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BOUNDS ON BASS NUMBERS AND THEIR DUAL

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Abstract. Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring. We establish some bounds for the sequence of Bass numbers and their dual for a finitely generated R-module.

Introduction

Throughout this paper, $(R, \mathfrak{m}, k)$ is a non-trivial commutative Noetherian local ring with unique maximal ideal $\mathfrak{m}$ and residue field k. Several authors have obtained results on the growth of the sequence of Betti numbers $\{\beta_n(k)\}$ (e.g., see [9] and [1]). In [10] Ramras gives some bounds for the sequence $\{\beta_n(M)\}$ when M is a finitely generated non-free R-module. In this paper, we seek to give some bounds for the sequence of Bass numbers.

For a finitely generated R-module M, let

$$0 \to M \to E^0 \to E^1 \to \cdots \to E^i \to \cdots$$

be a minimal injective resolution of M. Then, $\mu^i(M)$ denotes the number of indecomposable components of E^i isomorphic to the injective envelope $\mathrm{E}(k)$ and is called *Bass number* of M. This is a dual notion of Betti number. For a prime ideal $\mathfrak{p}$, $\mu^i(\mathfrak{p}, M)$ denotes the number of indecomposable components of E^i isomorphic to the injective envelope $\mathrm{E}(R/\mathfrak{p})$. It is known that $\mu^i(M)$ is finite and is equal to the dimension of $\mathrm{Ext}\,^i_R(R/\mathfrak{m}, M)$ considered as a vector space over $R/\mathfrak{m}$ (note that $\mu^i(\mathfrak{p}, M) = \mu^i(M_\mathfrak{p})$). These numbers play important role in understanding the injective resolution of M, and are the subject of further work. For example, the ring R of dimension d is Gorenstein if and only if R is Cohen-Macaulay and the dth Bass number $\mu^d(R)$ is 1. This was proved by Bass in [2]. Vasconcelos conjectured that one could delete the hypothesis that R be Cohen-Macaulay. This was proved by Paul Roberts in [12].

For a finitely generated R-module M, it turns out that the least i for which $\mu^i(M) > 0$ is the depth of M, while the largest i with $\mu^i(M) > 0$ is the injective

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dimension $\operatorname{inj.dim}_R M$ of M (which might be infinite), cf. [2] and [8]. In [8] Foxby asked the question: Is $\mu^i(M) > 0$ for all i with $\operatorname{depth}_R M \leq i \leq \operatorname{inj.dim}_R M$? In [7], Fossum, Foxby, Griffith, and Reiten answered this question in the affirmative (see also [11]).

A homomorphism $\varphi\colon F \to M$ with a flat R-module F is called a flat precover of the R-module M provided $\operatorname{Hom}_R(G,F) \to \operatorname{Hom}_R(G,M) \to 0$ is exact for all flat R-modules G. If in addition any homomorphism $f : F \to F$ such that $f\varphi = \varphi$ is an automorphism of F, then $\varphi\colon F \to M$ is called a flat cover of M. A minimal flat resolution of M is an exact sequence $\cdots \to F_i \to F_{i-1} \to \cdots \to F_0 \to M \to 0$ such that F_i is a flat cover of $\operatorname{Im}(F_i \to F_{i-1})$ for all $i > 0$. A module C is called cotorsion if $\operatorname{Ext}_R^1(F,C) = 0$ for any flat R-module F. A flat cover of a cotorsion module is cotorsion and flat, and the kernel of a flat cover is cotorsion. In [4], Enochs showed that a flat cotorsion module F is uniquely a product $\prod T_{\mathfrak{p}}$, where $T_{\mathfrak{p}}$ is the completion of a free $R_{\mathfrak{p}}$-module, $\mathfrak{p} \in \operatorname{Spec} R$. Therefore, for $i > 0$ he defined $\pi_i(\mathfrak{p}, M)$ to be the cardinality of a basis of a free $R_{\mathfrak{p}}$-module whose completion is $T_{\mathfrak{p}}$ in the product $F_i = \prod T_{\mathfrak{p}}$. For $i = 0$ define $\pi_0(\mathfrak{p}, M)$ similarly by using the pure injective envelope of F_0. In some sense these invariants are dual to the Bass numbers. In [6], Enochs and Xu proved that for a cotorsion R-module M which possesses a minimal flat resolution, $\pi_i(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})} \operatorname{Tor}_i^R\big(k(\mathfrak{p}), \operatorname{Hom}_R(R_{\mathfrak{p}}, M)\big)$. Here $k(\mathfrak{p})$ denotes the quotient field of $R/\mathfrak{p}$. Note that in [3] the authors show that every module has a flat cover, see also [13] and [5].

