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# JET ISOMORPHISM FOR CONFORMAL GEOMETRY

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Abstract. Jet isomorphism theorems for conformal geometry are discussed. A new proof of the jet isomorphism theorem for odd-dimensional conformal geometry is outlined, using an ambient realization of the conformal deformation complex. An infinite order ambient lift for conformal densities in the case in which harmonic extension is obstructed is described. A jet isomorphism theorem for even dimensional conformal geometry is formulated using the inhomogeneous ambient metrics recently introduced by the author and K. Hirachi.

## Introduction

Local invariants of a metric in Riemannian geometry are quantities expressible in local coordinates in terms of the metric and its derivatives and which have an invariance property under changes of coordinates. It is a fundamental fact that such invariants may be written in terms of the curvature tensor of the metric and its covariant derivatives. In this form, they can be identified with invariants of the orthogonal group acting algebraically on the space of possible curvature tensors and derivatives. We refer to the result asserting that the space of infinite order jets of metrics modulo coordinate changes is isomorphic to a space of curvature tensors and derivatives modulo the orthogonal group as a jet isomorphism theorem. Such results recast the study and classification of local geometric invariants in purely algebraic terms, in which form the methods of invariant theory and representation theory can be brought to bear.

The goal of this paper is to describe analogous jet isomorphism theorems in the context of conformal geometry. In conformal geometry one is given a metric only up to scale. The results in the conformal case provide a tensorial description of the space of jets of metrics modulo changes of coordinates and conformal factor. The motion group of the flat model is the conformal group $G = O(n+1,1)/\{\pm I\}$ acting projectively on the sphere $S^n$ and the role of the orthogonal group in Riemannian geometry is played by the parabolic subgroup $P \subset G$ preserving a null line. Since

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$P$ is a matrix group in $n+2$ dimensions, its natural tensor representations are on tensor powers of $\mathbb{R}^{n+2}$. Thus one expects the appearance of tensors in $n+2$ dimensions in conformal jet isomorphism theorems.

When $n$ is odd, the ambient metric construction of [9] associates to a conformal Riemannian manifold $(M,[g])$ of dimension $n$ an infinite order jet of a Lorentzian metric $\widetilde{g}$ along a hypersurface in a space $\widetilde{\mathcal{G}}$ of dimension $n+2$, uniquely determined up to diffeomorphism. The tensors in the odd-dimensional conformal jet isomorphism theorem are the curvature tensor and its covariant derivatives for the ambient metric. They satisfy extra identities beyond those satisfied by the derivatives of curvature of a general metric as a consequence of the Ricci-flatness and homogeneity conditions satisfied by an ambient metric. The elaboration of these identities and a formulation and proof of a conformal jet isomorphism theorem in odd dimensions are given in [10]. The algebra of the proof is more involved than in the Riemannian case because one is comparing tensors in different dimensions.

In even dimensions, the ambient metric construction is obstructed at finite order, so this gives only a finite order version of a jet isomorphism theorem. An extension of the ambient metric construction to all orders in even dimensions has recently been described in [14]. Based on this, joint work in preparation with K. Hirachi formulates and proves an infinite order version of the jet isomorphism theorem in even dimensions. The method of proof of the jet isomorphism theorem in this work is different than that used in [10]; it relies on an ambient lift of the conformal deformation complex on $G/P$. This same method also can be used to give another proof of the odd-dimensional jet isomorphism theorem. This proof in the odd-dimensional case will be sketched in §2 below and the details will be the subject of [15].

In the even-dimensional case, an ambient metric depends not only on the conformal manifold $(M,[g])$, but also on the choice of a trace-free symmetric 2-tensor called the ambiguity tensor. Likewise, the even-dimensional jet isomorphism theorem provides a tensorial description of an enlargement of the space of jets of metrics by the space of jets of the ambiguity.

The results concerning the ambient lift of the deformation complex are perhaps of independent interest. In odd dimensions, all the spaces occuring in the complex except for the next to last have isomorphic realizations in terms of infinite order jets along a hypersurface of tensors defined on the ambient space, and the maps in the complex simplify when written in these realizations. The situation in even dimensions is more complicated owing to the existence of ambiguities in the lifts, but it is nonetheless possible to prove results concerning ambient realization including ambiguities which can be used to prove the jet isomorphism theorem in even dimensions.

