemma 1.4.

$$\forall \tau > 0,\ \exists\, c_\tau > 0 \quad \textit{such that} \quad \|u_i^n\|_{L^\infty(\tau,T;L^\infty(\Omega))} \le c_\tau\,. \tag{1.7}$$

Proof. For $n = 0$, (1.7) is proved in [21], so suppose (1.7) for $(n-1)$.

Multiplying (1.1) by $|b_1(u_1^n)|^k\, b_1(u_1^n)$, k integer, and integrating over Ω to obtain:

$$
\begin{aligned}
\frac{1}{k+2} \int_\Omega |b_1(u_1^n)|^{k+2}\,dx &+ (k+1) \int_\Omega b_1'(u_1^n)\, |b_1(u_1^n)|^k\, a_1(\nabla u_1)\cdot\nabla u_1\,dx\\
&+ \int_\Omega f_1(u_1^n) b_1(u_1^n)\, |b_1(u_1^n)|^k\,dx = \int_\Omega \delta_1 \frac{\partial H}{\partial u_1}(u_1^{n-1}, u_2^{n-1}) b_1(u_1^n)\, |b_1(u_1^n)|^k\,dx\,.
\end{aligned}
$$

We note that

$$
\begin{aligned}
&\int_\Omega b_1'(u_1^n)\, |b_1(u_1^n)|^k\, a_1(\nabla u_1)\cdot\nabla u_1\,dx \ge 0\,,\\
&\int_\Omega \delta_1 \frac{\partial H}{\partial u_1}(u_1^{n-1}, u_2^{n-1}) b_1(u_1^n)\, |b_1(u_1^n)|^k\,dx \le c \int_\Omega |b_1(u_1^n)|^{k+1}\,dx\,.
\end{aligned}
$$

We thus have

$$\frac{1}{k+2}\int_\Omega |b_1(u_1^n)|^{k+2}\,dx + c\int_\Omega |b_1(u_1^n)|^{k+p_1+2}\,dx \le c'\int_\Omega |b_1(u_1^n)|^{k+1}\,dx\,.$$

Setting $y_{k,n}(t) = \|b_1(u_1^n)\|_{L^{k+2}(\Omega)}$ and using Holder inequality on both sides, we have the existence of two constants $\lambda > 0$ and $\delta > 0$ such that

$$\frac{dy_{k,n}(t)}{dt} + \lambda y_{k,n}^{p_1+1}(t) \le \delta,$$

which implies from Lemma 1.1 that $\forall\ t \ge \tau > 0$

$$y_{k,n}(t) \le (\frac{\delta}{\lambda})^{\frac{1}{p_1}} + \frac{1}{[\lambda(p_1)t]^{\frac{1}{p_1}}}\,.$$

As $k \to \infty$, we obtain

$$|u_1^n(t)| \le c_\tau \quad \forall\ t \ge \tau > 0\,.$$

The same holds also for u_2^n.

Lemma 1.5. $\forall \tau > 0,\ \exists\ c_i = c_i\,(\tau, \varphi_0, \psi_0) > 0$:

$$\|u_i^n\|_{L^2(0,T;H_0^1(\Omega))} \le c\,,$$
$$\|u_i^n\|_{L^\infty(\tau,T;H_0^1(\Omega)\cap L^\infty(\Omega))} \le c'\,,$$
$$\sum_{i=1}^{2}\Big[\int_0^T\int_\Omega |\nabla u_i^n|^2\,dx + c'\int_0^T\int_\Omega |u_i^n|^{p_i}\,dx\Big] \le c''\,.$$

Proof. Multiplying (1.1) by u_1^n and (1.4) by u_2^n, and adding, we get:

$$\frac{d}{dt}\sum_{i=1}^{2}\Big[\int_\Omega \Psi_i^*\big(b_i(u_i^n)\big)\,dx\Big] + \sum_{i=1}^{2}\int_\Omega |\nabla u_i^n|^2\,dx + c\sum_{i=1}^{2}\int_\Omega |u_i^n|^{p_i+2}\,dx \le c'\,. \tag{1.8}$$

But

$$|\varphi_0|_{L^2(\Omega)} + |\psi_0|_{L^2(\Omega)} \le c \Rightarrow \int_\Omega \Psi_1^*\big(b_1(\varphi_0)\big)\,dx + \int_\Omega \Psi_2^*\big(b_2(\psi_0)\big)\,dx \le c\,,$$

so we deduce that:

$$\sum_{i=1}^{2}\int_0^T\int_\Omega |\nabla u_i^n|^2\,dx + c\sum_{i=1}^{2}\int_0^T\int_\Omega |u_i^n|^{p_i+2}\,dx \le c'\,.$$

Whence Lemma 1.5.