In this paper, we study the sequence of Bass numbers $\mu^i(\mathfrak{p}, M)$ and its dual $\pi_i(\mathfrak{p}, M)$. Among the other things we establish the following bounds:

(1) $\mu^2(M)/\mu^1(M) \leq \ell(R)$ and $\mu^{n+1}(M)/\mu^n(M) < \ell(R)$ for any $n \geq 2$,

(2) $\mu^n(M)/\mu^{n+1}(M) < \ell(R)/\ell\big(\operatorname{Soc}(R)\big)$ for any $n \geq 1$,

where $\ell(*)$ refers to the length of $*$.

1. Main results

The following lemma is the key to our main result.

Lemma 1.1. *Let $\mathfrak{p}$ be a prime ideal of R and let L be an $R_{\mathfrak{p}}$-module of finite length. Then the following hold:*

(a) *For any module M and any non-negative integer n,*

$$\ell\big(\operatorname{Ext}_{R_{\mathfrak{p}}}^{n+1}(L,M)\big) - \ell\big(\operatorname{Ext}_{R_{\mathfrak{p}}}^{n}(L,M)\big) \geq \mu^{n+1}(\mathfrak{p}, M) - \ell(L)\mu^n(\mathfrak{p}, M)\,.$$

(b) *For any cotorsion R-module M and any non-negative integer n,*

$$\ell\big(\operatorname{Tor}^{R_{\mathfrak{p}}}_{n+1}(L,M)\big) - \ell\big(\operatorname{Tor}^{R_{\mathfrak{p}}}_{n}(L,M)\big) \geq \pi_{n+1}(\mathfrak{p}, M) - \ell(L)\pi_n(\mathfrak{p}, M)\,.$$

Proof. (a) We proceed by induction on $s = \ell(L)$. If $s = 1$, then $L \cong k(\mathfrak{p})$, and

$$\ell\big(\operatorname{Ext}_{R_{\mathfrak{p}}}^{n+1}(k(\mathfrak{p}),M)\big) - \ell\big(\operatorname{Ext}_{R_{\mathfrak{p}}}^{n}(k(\mathfrak{p}),M)\big) = \mu^{n+1}(\mathfrak{p}, M) - \mu^n(\mathfrak{p}, M)\,.$$

Now assume that $s > 1$. Then there is a submodule K of L with $\ell(K) = s - 1$ such that the sequence $0 \to k(\mathfrak{p}) \to L \to K \to 0$ is exact. The corresponding long exact sequence for $\operatorname{Ext}_{R_{\mathfrak{p}}}(-, M)$ gives the exact sequence

$$\begin{aligned}\operatorname{Ext}^n_{R_{\mathfrak{p}}}(K, M) \to \operatorname{Ext}^n_{R_{\mathfrak{p}}}(L, M) \to \operatorname{Ext}^n_{R_{\mathfrak{p}}}(k(\mathfrak{p}), M)&\\ \to \operatorname{Ext}^{n+1}_{R_{\mathfrak{p}}}(K, M) \to \operatorname{Ext}^{n+1}_{R_{\mathfrak{p}}}(L, M)\,.&\end{aligned}$$

It follows that

$$\begin{aligned}\ell\big(\operatorname{Ext}^{n+1}_{R_{\mathfrak{p}}}(L, M)\big) - \ell\big(\operatorname{Ext}^n_{R_{\mathfrak{p}}}(L, M)\big) &\geq \ell\big(\operatorname{Ext}^{n+1}_{R_{\mathfrak{p}}}(K, M)\big)\\ &\quad - \ell\big(\operatorname{Ext}^n_{R_{\mathfrak{p}}}(K, M)\big) - \mu^n(\mathfrak{p}, M)\\ &\geq \mu^{n+1}(\mathfrak{p}, M) - \ell(K)\mu^n(\mathfrak{p}, M) - \mu^n(\mathfrak{p}, M)\\ &= \mu^{n+1}(\mathfrak{p}, M) - \ell(L)\mu^n(\mathfrak{p}, M)\,,\end{aligned}$$

where the first inequality follows from the property of length and the equality $\operatorname{Ext}^n_{R_{\mathfrak{p}}}(k(\mathfrak{p}), M) = \mu^n(\mathfrak{p}, M)$, also the second inequality follows by the induction hypothesis.