In §1, we first recall the jet isomorphism theorem for pseudo-Riemannian geometry. We then show how the space of jets of metrics modulo changes of coordinates and conformal factor has a natural action of $P$. In both cases, our presentation is in terms of a quotient of the space of jets of all metrics as in [15] rather than in terms of metrics in geodesic normal coordinates as in [10]. Next we formulate the odd-dimensional conformal jet isomorphism theorem. We also briefly review the ambient metric construction and show how in odd dimensions it gives rise to the map from jets of metrics to the algebraic space of ambient curvature tensors and their covariant derivatives. In §2, we begin by introducing the conformal deformation complex. We then formulate Theorem 2.1, which gives the ambient realization in the unobstructed cases for sections of homogeneous bundles on $G/P$ with symmetries defined by a Young diagram with no more than two columns. This generalizes to tensors results of [6] for scalars. We sketch the proof for scalars and 1-forms. A consequence of Theorem 2.1 is the ambient realization of the deformation complex for $n$ odd. We conclude §2 by sketching the proof of the odd-dimensional jet isomorphism theorem, using this ambient realization and the exactness of the deformation complex on jets. In §3, we first discuss the ambient lift in the simplest obstructed case: that of densities whose weight is such that smooth harmonic extension is obstructed. We indicate how an infinite order harmonic extension involving a log term can always be found, albeit with an ambiguity, and how to reformulate the harmonic extension in terms of the smooth part. These considerations result in Theorem 3.3, the substitute ambient lift theorem for scalars in the obstructed cases. The discussion of the scalar case illustrates the phenomena which occur for jets of conformal structures in even dimensions. We next formulate the conformal jet isomorphism theorem in even dimensions. Then we outline the extension of the ambient metric construction to all orders and indicate by analogy with §1 how it gives rise to the map from jets of metrics and ambiguity tensors to ambient curvature tensors and briefly indicate what is involved in the proof of the even-dimensional jet isomorphism theorem.

A jet isomorphism theorem for a parabolic geometry was first considered in [8], and the entire perspective explicated here owes much to this pioneering work. The idea of incorporating an ambiguity into a jet isomorphism theorem was introduced in [16]. A different approach for conformal geometry using tractor calculus rather than the ambient metric to construct curvature tensors is given in [12].

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## 1. Jet isomorphism, odd dimensions

We begin by reviewing the jet isomorphism theorem for pseudo-Riemannian geometry. Fix a signature $(p, q)$, $p + q = n \geq 2$, and a reference quadratic form $h_{ij}$ on $\mathbb{R}^n$ of signature $(p, q)$ (one typically takes $h_{ij} = \delta_{ij}$ in the positive definite case). By a change of coordinates, any metric of signature $(p, q)$ can be made to equal $h_{ij}$ at the origin. It is convenient to include this normalization in our definition. So we set

$$\mathcal{M} = \{\text{Jets of metrics } g_{ij} \text{ such that } g_{ij}(0) = h_{ij}\}\,.$$

Here jets means infinite order jets of smooth metrics at the origin in $\mathbb{R}^n$. We can identify an element of $\mathcal{M}$ with the list $(\partial^\alpha g_{ij}(0))_{|\alpha|\geq 1}$. Define also

$$\mathrm{Diff} = \{\text{Jets of local diffeomorphisms } \varphi \text{ of } \mathbb{R}^n \text{ satisfying } \varphi(0) = 0\}\,.$$

Then Diff is a group under composition. Since we have normalized our metric at the origin, we need to restrict to diffeomorphisms which preserve the normalization. So we define the subgroup ODiff $\subset$ Diff by

$$\mathrm{ODiff} = \{\varphi \in \mathrm{Diff} : \varphi'(0) \in O(h)\}$$

and the normal subgroup $\mathrm{Diff}_0 \subset \mathrm{ODiff}$ by

$$\mathrm{Diff}_0 = \{\varphi : \varphi'(0) = I\}\,.$$

Then ODiff acts on $\mathcal{M}$ on the left by $\varphi.g = (\varphi^{-1})^* g$. We can view $O(h)$ as the subgroup of ODiff consisting of linear transformations. Then $O(h)$ is the isotropy group in ODiff of the flat metric $h \in \mathcal{M}$, and $\mathrm{ODiff} = O(h) \cdot \mathrm{Diff}_0$.

Since $\mathrm{Diff}_0$ is a normal subgroup of ODiff, there is an induced action of $O(h)$ on the orbit space $\mathcal{M}/\mathrm{Diff}_0$ (we write the quotient on the right even though this is a left action). Local invariants of pseudo-Riemannian metrics can be thought of as functions on $\mathcal{M}$ which are invariant under the action of $\mathrm{Diff}_0$ and equivariant under $O(h)$; such a function determines an assignment to each metric on an arbitrary manifold of a section of the associated bundle by evaluation at each point in local coordinates. The jet isomorphism theorem for pseudo-Riemannian geometry provides an $O(h)$-equivariant description of the space $\mathcal{M}/\mathrm{Diff}_0$ in terms of curvature tensors and their covariant derivatives.