From Lemma 1.4 and Lemma 1.5, there is a subsequence u_i^n $(i = 1, 2)$ with the following properties:

$$u_i^n \to u_i \quad \text{weakly in} \quad L^2\big(0,T;H_0^1(\Omega)\big)\cap L^{p_i+1}\big(0,T;L^{p_i+1}(\Omega)\big)\,,$$
$$b_i(u_i^n) \to \chi_i \quad \text{weakly in} \quad L^2\big(0,T;L^2(\Omega)\big)\,,$$
$$b_i(u_i^n) \to \chi_i \quad \text{strongly in} \quad L^2\big(\tau,T;H^{-1}(\Omega)\big)$$

(by the compactness result of Aubin (see [24]). By Lemma 7 in [6], we have $\chi_i = b_i(u_i)$. Moreover,

$$\delta_i \frac{\partial H}{\partial u_i}(u_1^{n-1}, u_2^{n-1}) - f_i(\cdot, u_i^n) \to \delta_i \frac{\partial H}{\partial u_i}(u_1, u_2) - f_i(\cdot, u_i)$$

in $L^r\big(\tau, T; L^r(\Omega)\big)$, $\forall\, r \geq 1$, $\forall\, \tau \geq 1$. Taking the limit as n goes to $+\infty$, we deduce that (u_1, u_2) is a weak solution of $(\mathcal{P})$. $\square$

1.3. **Uniqueness.**
Let (A1) to (A6) be satisfied. Then $(\mathcal{P})$ has a unique solution (u, v) in Q_T.

Proof. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be solutions of $(\mathcal{P})$, we have:

$$\begin{aligned} &\frac{\partial(b_1(u_1) - b_1(v_1))}{\partial t} - \operatorname{div}(a_1(\nabla u_1) - a_1(\nabla v_1)) + f_1(u_1) - f_1(v_1) \\ &\qquad = \delta_1 \frac{\partial H}{\partial u_1}(u) - \delta_1 \frac{\partial H}{\partial u_1}(v)\,, \end{aligned} \tag{1.9}$$

$$\begin{aligned} &\frac{\partial(b_2(u_1) - b_2(u_2))}{\partial t} - \operatorname{div}(a_2(\nabla u_2) - a_2(\nabla v_2)) + f_2(u_2) - f_2(v_2) \\ &\qquad = \delta_2 \frac{\partial H}{\partial u_2}(u) - \delta_2 \frac{\partial H}{\partial u_2}(v)\,. \end{aligned} \tag{1.10}$$

The difference $w_i = u_i - v_i$ satisfies

$$\begin{aligned} &b_i'(u_i) w_i' - \operatorname{div}(a_i(\nabla u_i) - a_i(\nabla v_i)) + f_i(u_i) - f_i(v_i) \\ &\qquad = [b_i'(v_i) - b_i'(u_i)]\, v_i' + \delta_i \frac{\partial H}{\partial u_i}(u) - \delta_i \frac{\partial H}{\partial u_i}(v) \end{aligned} \tag{1.11}$$

(1.11) becomes

$$\begin{aligned} &b_i'(u_i) w_i' - \operatorname{div}(a_i(\nabla u_i) - a_i(\nabla v_i)) + f_i(u_i) - f_i(v_i) \\ &\qquad = [b_i'(v_i) - b_i'(u_i)]\, v_i' + \delta_i \frac{\partial H}{\partial u_i}(u) - \delta_i \frac{\partial H}{\partial u_i}(v)\,. \end{aligned} \tag{1.12}$$

We multiply (1.12) by $\frac{w_i}{b_i'(u_i)}$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{2} |w_i|^2_{L^2(\Omega)} + \sum_{i=1}^{2}\Big((a_i(\nabla u_i) - a_i(\nabla v_i), \nabla(\frac{w_i}{b_i'(u_i)})\Big)_{L^2(\Omega)} \\ &\qquad + \sum_{i=1}^{2}\Big(f_i(u_i) - f_i(v_i), \frac{w_i}{b_i'(u_i)}\Big) = \sum_{i=1}^{2}\Big(\frac{b_i'(v_i) - b_i'(u_i)}{b_i'(u_i)} v_i', w_i\Big)_{L^2(\Omega)} \\ &\qquad + \sum_{i=1}^{2}\Big(\delta_i \frac{\partial H}{\partial u}(u) - \delta_i \frac{\partial H}{\partial u_i}(v), \frac{w_i}{b_i'(u_i)}\Big)_{L^2(\Omega)}\,. \end{aligned} \tag{1.13}$$