(b) We proceed by induction on $s = \ell(L)$. If $s = 1$, then $L \cong k(\mathfrak{p})$, and we have

$$\ell\big(\operatorname{Tor}^{R_{\mathfrak{p}}}_{n+1}(k(\mathfrak{p}), M)\big) - \ell\big(\operatorname{Tor}^{R_{\mathfrak{p}}}_n(k(\mathfrak{p}), M)\big) = \pi_{n+1}(\mathfrak{p}, M) - \ell(L)\pi_n(\mathfrak{p}, M)\,.$$

Now assume that $s > 1$. Then there is an $R_{\mathfrak{p}}$- submodule K of L with $\ell(K) = s - 1$ such that the sequence $0 \to k(\mathfrak{p}) \to L \to K \to 0$ is exact. Set $N = \operatorname{Hom}_R(R_{\mathfrak{p}}, M)$. The corresponding long exact sequence for $\operatorname{Tor}^{R_{\mathfrak{p}}}(-, N)$ leads to the exact sequence

$$\begin{aligned}\operatorname{Tor}^{R_{\mathfrak{p}}}_{n+1}(L, N) \to \operatorname{Tor}^{R_{\mathfrak{p}}}_{n+1}(K, N) \to \operatorname{Tor}^{R_{\mathfrak{p}}}_n(k(\mathfrak{p}), N)&\\ \to \operatorname{Tor}^{R_{\mathfrak{p}}}_n(L, N) \to \operatorname{Tor}^{R_{\mathfrak{p}}}_n(K, N)\,.&\end{aligned}$$

It follows that

$$\begin{aligned}\ell\big(\operatorname{Tor}^{R_{\mathfrak{p}}}_{n+1}(L, N)\big) - \ell\big(\operatorname{Tor}^{R_{\mathfrak{p}}}_n(L, N)\big) &\geq \ell\big(\operatorname{Tor}^{R_{\mathfrak{p}}}_{n+1}(K, N)\big)\\ &\quad - \ell\big(\operatorname{Tor}^{R_{\mathfrak{p}}}_n(K, N)\big) - \pi_n(M)\\ &\geq \pi_{n+1}(M) - \ell(K)\pi_n(M) - \pi_n(M)\\ &= \pi_{n+1}(M) - \ell(L)\pi_n(M)\,,\end{aligned}$$

where the second inequality follows by the induction hypothesis. □

Corollary 1.2. *Let R be a zero dimensional ring and let M be an R-module. For any prime ideal $\mathfrak{p}$ and any integer $n \geq 1$ the following hold:*

(a)

$$\mu^{n+1}(\mathfrak{p}, M) \leq \ell(R_{\mathfrak{p}})\mu^n(\mathfrak{p}, M)\,.$$

(b) *If M is a cotorsion R-module, then*

$$\pi_{n+1}(\mathfrak{p}, M) \leq \ell(R_{\mathfrak{p}})\pi_n(\mathfrak{p}, M)\,.$$

Proof. (a) Replace the module L in Lemma 1.1(a) with $R_{\mathfrak{p}}$ and note that $\operatorname{Ext}^i_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}, -) = 0$ for all $i \geq 1$.

(b) Replace the module L in Lemma 1.1(b) with $R_{\mathfrak{p}}$ and note that $\operatorname{Tor}^{R_{\mathfrak{p}}}_i(R_{\mathfrak{p}}, -) = 0$ for any $i \geq 1$. □

Proposition 1.3. *Let R be a zero dimensional ring. Then the following hold:*

(a) *Let M be an R-module. For any integer $n \geq 1$ and prime ideal $\mathfrak{p}$,*

$$\mu^{n+1}(\mathfrak{p}, M) \leq \ell(R_{\mathfrak{p}})\mu^n(\mathfrak{p}, M)\,.$$

(b) *Let M be a cotorsion R-module. For any $\mathfrak{p} \in \operatorname{Spec} R$ and any $n \geq 2$,*

$$\pi_{n+1}(\mathfrak{p}, M) + \ell\big(\operatorname{Soc}(R)\big)\pi_{n-1}(\mathfrak{p}, M) \leq \ell(R_{\mathfrak{p}})\pi_n(\mathfrak{p}, M)\,.$$

Proof. (a) It is clear from Lemma 1.1(a).