**Definition 1.1.** The space $\mathcal{R} \subset \prod_{r=0}^\infty \bigwedge^2 \mathbb{R}^{n*} \otimes \bigwedge^2 \mathbb{R}^{n*} \otimes \bigotimes^r \mathbb{R}^{n*}$ is the set of lists $(R^{(0)}, R^{(1)}, R^{(2)}, \dots)$ with $R^{(r)} \in \bigwedge^2 \mathbb{R}^{n*} \otimes \bigwedge^2 \mathbb{R}^{n*} \otimes \bigotimes^r \mathbb{R}^{n*}$, such that:

(1) $R_{i[jkl],m_1\dots m_r} = 0$
(2) $R_{ij[kl,m_1]m_2\dots m_r} = 0$
(3) $R_{ijkl,m_1\dots[m_{s-1}m_s]\dots m_r} = Q^{(s)}_{ijklm_1\dots m_r}(R)$.

Here the comma after the first four indices is just a marker separating these indices. $Q^{(s)}_{ijklm_1\cdots m_r}(R)$ denotes the quadratic expression in the $R^{(r')}$ with $r' \leq r-2$ which one obtains by covariantly differentiating the usual Ricci identity for commuting covariant derivatives, expanding the differentiations using the Leibnitz rule, and then setting equal to $h$ the metric which contracts the two factors in each term. We have suppressed the ${}^{(r)}$ on the $R^{(r)}$ since the value of $r$ is evident from the list of indices.

The action of $O(h)$ on $\mathbb{R}^n$ induces actions on the spaces of tensors in the usual way and therefore also on $\prod_{r=0}^\infty \bigwedge^2 \mathbb{R}^{n*} \otimes \bigwedge^2 \mathbb{R}^{n*} \otimes \bigotimes^r \mathbb{R}^{n*}$. Since $\mathcal{R}$ is an $O(h)$-invariant subset of this product, $\mathcal{R}$ has a natural $O(h)$ action.

Evaluation of the covariant derivatives of curvature of a metric at the origin induces a polynomial map $\mathcal{M} \to \mathcal{R}$. Since the covariant derivatives of curvature are tensors, it follows that this map passes to the quotient, and so defines a map $\mathcal{M}/\mathrm{Diff}_0 \to \mathcal{R}$ which is $O(h)$-equivariant. The pseudo-Riemannian jet isomorphism theorem is then the following.

**Theorem 1.2.** *The map $\mathcal{M}/\mathrm{Diff}_0 \to \mathcal{R}$ is an $O(h)$-equivariant bijection with polynomial inverse.*

The proof proceeds via the introduction of geodesic normal coordinates. These provide a slice for the action of $\mathrm{Diff}_0$ on $\mathcal{M}$: each orbit in $\mathcal{M}/\mathrm{Diff}_0$ is represented by a unique jet of a metric for which the background coordinates on $\mathbb{R}^n$ are geodesic normal coordinates to infinite order at the origin. A linearization argument reduces the theorem to showing that the linearized map restricted to infinitesimal jets of metrics in normal coordinates is a vector space isomorphism. The linearized map can be explicitly identified as the direct sum over $r$ of intertwining maps between two equivalent realizations corresponding to different Young projectors of irreducible representations of $GL(n, \mathbb{R})$. See [7] for the analysis of the linearized map in a similar context.

This jet isomorphism theorem is fundamental in consideration of local pseudo-Riemannian invariants. It shows that such invariants correspond exactly to $O(h)$-invariants of $\mathcal{R}$. Weyl's classical invariant theory for $O(h)$ completely describes such invariants.

Next we pass to the conformal analogue. We begin with a discussion of the conformal group and its parabolic subgroup $P$ which plays the role in conformal geometry of the group $O(h)$ in pseudo-Riemannian geometry. In the conformal case we assume that $n \geq 3$.

Define a quadratic form $\widetilde{h}$ of signature $(p+1, q+1)$ on $\mathbb{R}^{n+2}$ by

$$\widetilde{h}_{IJ} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & h_{ij} & 0 \\ 1 & 0 & 0 \end{pmatrix} .$$

On $\mathbb{R}^{n+2}$ we use as coordinates $x^I = (x^0, x^i, x^\infty)$. The null cone of $\widetilde{h}$ is

$$\mathcal{N} = \{x \in \mathbb{R}^{n+2} \setminus \{0\} : \widetilde{h}_{IJ} x^I x^J = 0\} ,$$

whose projectivization is the quadric

$$\mathcal{Q} = \{[x] \in \mathbb{P}^{n+1} : x \in \mathcal{N}\} ,$$

with projection $\pi : \mathcal{N} \to \mathcal{Q}$. If $x \in \mathcal{N}$, the metric $\widetilde{h}_{IJ} dx^I dx^J$ on $\mathbb{R}^{n+2}$ is degenerate when restricted to $T_x\mathcal{N}$: $\widetilde{h}(X, Y) = 0$ for all $Y \in T_x\mathcal{N}$, where $X = x^I \partial_I$ is the Euler field. Consequently, $\widetilde{h}|_{T_x\mathcal{N}}$ determines a nondegenerate quadratic form on $T_x\mathcal{N}/\operatorname{span} X \cong T_{\pi(x)}\mathcal{Q}$. As $x$ varies over a line in $\mathcal{N}$, the resulting quadratic forms on $T_{\pi(x)}\mathcal{Q}$ define a metric up to scale, i.e. a conformal class of metrics on $\mathcal{Q}$ of signature $(p, q)$.