With the assumptions and standard arguments, we get

$$\sum_{i=1}^{2}(a_i\Big(\nabla u_i) - a_i(\nabla v_i), \frac{1}{b_i'(u_i)}\nabla w_i\Big) \geq c\sum_{i=1}^{2}|\nabla w_i|^2 \ , \tag{1.14}$$

$$\sum_{i=1}^{2}\Big|\Big(a_i(\nabla u_i) - a_i(\nabla v_i), \frac{b_i''(u_i)}{b_i'(u_i)}w_i\nabla u_i\Big)\Big|$$

$$\leq c\sum_{i=1}^{2}|\nabla u_i|_{L^4(\Omega)}\,|w_i|_{L^4(\Omega)}\,\|w_i\|_{H_0^1(\Omega)} \ , \tag{1.15}$$

$$\sum_{i=1}^{2}\Big|\Big(f_i(u_i) - f_i(v_i), \frac{w_i}{b_i'(u_i)}\Big)\Big| \leq c\sum_{i=1}^{2}|w_i|^2 \ , \tag{1.16}$$

$$\sum_{i=1}^{2}\Big(\frac{b_i'(v_i) - b_i'(u_i)}{b_i'(u_i)}v_i', \frac{w_i}{b_i'(u_i)}\Big)_{L^2(\Omega)} \tag{1.17}$$

$$\leq c\sum_{i=1}^{2}|b_i'(v_i) - b_i'(u_i)|_{L^4(\Omega)}\,|v_i'|_{L^2(\Omega)}\,\|w_i\|_{L^4(\Omega)}$$

$$\sum_{i=1}^{2}\Big|\Big(\delta_i\frac{\partial H}{\partial u_i}(u) - \delta_i\frac{\partial H}{\partial u_i}(v), \frac{w_i}{b_i'(u_i)}\Big)\Big| \leq c\sum_{i=1}^{2}|w_i|^2 \ . \tag{1.18}$$

Since $(u, v) \in (H^2(\Omega))^2$ and $H^1(\Omega) \hookrightarrow L^4(\Omega)$

$$\frac{1}{2}\frac{d}{dt}\Big[\sum_{i=1}^{2}|w_i|^2\Big] + \frac{1}{k}\sum_{i=1}^{2}\|w_i\|_{H_0^1(\Omega)}^2 \leq c\sum_{i=1}^{2}|w_i|^2 + \frac{1}{2k}\sum_{i=1}^{2}\|w_i\|_{H_0^1(\Omega)}^2 \ . \tag{1.19}$$

We finally deduce from Gronwall's lemma,

$$\sum_{i=1}^{2}|w_i|^2 \leq \sum_{i=1}^{2}|w_i(0)|^2\exp(2cT)\,, \quad \forall t \in (0, T)\,.$$

Thus, we deduce that $u_1 = v_1$ and $u_2 = v_2$. □

2. Global attractor

Proposition 1. *Assuming that* (A1)–(A6) *hold, then the solution* (u_1, u_2) *of system* $(\mathcal{P})$ *satisfies*

$$|u_1(t)|_{L^\infty(\Omega)} + |u_2(t)|_{L^\infty(\Omega)} \leq c(r)\,, \qquad \forall t \geq r\,, \tag{2.0}$$

$$\sum_{i=1}^{2}|\nabla u_i|^2 \leq c(r)\,, \qquad \forall t \geq t_0 + r\,. \tag{2.1}$$

Proof. Reasoning as in the proof of Lemma 1.4, we also have (2.0).

Multiplying the first equation of $(\mathcal{P})$ by u_1 and the second by u_2, we get

$$\frac{d}{dt}\sum_{i=1}^{2}\int_\Omega \Psi_i(u_i)\,dx + \sum_{i=1}^{2}\big(a_i(\nabla u_i),\nabla u_i\big) + \sum_{i=1}^{2}\big(f_i(u_i),u_i\big) = -\sum_{i=1}^{2}\delta_i\frac{\partial H}{\partial u_i}(u)u_i\,dx\,. \tag{2.2}$$

We note that it follows from (A2) that

$$\big(a_i(\nabla u_i),\nabla u_i\big) \geq c_i\left|\nabla u_i\right|^2\,. \tag{2.3}$$

We deduce from (A3) that

$$\big(f_i(u_i),u_i\big) \geq c\left|u_i\right|^{p_i+2} - c'\,. \tag{2.4}$$

For fixed $r>0$ and $\tau>0$, integrate (2.2) on $]t,t+r[$

$$\sum_{i=1}^{2}\int_t^{t+r}\left|\nabla u_i\right|^2\,ds \leq c(\tau),\quad \forall t\geq\tau>0\,,$$

$$\sum_{i=1}^{2}\int_t^{t+r}\left|u_i\right|^{p_i+2}\,ds \leq c(\tau)\,. \tag{2.5}$$