(b) Assume that $\mathfrak{p} \in \operatorname{Spec} R$ and set $I = \operatorname{Soc}(R_{\mathfrak{p}})$, $N = \operatorname{Hom}_R(R_{\mathfrak{p}}, M)$. From the exact sequence

$$0 \to I \to R_{\mathfrak{p}} \to R_{\mathfrak{p}}/I \to 0\,,$$

it follows that for any $n \geq 1$,

$$\operatorname{Tor}_{n+1}^{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/I, N) \cong \operatorname{Tor}_n^R(I, N) \cong \oplus \operatorname{Tor}_n^R(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, N)\,,$$

where the numbers of copies in the direct sum is $\ell(I)$. Hence

$$\ell\big(\operatorname{Tor}_{n+1}^{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/I, N)\big) = \ell(I)\pi_n(\mathfrak{p}, M) \quad \text{for} \quad n \geq 1\,.$$

Thus, by Lemma 1.1(b), for $n \geq 2$,

$$\ell(I)\big(\pi_n(\mathfrak{p}, M) - \pi_{n-1}(\mathfrak{p}, M)\big) \geq \pi_{n+1}(\mathfrak{p}, M) - \ell(R_{\mathfrak{p}}/I)\pi_n(\mathfrak{p}, M)\,.$$

Therefore, $\ell(I)\pi_{n-1}(\mathfrak{p}, M) + \pi_{n+1}(\mathfrak{p}, M) \leq \ell(R_{\mathfrak{p}})\pi_n(M)$. ☐

Theorem 1.4. *Let R be a zero dimensional local ring. For any finitely generated non-injective R-module M the following hold:*

(1) $\mu^{n+1}(M)/\mu^n(M) < \ell(R)$ *for any* $n \geq 2$,

(2) $\mu^n(M)/\mu^{n+1}(M) < \ell(R)/\ell\big(\operatorname{Soc}(R)\big)$ *for any* $n \geq 1$.

Proof. Let $I = \operatorname{Soc}(R)$. From the exact sequence

$$0 \to I \to R \to R/I \to 0\,,$$

it follows that for any $n \geq 1$,

$$\operatorname{Ext}_R^{n+1}(R/I, M) \cong \operatorname{Ext}_R^n(I, M) \cong \oplus \operatorname{Ext}_R^n(R/\mathfrak{m}, M)\,,$$

where the numbers of copies in the direct sum is $\ell(I)$. Hence

$$\ell\big(\operatorname{Ext}_R^{n+1}(R/I, M)\big) = \ell(I)\mu^n(M) \quad \text{for} \quad n \geq 1\,.$$

Thus, by Lemma 1.1, for $n \geq 2$,

$$\ell(I)\big(\mu^n(M) - \mu^{n-1}(M)\big) \geq \mu^{n+1}(M) - \ell(R/I)\mu^n(M)\,.$$

Therefore, $\ell(I)\mu^{n-1}(M) + \mu^{n+1}(M) \leq \ell(R)\mu^n(M)$. By [7, Theorem 1.1], $\mu^i(M) > 0$ for $\operatorname{depth}_R M \leq i \leq \operatorname{inj.dim}_R M$. Since R is Artinian, $\operatorname{depth}_R M = 0$. Thus for any n, $n \geq 2$, $\mu^n(M)$ and $\mu^{n-1}(M)$ are positive integer and hence $\mu^{n+1}(M)/\mu^n(M) < \ell(R)$. Moreover, if $2 \leq n$, then $\mu^n(M)$ and $\mu^{n+1}(M)$ are positive integers and thus $\mu^{n-1}(M)/\mu^n(M) < \ell(R)/\ell\big(\operatorname{Soc}(R)\big)$. ☐

Corollary 1.5. *Let R be a zero dimensional ring. Let M be a finitely generated R-module. For any prime ideal $\mathfrak{p}$ with $M_{\mathfrak{p}}$ non-injective $R_{\mathfrak{p}}$-module, the following hold:*

(1) $\mu^{n+1}(\mathfrak{p}, M)/\mu^n(\mathfrak{p}, M) < \ell(R_\mathfrak{p})$ *for any* $n \geq 2$,

(2) $\mu^n(\mathfrak{p}, M)/\mu^{n+1}(\mathfrak{p}, M) < \ell(R_\mathfrak{p})/\ell\big(\operatorname{Soc}(R_\mathfrak{p})\big)$ *for any* $n \geq 1$.

Remark 1.6. To the best of the knowledge of the authors, there is no condition (yet!) which implies that $\pi_n(\mathfrak{p}, M) > 0$. This is the reason that we could not give a similar result as Theorem 1.4 for the dual notion of Bass numbers.

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