The conformal group is $G = O(\widetilde{h})/\{\pm I\}$. The linear action of $O(\widetilde{h})$ on $\mathbb{R}^{n+2}$ preserves $\mathcal{N}$, so there is an induced action of $G$ on $\mathcal{Q}$ which is transitive. Since $O(\widetilde{h})$ acts by isometries of the metric $\widetilde{h}_{IJ} dx^I dx^J$ on $\mathbb{R}^{n+2}$, the induced action of $G$ on $\mathcal{Q}$ is by conformal transformations. Define the subgroup $P \subset G$ to be the isotropy group of $[e_0] \in \mathcal{Q}$, so that $\mathcal{Q} = G/P$. Then $P$ can be identified with the subgroup $P = \{p \in O(\widetilde{h}) : pe_0 = ae_0,\ a > 0\}$. The first column of $p$ is $ae_0$;

combining this with the fact that $p \in O(\widetilde{h})$, one finds that $p \in P$ is of the form

$$p = \begin{pmatrix} a & b_j & c \\ 0 & m^i{}_j & d^i \\ 0 & 0 & a^{-1} \end{pmatrix}, \tag{1.1}$$

where

$$a > 0,\ b_j \in \mathbb{R}^{n*},\ m^i{}_j \in O(h) \quad \text{and} \quad c = -\frac{1}{2a} b_j b^j,\ d^i = -\frac{1}{a} m^{ij} b_j\,.$$

Lower case indices are raised and lowered using $h$. The parameters $a > 0$, $b_j \in \mathbb{R}^{n*}$ and $m^i{}_j \in O(h)$ are free, so that $P$ can be written as the product of its subgroups:

$$P = \mathbb{R}_+ \cdot \mathbb{R}^n \cdot O(h)\,.$$

Since $\widetilde{h}_{IJ} x^I x^J = 2x^0 x^\infty + |x|^2$, where $|x|^2 = h_{ij} x^i x^j$, the intersection of $\mathcal{Q}$ with the cell $\{[x^I] : x^0 \neq 0\}$ can be identified with $\mathbb{R}^n$ via the inclusion $i : \mathbb{R}^n \to \mathcal{Q}$ defined by

$$i(x) = \begin{bmatrix} 1 \\ x \\ -\frac{1}{2}|x|^2 \end{bmatrix}.$$

In this identification, the conformal structure on $\mathcal{Q}$ is represented by the flat metric $h_{ij} dx^i dx^j$ on $\mathbb{R}^n$. For $p \in P$, we will denote by $\varphi_p$ the corresponding conformal transformation on $\mathbb{R}^n$, and by $\Omega_p$ the conformal factor, so that

$$\varphi_p^* h = \Omega_p^2\, h\,.$$

These are given explicitly by:

$$\big(\varphi_p(x)\big)^i = \frac{m^i{}_j x^j - \frac{1}{2}|x|^2 d^i}{a + b_j x^j - \frac{1}{2} c |x|^2}\,, \qquad \Omega_p = \big(a + b_j x^j - \tfrac{1}{2} c |x|^2\big)^{-1}.$$

Observe that

$$\varphi_p'(0) = a^{-1} m^i{}_j, \qquad \Omega_p(0) = a^{-1}\,, \qquad \Omega_p'(0) = -a^{-2} b_j\,. \tag{1.2}$$

We now consider changing the metric by rescaling as well as by diffeomorphism. Set

$$C_+^\infty = \{\text{Jets at } 0 \in \mathbb{R}^n \text{ of smooth positive functions } \Omega\}\,.$$

Then $C_+^\infty$ is a group under multiplication. Consider the semidirect product group $\mathrm{Diff} \ltimes C_+^\infty$, where the product is defined so that $g \to (\varphi^{-1})^*(\Omega^2 g)$ is an action. This product is given explicitly by:

$$(\varphi_1, \Omega_1) \cdot (\varphi_2, \Omega_2) = \big(\varphi_1 \circ \varphi_2, (\Omega_1 \circ \varphi_2)\, \Omega_2\big)\,.$$

As before, we need to preserve the normalization $g_{ij}(0) = h_{ij}$. So we define the subgroup $\mathrm{CDiff} \subset \mathrm{Diff} \ltimes C_+^\infty$ by

$$\mathrm{CDiff} = \{(\varphi, \Omega) : (\Omega^{-1} \varphi')(0) \in O(h)\}$$

and we set

$$\mathrm{CDiff}_0 = \{(\varphi, \Omega) : \varphi'(0) = I\,, \quad \Omega(0) = 1\,, \quad d\Omega(0) = 0\} \subset \mathrm{CDiff}\,.$$

Then CDiff acts on $\mathcal{M}$ by: $(\varphi, \Omega).g = (\varphi^{-1})^*(\Omega^2 g)$. We can view $P \subset \mathrm{CDiff}$ by $p \mapsto (\varphi_p, \Omega_p)$. Then $P$ is the isotropy group of the flat metric $h \in \mathcal{M}$ under the CDiff action.