Multiplying the first equation of $(\mathcal{P})$ by $\frac{1}{\delta_1}(u_1)_t$ and the second by $\frac{1}{\delta_2}(u_2)_t$, we get

$$\sum_{i=1}^{2}\Big(\frac{1}{\delta_i}a_i(\nabla u_i),\nabla\frac{\partial u_i}{\partial t}\Big) + \sum_{i=1}^{2}\Big(\frac{1}{\delta_i}b_i'(u_i),\Big(\frac{\partial u_i}{\partial t}\Big)^2\Big) + \frac{1}{\delta_i}\frac{d}{dt}\int_\Omega F_i(u_i)\,dx = \int_\Omega\left[H(u_1(T),u_2(T)) - H(\varphi_0,\psi_0)\right]\,dx\,. \tag{2.6}$$

We note that

$$\Big(\frac{1}{\delta_i}b_i'(u_i),\Big(\frac{\partial u_i}{\partial t}\Big)^2\Big) \geq c\Big|\frac{\partial u_i}{\partial t}\Big|^2, \tag{2.7}$$

$$\Big(\frac{1}{\delta_i}a_i(\nabla u_i),\nabla\frac{\partial u_i}{\partial t}\Big) = \frac{1}{\delta_i}\frac{d}{dt}\int_\Omega A_i(\nabla u_i)\,dx\,, \tag{2.8}$$

$$\alpha_i\left|\nabla u_i\right|^2 - c \leq \int_\Omega A_i(\nabla u_i)\,dx \leq d_i\left|\nabla u_i\right|^2 + c'\,, \tag{2.9}$$

$$c\left|u_i\right|^{p_i+2} - c \leq \int_\Omega F_i(u_i)\,dx \leq c\left|u_i\right|^{p_i+2} + c\,. \tag{2.10}$$

We finally deduce from (2.5)–(2.10) and the uniform Gronwall's lemma that, for $r>0$,

$$\sum_{i=1}^{2}\left|\nabla u_i\right|^2 \leq c(\tau)\,,\quad \forall t\geq t_0+r\,. \tag{2.11}$$

□

Remark 1. By Proposition 1 we deduce that there exist absorbing sets in $L^{\sigma_1}(\Omega)\times L^{\sigma_1}(\Omega)$ for any $\sigma_i\colon 1 \le \sigma_i \le +\infty$ and absorbing sets in $\left(H_0^1(\Omega)\right)^2$; then assumptions (1.1),(1.4) and (1.12) in Theorem 1.1 [27, p. 23], are satisfied with $U=\left[L^2(\Omega)\right]^2$, so we have the following

Theorem 2. *Assuming that* (A1)–(A6) *are satisfied, then the semi-group* $S(t)$ *associated with the boundary value problem* $(\mathcal{P})$ *possesses a maximal attractor*$\mathcal{A}$*, which is bounded in* $\left([H_0^1(\Omega)\cap L^\infty(\Omega)]\right)^2$*, compact and connected in* $\left[L^2(\Omega)\right]^2$*. Its domain of attraction is the whole space* $\left[L^2(\Omega)\right]^2$*.*

Proposition 2. *We assume that* $(\varphi_0,\psi_0)\in\mathcal{A}$*. Then, for every* $t>0$, $\left(\frac{\partial u_1}{\partial t},\frac{\partial u_2}{\partial t}\right)\in\left(H_0^1(\Omega)\right)^2$ *and* $\mathcal{A}\in\left(H^2(\Omega)\right)^2$*.*

Proof. Differentiating equation

$$\frac{\partial b_i(u_i)}{\partial t}-\operatorname{div}\left[a_i(\nabla u_i)\right]+f_i(u_i)=\delta_i\frac{\partial H}{\partial u_i}(u_1,u_2)\,.$$

Setting $\theta_i=\frac{\partial u_i}{\partial t}$, we get

$$b_i'(u_i)\theta_i'+b_i''(u_i)(\theta_i)^2-\operatorname{div}\left(d_s a_i(\nabla u_i).\nabla\theta_i\right)+f_i'(u_i)\theta_i=\sum_{j=1}^{2}\delta_i\frac{\partial^2H(u)}{\partial u_i\partial u_j}\theta_j\,. \tag{2.12}$$