If $(\varphi, \Omega) \in \mathrm{CDiff}$, (1.2) shows that there is a unique $p \in P$ such that to second order we have $\varphi = \varphi_p$ and $\Omega = \Omega_p$. This defines a homomorphism $\mathrm{CDiff} \to P$ with kernel $\mathrm{CDiff}_0$. Thus $\mathrm{CDiff}_0$ is a normal subgroup of CDiff and $\mathrm{CDiff} = P \cdot \mathrm{CDiff}_0$. Moreover, $\mathcal{M}/\mathrm{CDiff}_0$ has a natural left $P$-action.

Just as in the pseudo-Riemannian case, local conformal invariants correspond precisely to $P$-invariants of $\mathcal{M}/\mathrm{CDiff}_0$. The conformal jet isomorphism theorem provides a tensorial description of $\mathcal{M}/\mathrm{CDiff}_0$. Since $P \subset GL(n+2, \mathbb{R})$, $P$ acts on tensor powers of $\mathbb{R}^{n+2}$, not $\mathbb{R}^n$. Thus one anticipates a description as a $P$-space in terms of tensors in $n+2$ dimensions.

A significant difference from the pseudo-Riemannian case which will appear below is the fact that the structure of $\mathcal{M}/\mathrm{CDiff}_0$ depends in a fundamental way on whether $n$ is even or odd. This is not evident at a superficial level. The tangent space $T(\mathcal{M}/\mathrm{CDiff}_0)$ is isomorphic to the quotient of a particular dual generalized Verma module, the jets of trace-free symmetric 2-tensors of weight 2, by the image of the conformal Killing operator acting on jets of vector fields. (See Lemma 2.3 below.) These spaces have natural structures as $(\mathfrak{g}, P)$-modules, where $\mathfrak{g}$ denotes the Lie algebra of $G$. As a $(\mathfrak{g}, P)$-module, this quotient is irreducible if $n$ is odd, but has a unique proper $(\mathfrak{g}, P)$-submodule with irreducible quotient if $n$ is even. Geometrically, the distinction is exhibited by the existence of the ambient obstruction tensor, a conformally invariant natural tensor which exists only in even dimensions.

Next we formulate the jet isomorphism theorem for conformal geometry for $n$ odd.

**Definition 1.3.** Let $n \geq 3$ be odd. The space $\widetilde{\mathcal{R}} \subset \prod_{r=0}^{\infty} \bigwedge^2 \mathbb{R}^{n+2*} \otimes \bigwedge^2 \mathbb{R}^{n+2*} \otimes \bigotimes^r \mathbb{R}^{n+2*}$ is the set of lists $\big(\widetilde{R}^{(0)}, \widetilde{R}^{(1)}, \widetilde{R}^{(2)}, \dots\big)$ with $\widetilde{R}^{(r)} \in \bigwedge^2 \mathbb{R}^{n+2*} \otimes \bigwedge^2 \mathbb{R}^{n+2*} \otimes \bigotimes^r \mathbb{R}^{n+2*}$, such that:

(1) $\widetilde{R}_{I[JKL],M_1\dots M_r} = 0$

(2) $\widetilde{R}_{IJ[KL,M_1]M_2\dots M_r} = 0$

(3) $\widetilde{R}_{IJKL,M_1\dots[M_{s-1}M_s]\dots M_r} = \widetilde{Q}^{(s)}_{IJKLM_1\dots M_r}(\widetilde{R})$

(4) $\widetilde{h}^{IK}\widetilde{R}_{IJKL,M_1\dots M_r} = 0$

(5) $\widetilde{R}_{IJK0,M_1\dots M_r} = -\sum_{s=1}^{r} \widetilde{R}_{IJKM_s,M_1\dots\widehat{M_s}\dots M_r}$.

Here $\widetilde{Q}^{(s)}_{IJKLM_1\dots M_r}(\widetilde{R})$ is the same quadratic expression in the $\widetilde{R}^{(r')}$, $r' \leq r-2$, as in Definition 1.1, except that now the tensors are the $\widetilde{R}^{(r')}$ instead of the $R^{(r')}$ and the contractions are taken with respect to $\widetilde{h}$ instead of $h$. Condition (5) in case $r = 0$ is interpreted as $\widetilde{R}_{IJK0} = 0$.

Conditions (1)–(4) are invariant under all of $O(\widetilde{h})$. Condition (5) is certainly not invariant under $O(\widetilde{h})$, but it is almost invariant under $P$. Recall that $p \in P$ given by (1.1) satisfies $pe_0 = ae_0$, $a > 0$. So (5) is invariant under $P$ modulo the rescaling of $e_0$. To correct the scaling, for $w \in \mathbb{C}$ define the character $\sigma_w \colon P \to \mathbb{C}$

by $\sigma_w(p) = a^{-w}$. Then if we define the $P$ action by viewing

$$\widetilde{\mathcal{R}} \subset \prod_{r=0}^{\infty} \bigwedge^2 \mathbb{R}^{n+2*} \otimes \bigwedge^2 \mathbb{R}^{n+2*} \otimes \bigotimes^r \mathbb{R}^{n+2*} \otimes \sigma_{-2-r}\,, \tag{1.3}$$

then the scaling of the $\widetilde{R}^{(r)}$ cancels the scaling of $e_0$ and condition (5) becomes invariant under $P$. Thus $\widetilde{\mathcal{R}}$ becomes a $P$-space. (In the factor $\sigma_{-2-r}$, the $-2$ could be replaced by any other number and condition (5) would still be $P$-invariant. The choice of $-2$ is necessary for the map $c$ below to be $P$-equivariant.) The conformal jet isomorphism theorem for $n$ odd is then the following.

**Theorem 1.4.** *If $n$ is odd, then there is a $P$-equivariant polynomial bijection $c\colon \mathcal{M}/\mathrm{CDiff}_0 \to \widetilde{\mathcal{R}}$ with polynomial inverse.*

If $n$ is even, the analogous statement holds only for truncated jets: there is a bijection from $(n-1)$-jets of metrics mod $\mathrm{CDiff}_0$ to a correspondingly truncated version of the space $\widetilde{\mathcal{R}}$. An infinite order extension of this result for $n$ even will be discussed in §3.

The jet isomorphism theorem reduces the study of conformal invariants to the study of $P$-invariants of $\widetilde{\mathcal{R}}$. This is important because algebraic tensorial operations can be utilized to construct and study conformal invariants.

Next we discuss the origin of the space $\widetilde{\mathcal{R}}$ and the construction of the map $c$. As described above, the conformal geometry of the quadric $\mathcal{Q}$ naturally arises from the metric $\widetilde{h}_{IJ}dx^I dx^J$ on $\mathbb{R}^{n+2}$. In [9], a version of the metric $\widetilde{h}$ for a general conformal manifold, called the ambient metric, was introduced. The tensors $\widetilde{R}^{(r)}$ arise as the iterated covariant derivatives of the curvature tensor of the ambient metric.

Suppose that $M$ is a smooth manifold with a conformal class $[g]$ of metrics of signature $(p,q)$. The metric bundle $\mathcal{G}$ of $[g]$ is $\mathcal{G} = \{(x, t^2 g(x)) : x \in M,\, t > 0\} \subset \bigodot^2 T^*M$, where $g$ is a metric in the conformal class. The fiber variable $t$ on $\mathcal{G}$ is associated to the metric $g$ and provides an identification $\mathcal{G} \cong \mathbb{R}_+ \times M$. There is a tautological symmetric 2-tensor $\mathbf{g_0}$ on $\mathcal{G}$ defined by $\mathbf{g_0}(Y,Z) = \underline{g}(\pi_* Y, \pi_* Z)$, where $\pi\colon \mathcal{G} \to M$ is the projection and $Y$, $Z$ are tangent vectors to $\mathcal{G}$ at $(x,\underline{g}) \in \mathcal{G}$. The family of dilations $\delta_s\colon \mathcal{G} \to \mathcal{G}$ defined by $\delta_s(x,\underline{g}) = (x, s^2\underline{g})$ defines an $\mathbb{R}_+$ action on $\mathcal{G}$, and one has $\delta_s^*\mathbf{g_0} = s^2\mathbf{g_0}$. We denote by $T = \frac{d}{ds}\delta_s|_{s=1}$ the vector field on $\mathcal{G}$ which is the infinitesimal generator of the dilations $\delta_s$. Note that $\mathbf{g_0}$ is degenerate: $\mathbf{g}_0(T,Y) = 0$ for all $Y \in T\mathcal{G}$. In the case that $(M,[g])$ is the quadric $\mathcal{Q}$ with its conformal structure defined above, $\mathcal{G}$ can be identified with $\mathcal{N}/\{\pm I\}$, $\mathbf{g_0}$ with $\widetilde{h}|_{T\mathcal{N}}$, and $T$ with $X$.