Now multiplying (2.12) by θ_i, and integrating over Ω, we obtain, thanks to (A2),

$$\begin{aligned}\sum_{i=1}^{2}\int_\Omega b_i'(u_i)\theta_i'\theta_i\,dx+\sum_{i=1}^{2}\int_\Omega b_i''(u_i)(\theta_i)^3\,dx+c\sum_{i=1}^{2}\left|\nabla\theta_i\right|^2&\\+\sum_{i=1}^{2}\int_\Omega f_i'(u_i)\left|\theta_i\right|^2\,dx\le\int_\Omega\Big(\sum_{i=1}^{2}\sum_{j=1}^{2}\delta_i\frac{\partial^2H(u)}{\partial u_i\partial u_j}\theta_i\Big)\theta_i\,dx\,,&\end{aligned}\tag{2.13}$$

the L^∞ estimate and hypothesis imply successively

$$\frac{d}{dt}\int_\Omega b_i'(u_i)\left|\theta_i'\right|^2dx=\int_\Omega b_i'(u_i)\theta_i'\theta_i\,dx+\frac12\int_\Omega b_i''(u_i)(\theta_i)^3\,dx\,, \tag{2.14}$$

$$\int_\Omega\Big(\sum_{i=1}^{2}\sum_{j=1}^{2}\delta_i\frac{\partial^2H(u)}{\partial u_i\partial u_j}\theta_j\Big)\theta_i\,dx\le M\sum_{i=1}^{2}\left|\theta_i\right|^2, \tag{2.15}$$

$$\sum_{i=1}^{2}\int_\Omega f_i'(u_i)\left|\theta_i\right|^2\,dx\ge -c\sum_{i=1}^{2}\left|\theta_i\right|^2. \tag{2.16}$$

We deduce from (A1) that

$$\begin{aligned}\Big|\int_\Omega b''(u_i)(u_i')^3dx\Big|&\le c\left|\theta_i\right|^3_{L^3(\Omega)}\le c\left\|\theta\right\|^3_{H^{\frac13}(\Omega)}\\&\le c\left|\theta_i\right|^2\left|\nabla\theta_i\right|\le M\left|\nabla\theta_i\right|^2+M'\left|\theta_i\right|^4.\end{aligned}\tag{2.17}$$

By (2.13)–(2.17) becomes

$$\frac{d}{dt}\Big[\sum_{i=1}^{2}\int_\Omega b_i'(u_i)\,|\theta_i'|^2\,dx\Big] \le c\sum_{i=1}^{2}|\theta_i|^4 + c'\sum_{i=1}^{2}|\theta_i|^2\,. \tag{2.18}$$

Setting $y = \sum_{i=1}^{2}\int_\Omega b_i'(u_i)\,|\theta_i'|^2\,dx$, we obtain

$$\frac{dy}{dt} \le cy^2 + c'. \tag{2.19}$$

We have, owing to estimates (2.9) and (2.10)

$$\int_\tau^{\tau+r} y dt \le c_\tau\,, \quad \text{for any} \quad \tau \ge t_0\,. \tag{2.20}$$

The uniform Gronwall's lemma gives

$$y(t) = \sum_{i=1}^{2}\int_\Omega b_i'(u_i)\,|\theta_i'|^2\,dx \le c(r)\,, \quad \text{for any} \quad t \ge r\,. \tag{2.21}$$

Now, by (2.21) and hypothesis (A2), we get

$$\sum_{i=1}^{2}\int_\Omega \theta_i^2 dx \le c(t_0)\,, \quad \text{for any} \quad t \ge t_0\,.$$

Then, for every $t>0$, $\big(\frac{\partial u_1}{\partial t}, \frac{\partial u_2}{\partial t}\big) \in \big(L^2(\Omega)\big)^2$.

But we have

$$(\mathbf{E}) \qquad \begin{cases} -\operatorname{div}\big(a_i(\nabla u_i)\big) = \phi_i & \text{in} \quad \Omega\,,\\ u_i = 0 & \text{on} \quad \partial\Omega\,,\end{cases}$$

where $\phi_i(x,t) = -f_i(u_i) - b_i'(u_i)\theta_i + \delta_i \frac{\partial H}{\partial u_i}(u_1,u_2) \in L^2(\Omega)$, for every $t>0$.

It follows from Lemma 1.3 that the problem ($\mathbf{E}$) possesses a unique solution (u,v) such that $(u,v) \in \big(H_0^1(\Omega)\cap H^2(\Omega)\big)^2$. □

3. Dimension of the global attractor $\mathcal{A}$

Proposition 3. *Let $(\varphi_0,\psi_0) \in \mathcal{A}$; $u=(u_1,u_2)$ and $v=(v_1,v_2)$ be two solutions of $(\mathcal{P})$. Then,*

$$\begin{aligned}\frac{d}{dt}\Big[\sum_{i=1}^{2}\int_\Omega b_i'(u_i)\,|u_i-v_i|^2\,dx\Big] &+ M_1\sum_{i=1}^{2}\nabla\,|u_i-v_i|^2\\ &\le M_2\sum_{i=1}^{2}\int_\Omega b_i'(u_i)\,|u_i-v_i|^2\,dx\,.\end{aligned} \tag{3.1}$$