The ambient space is defined to be $\widetilde{\mathcal{G}} = \mathcal{G} \times \mathbb{R}$; the coordinate in the $\mathbb{R}$ factor is typically written $\rho$. The dilations $\delta_s$ extend to $\widetilde{\mathcal{G}}$ acting on the $\mathcal{G}$ factor and we denote also by $T$ the infinitesimal generator of the $\delta_s$ on $\widetilde{\mathcal{G}}$. We embed $\mathcal{G}$ into $\widetilde{\mathcal{G}}$ by $\iota\colon z \to (z,0)$ for $z \in \mathcal{G}$, and we identify $\mathcal{G}$ with its image under $\iota$. We say that a subset of $\widetilde{\mathcal{G}}$ is homogeneous if it is invariant under the $\delta_s$ for all $s > 0$. We say that a map between homogeneous subsets of $\widetilde{\mathcal{G}}$ is homogeneous if it commutes with the $\delta_s$.

**Definition 1.5.** Let $n$ be odd. An ambient metric $\widetilde{g}$ for $(M,[g])$ is a smooth metric of signature $(p+1,q+1)$ on a homogeneous neighborhood of $\mathcal{G}$ in $\mathcal{G}\times\mathbb{R}$ satisfying:

(1) $\delta_s^*\widetilde{g}=s^2\widetilde{g}$
(2) $\iota^*\widetilde{g}=\mathbf{g_0}$
(3) $\operatorname{Ric}(\widetilde{g})=0$ to infinite order along $\mathcal{G}$.

The main result concerning existence and uniqueness of the ambient metric for $n$ odd is:

**Theorem 1.6.** *If $n$ is odd, there exists an ambient metric for $(M,[g])$. It is unique up to:*

(a) *Pullback by a homogeneous diffeomorphism $\Phi$ satisfying $\Phi|_{\mathcal{G}}=I$, and*
(b) *Addition of a tensor homogeneous of degree 2 which vanishes to infinite order along $\mathcal{G}$.*

The proof proceeds by the introduction of a gauge normalization to break the diffeomorphism invariance together with a formal power series analysis of the equations $\operatorname{Ric}(\widetilde{g})=0$. See [10].

When $n\geq 4$ is even, there is an obstruction at order $n/2$ to existence of a metric satisfying (1)–(3), which is a conformally invariant natural tensor called the ambient obstruction tensor. However, there is a solution up to this order, again unique up to homogeneous diffeomorphism and up to a term vanishing to order $n/2$.

The solution has an extra geometric property: for each $p\in\widetilde{\mathcal{G}}$, the parametrized dilation orbit $s\to\delta_s p$ is a geodesic for $\widetilde{g}$ (to infinite order along $\mathcal{G}$).

The diffeomorphism ambiguity in $\widetilde{g}$ can be fixed by the choice of a metric $g$ in the conformal class. As described above, the choice of such a metric determines an identification $\mathcal{G}\cong\mathbb{R}_+\times M$, and therefore an identification $\widetilde{\mathcal{G}}\cong\mathbb{R}_+\times M\times\mathbb{R}$.

**Definition 1.7.** A metric $\widetilde{g}$ satisfying conditions (1) and (2) in Definition 1.5 is said to be in normal form relative to $g$ if in the identification $\widetilde{\mathcal{G}}\cong\mathbb{R}_+\times M\times\mathbb{R}$ induced by $g$, one has

(1) $\widetilde{g}=2t\,dt\,d\rho+\mathbf{g_0}$ at $\rho=0$, and
(2) The curve $\rho\to(t,x,\rho)$ is a geodesic for $\widetilde{g}$ for each choice of $(t,x)\in\mathbb{R}_+\times M$.

If $n$ is odd, an ambient metric $\widetilde{g}$ can always be found which is in normal form relative to $g$, and it is uniquely determined up to $O(\rho^\infty)$. Each term in the Taylor expansion at $\rho=0$ of such a $\widetilde{g}$ in normal form relative to $g$ is given by a polynomial expression in $g^{-1}$ and in derivatives of $g$.

An analogue in conformal geometry of the curvature tensor and its covariant derivatives for a pseudo-Riemannian metric are the restrictions to $\mathcal{G}$ of the curvature tensor and its covariant derivatives of the ambient metric. These can be interpreted as sections of weighted tensor powers of the cotractor bundle associated to the conformal structure; see [4], [2] and [10]. For our purposes, the map