Proof. Wet set $w_i = u_i - v_i$, we have

$$\frac{d}{dt}\left(b_i(u_i) - b_i(v_i)\right) - \operatorname{div}\big(a_i(\nabla u_i) - a_i(\nabla v_i)\big) + f_i(u_i) - f_i(v_i) = \delta_i \frac{\partial H}{\partial u_i}(u) - \delta_i \frac{\partial H}{\partial u_i}(v)\,. \tag{3.2}$$

Multiplying (3.2) by w_i and integrating over Ω to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega b_i'(u_i)w_i^2\,dx\Big) + (a_i(\nabla u_i) - a_i(\nabla v_i), \nabla w_i) + (f_i(u_i) - f_i(v_i), w_i) \\ = \Big(\delta_i \frac{\partial H}{\partial u_i}(u) - \delta_i \frac{\partial H}{\partial u_i}(v), w_i\Big) + \frac{1}{2}\int_\Omega b_i''(u_i)\frac{\partial u_i}{\partial t} w_i^2\,dx \\ - \int_\Omega \big(b_i'(u_i) - b_i'(v_i)\big)\frac{\partial v_i}{\partial t} w_i\,dx.\end{aligned} \tag{3.3}$$

By the assumption above, we have

$$\sum_{i=1}^{2}\big(a_i(\nabla u_i) - a_i(\nabla v_i), \nabla w_i\big) \geq c\sum_{i=1}^{2}|\nabla w_i|^2\,, \tag{3.4}$$

$$\begin{aligned}\Big|\frac{1}{2}\sum_{i=1}^{2}\int_\Omega b_i''(u_i)\frac{\partial u_i}{\partial t}w_i^2\,dx - \sum_{i=1}^{2}\int_\Omega \big(b_i'(u_i) - b_i'(v_i)\big)\frac{\partial v_i}{\partial t}w_i\,dx\Big| \\ \leq c\sum_{i=1}^{2}\int_\Omega \Big(\Big|\frac{\partial u_i}{\partial t}\Big| + \Big|\frac{\partial v_i}{\partial t}\Big|\Big)|w_i|^2 \\ \leq c\sum_{i=1}^{2}\int_\Omega \Big(\Big|\frac{\partial u_i}{\partial t}\Big| + \Big|\frac{\partial v_i}{\partial t}\Big|\Big)|w_i|^2 \\ \leq c\sum_{i=1}^{2}\Big(\Big|\frac{\partial u_i}{\partial t}\Big| + \Big|\frac{\partial v_i}{\partial t}\Big|\Big)\|w_i\|^2_{L^4(\Omega)} \\ \leq c\sum_{i=1}^{2}|w_i|\ \ |\nabla w_i| \leq \frac{M}{2}\sum_{i=1}^{2}|\nabla w_i|^2 + c\sum_{i=1}^{2}|w_i|^2\ ,\end{aligned} \tag{3.5}$$

$$\sum_{i=1}^{2}|(f_i(u_i) - f_i(v_i), w_i)| \leq c\sum_{i=1}^{2}|w_i|^2\ , \tag{3.6}$$

$$\sum_{i=1}^{2}\Big|\Big(\delta_i\frac{\partial H}{\partial u_i}(u) - \delta_i\frac{\partial H}{\partial u}(v), w_i\Big)\Big| \leq c\sum_{i=1}^{2}|w_i|^2\ . \tag{3.7}$$

We finally deduce from (3.4)–(3.7) that

$$\sum_{i=1}^{2}\frac{d}{dt}\int_\Omega\big(b_i'(u_i)w_i^2\big)\,dx\ + c\sum_{i=1}^{2}|\nabla w_i|^2 \leq c'\sum_{i=1}^{2}|u_i|^2\,. \tag{3.8}$$

Set $\omega_i = [b_i(u_i)]^{\frac{1}{2}}\, w_i$, we deduce from (3.1) and Gronwall's lemma that,

$$\sum_{i=1}^{2} |\omega_i(t_2)|^2 \le \exp\left(M_2(t_2-t_1\right)\sum_{i=1}^{2} |\omega_i(t_1)|^2$$

$$\le \exp(2)\sum_{i=1}^{2} |\omega_i(t_1)|^2\ , \quad \text{for} \quad 0 \le t_2 - t_1 \le \frac{2}{M_2}\,. \tag{3.9}$$

We fix $s \in \left(0, l = \frac{1}{M_2}\right)$ and integrate (3.1) over $t \in (s,l)$ to obtain, owing to (4.5)

$$\sum_{i=1}^{2} |\omega_i(l)|^2 + c\sum_{i=1}^{2}\int_s^{2l} |\nabla\omega_i|^2\, dt \le c'\sum_{i=1}^{2}\int_s^{2l} |\omega_i|^2\, dt + \sum_{i=1}^{2} |\omega_i(s)|^2$$

$$\le c'\sum_{i=1}^{2} |\omega_i(s)|^2 \le c'\sum_{i=1}^{2} |u_i(s)|^2\ , \tag{3.10}$$

which yields

$$\sum_{i=1}^{2}\int_l^{2l} |\nabla\omega_i|^2\, dt \le c'\sum_{i=1}^{2} |w_i(s)|^2\ . \tag{3.11}$$

Integrating (3.11) over $s \in (0,l)$,

$$\sum_{i=1}^{2}\int_l^{2l} |\nabla u_i - \nabla v_i|^2\, dt \le c\sum_{i=1}^{2}\int_0^{2l} |u_i - v_i|^2\ dt\,. \tag{3.12}$$

□

Proposition 4. *Let $(\varphi_0, \psi_0) \in \mathcal{A}$; (u_1,u_2) and (v_1,v_2) be two solutions of $(\mathcal{P})$. Then,*

$$\sum_{i=1}^{2}\left\| \frac{d}{dt}\,(u_i - v_i)\,\right\|_{L^2\left(l,2l;H^{-1}(\Omega)\right)} \le c\sum_{i=1}^{2} \|u_i - v_i\|_{L^2(0,l;H(\Omega))}\,. \tag{3.13}$$

Proof. We have

$$\left\| b^{'}(u_i)\frac{\partial w_i}{\partial t}\right\|_{L^2(l,2l;H^{-1}(\Omega))} = \sup\left|\int_l^{2l} < b^{'}(u_i)\frac{\partial w_i}{\partial t}, \varphi_i > dt\right|,$$

where $\varphi_i \in L^2(l,2l;V)), \|\varphi_i\|_{L^2(l,2l;V)} = 1$.

Noting that

$$b^{'}(u_i)\frac{\partial w_i}{\partial t} = \ \mathrm{div}\left(a_i(\nabla u_i) - a_i(\nabla v_i)\right) - \left(f_i(u_i) - f_i(v_i)\right)$$

$$+\,\delta_i\frac{\partial H}{\partial u_i}(u) - \delta_i\frac{\partial H}{\partial u_i}(v) - \left(b'(u_i) - b'(v_i)\right)\frac{\partial v_i}{\partial t}\,.$$

Furthermore,

$$\sum_{i=1}^{2}\int_l^{2l} |a_i(\nabla u_i) - a_i(\nabla v_i)|\,|\nabla\varphi_i|\ dt \le c\sum_{i=1}^{2} \|w_i\|_{L^2(l,2l;V)}\,, \tag{3.14}$$

$$\sum_{i=1}^{2}\int_{l}^{2l}\left|f_i(u_i)-f_i(v_i)\right|\,\left|\varphi_i\right|dt \le c\sum_{i=1}^{2}\left\|w_i\right\|_{L^2(l,2l;H)}$$

$$\le c\sum_{i=1}^{2}\|w_i\|_{L^2(l,2l;V)}\,, \tag{3.15}$$

$$\sum_{i=1}^{2}\int_{l}^{2l}\left|\Big(\delta_i\frac{\partial H}{\partial u_i}(u)-\delta_i\frac{\partial H}{\partial u}(v)\Big)\right|\,|\varphi_i|\;dt \le c\sum_{i=1}^{2}\|w_i\|_{L^2(l,2l;H)}$$

$$\le c\sum_{i=1}^{2}\|w_i\|_{L^2(l,2l;V)}\,, \tag{3.16}$$

$$\sum_{i=1}^{2}\int_{l}^{2l}dt\int_{\Omega}\left|\big(b'(u_i)-b'(v_i)\big)\right|\left|\frac{\partial v_i}{\partial t}\right||\varphi_i|\,dx \le c\sum_{i=1}^{2}\int_{l}^{2l}dt\int_{\Omega}|w_i|\left|\frac{\partial v_i}{\partial t}\right||\varphi_i|\;dx$$

$$\le c\sum_{i=1}^{2}\left\|\frac{\partial v_i}{\partial t}\right\|_{L^{+\infty}(l,2l;H)}\|w_i\|_{L^2(l,2l;V)} \le c\sum_{i=1}^{2}\|w_i\|_{L^2(l,2l;V)}\,. \tag{3.17}$$

We thus deduce from (3.14)–(3.17) that

$$\sum_{i=1}^{2}\left\|b^{'}(u_i)\frac{\partial w_i}{\partial t}\right\|_{L^2(l,2l;H^{-1}(\Omega))} \le \sum_{i=1}^{2}\int_{l}^{2l}|a_i(\nabla u_i)-a_i(\nabla v_i)|\,|\nabla\varphi_i|\;dt$$
$$+\sum_{i=1}^{2}\int_{l}^{2l}\left|f_i(u_i)-f_i(v_i)\right|\,|\varphi_i|\;dt+\sum_{i=1}^{2}\int_{l}^{2l}\left|\Big(\delta_i\frac{\partial H}{\partial u_i}(u)-\delta_i\frac{\partial H}{\partial u_i}(v)\Big)\right||\nabla\varphi_i|\,dt$$
$$+\sum_{i=1}^{2}\int_{l}^{2l}dt\int_{\Omega}\left|\big(b'(u_i)-b'(v_i)\big)\right|\left|\frac{\partial v_i}{\partial t}\right|\,|\varphi_i|\;dx \le c\sum_{i=1}^{2}\|w_i\|_{L^2(l,2l;V)}\,,$$

and by (3.12), we obtain

$$\sum_{i=1}^{2}\left\|b^{'}(u_i)\frac{\partial w_i}{\partial t}\right\|_{L^2(l,2l;H^{-1}(\Omega))} \le c\sum_{i=1}^{2}\|w_i\|_{L^2(0,l;H)}\,. \tag{3.18}$$

□

Theorem 3. *The global attractor $\mathcal{A}$ associated with $(\mathcal{P})$ has finite fractal dimension.*

Proof. It follows from Proposition 1 and Proposition 2 that the semigroup associated with $(\mathcal{P})$ possesses a bounded and positively invariant absorbing set $\mathcal{B}_2$ in $\big(H_0^1(\Omega)\cap H^2(\Omega)\big)^2$. More precisely, we will take $\mathcal{B}_2=\overline{\cup_{t\ge t_0}S(t)B_2}$, where $\mathcal{B}_2$ is a bounded absorbing set in $\big(H^2(\Omega)\big)^2$ and τ is such that $t\ge\tau$ implies $S(t)\mathcal{B}_2\subset\mathcal{B}_2$ and where the closure is taken in the topology of $\big(H^2(\Omega)\big)^2$. We use the l-trajectories method introduced by Málek and Pražák in [20]. Let l be fixed as defined above, we introduce the space of trajectories $\mathcal{X}_l=\big\{w=(u,v):(0,l)\to H^2, w$ is the solution of $(\mathcal{P})$ on $(0,l)\big\}$. We endow $\mathcal{X}_l$ with the topology of

$\left(L^2(0,l;H)\right)^2$. We then set $\mathcal{B}_l = \{w \in \mathcal{X}_l, w(0) = (\varphi_0, \psi_0) \in \mathcal{B}_2\}$. By construction $\mathcal{B}_2$ is weakly closed in $\left(H^2(\Omega)\right)^2$. This yields that $\mathcal{B}_l$ is a complete metric space, and we define the operators $\mathbb{L}_t \colon \mathcal{X}_l \to \mathcal{X}_l,\ t \geq 0$, by $\left(\mathbb{L}_t(\overline{w})\right)(s) = w(t+s), s \in [0,l]$, where w is the unique solution of $(\mathcal{P})$ such that $w_{|[0.l]} = \overline{w}$. We finally set $\mathcal{L} = \mathbb{L}_l$.

Finally if we express (3.12) in terms of l-trajectories, we get for all $w_1, w_2 \in \mathcal{B}_l$

$$\|\mathcal{L}w_1 - \mathcal{L}w_2\|_{L^2(0,l;V)} \leq c\,\|w_1 - w_2\|_{\mathcal{X}_l}\,, \tag{3.19}$$

and it follows from (3.13) that

$$\left\|\frac{d}{dt}\left(\mathcal{L}w_1 - \mathcal{L}w_2\right)\right\|_{L^2(0,l;H^{-1}(\Omega))} \leq c\,\|w_1 - w_2\|_{\mathcal{X}_l}\,. \tag{3.20}$$

Furthermore, $\mathcal{L}$ is Lipschitz from $\mathcal{B}_l$ onto $\mathcal{X}_l$, $\forall t \geq 0$, and the mapping $t \to \mathbb{L}_t w$ is Lipschitz, $\forall w \in \mathcal{X}_l$. Thanks to (3.19) and (3.20), it can be proved (see [20] for the details of the proof) that the semigroup $\mathbb{L}_t$ possesses an exponential attractor $\mathcal{M}_l$ on $\mathcal{B}_l$, that $\mathcal{M}_l$ is compact for the topology of $\mathcal{X}_l$, is positively invariant and has finite fractal dimension. ☐

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Equipe Architures des Systèmes, Université Hassan II Ain Chock
Ensem, BP. 8118, Oasis Casablanca, Morrocco
E-mail: `elouardi@ensem-uh2c.ac.ma`