$c$ in Theorem 1.4 can be defined directly as follows. For $g \in \mathcal{M}$, choose a metric also denoted $g$ defined near $0 \in \mathbb{R}^n$ with the prescribed Taylor expansion. There is an ambient metric $\widetilde{g}$ in normal form relative to $g$, uniquely determined to infinite order in $\rho$. Define the tensors $\widetilde{R}^{(r)}$ in Theorem 1.4 to be the iterated covariant derivatives of the curvature tensor of $\widetilde{g}$ evaluated at $t = 1$, $x = 0$ and $\rho = 0$. It can be shown that these covariant derivatives satisfy the relations (1)–(5) in Definition 1.3 which define $\widetilde{\mathcal{R}}$. Relations (1)–(3) hold for the covariant derivatives of curvature of any metric. Relation (4) follows from the fact that $\widetilde{g}$ is Ricci-flat to infinite order, and (5) is a consequence of the homogeneity of $\widetilde{g}$ and the fact that the dilation orbits are geodesics to infinite order. Now using the fact that the ambient curvature tensors are tensors on the ambient space, it can be shown that this map $\mathcal{M} \to \widetilde{\mathcal{R}}$ passes to a map $c\colon \mathcal{M}/\mathrm{CDiff}_0 \to \widetilde{\mathcal{R}}$ which is $P$-equivariant. Details can be found in [10].

The invertibility of $c$ in Theorem 1.4 is also proved in [10]. As in the pseudo-Riemannian case, one first constructs a slice for the $\mathrm{CDiff}_0$ action, using geodesic normal coordinates and a "conformal normal form" which normalizes away the freedom of the derivatives of the conformal factor of order two or more. (Actually, the formulation of the jet isomorphism theorem in [10] is in terms of this slice rather than in terms of the space $\mathcal{M}/\mathrm{CDiff}_0$.) A linearization argument reduces the matter to showing the invertibility of the linearization $dc$ of $c$ at the flat metric $h$. Then the main part of the proof consists of an algebraic study of the relations obtained by linearizing (1)–(5) (they are all already linear except for (4)) and a direct analysis of $dc$. A more conceptual proof of the invertibility of the linearized map will be outlined in the next section as an application of the results on the ambient realization of the deformation complex.

## 2. Ambient lift of deformation complex

In this section we introduce the conformal deformation complex and indicate how it may be realized ambiently in odd dimensions. We then sketch a proof of the invertibility of the map $c$ in Theorem 1.4 using the exactness of the deformation complex on jets together with this ambient realization. Details will appear in [15].

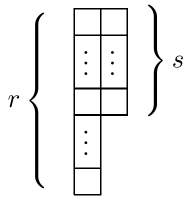
Recall from the previous section that the conformal group $G = O(\widetilde{h})/\{\pm I\}$ acts conformally on the quadric $\mathcal{Q}$ with isotropy group $P$ so that $\mathcal{Q} = G/P$, and that there is an embedding $i\colon \mathbb{R}^n \to \mathcal{Q}$ as an open dense subset on which the conformal structure is represented by the flat metric $h$. To each finite-dimensional representation of $P$ is associated a homogeneous vector bundle on $\mathcal{Q} = G/P$ and therefore also on $\mathbb{R}^n \hookrightarrow \mathcal{Q}$. Familiar examples include:

- $\mathcal{D}_w$, $w \in \mathbb{C}$: the bundle of conformal densities of weight $w$, induced by $\sigma_w$
- $T\mathcal{Q}$: the tangent bundle, induced by $p \mapsto a^{-1}m$
- $\bigwedge^r$, $0 \le r \le n$: the bundle of $r$-forms (the $r$-th exterior power of the cotangent bundle).

Set $\bigwedge^r(w) = \bigwedge^r \otimes \mathcal{D}_w$. We denote by $\mathcal{E}(w)$, $\mathcal{E}^r(w)$ the sheaf of germs of smooth sections of $\mathcal{D}_w$, $\bigwedge^r(w)$, resp., and by $\mathcal{E}_{\mathcal{U}}(w)$, $\mathcal{E}^r_{\mathcal{U}}(w)$ the space of sections on an open set $\mathcal{U} \subset \mathcal{Q}$.

For $0 \leq s \leq r \leq n$, define $\bigwedge^{r,s}$ to be the homogeneous bundle on $\mathcal{Q}$ of covariant tensors with Young symmetry given by the Young diagram

(2.1)



Explicitly, $\bigwedge^{r,s}$ is the subbundle of $\bigwedge^r \otimes \bigwedge^s$ consisting of those tensors

$$f_{i_1\ldots i_r j_1\ldots j_s} = f_{[i_1\ldots i_r][j_1\ldots j_s]} \in \bigwedge^r \otimes \bigwedge^s$$

which satisfy

$$f_{[i_1\ldots i_r j_1]j_2\ldots j_s} = 0\,.$$

Note that $\bigwedge^{r,0} = \bigwedge^r$ and that $\bigwedge^{1,1} = \bigodot^2$ is the bundle of symmetric 2-tensors. We denote by $\bigwedge^{r,s}_0 \subset \bigwedge^{r,s}$ the subbundle of tensors which are trace-free with respect to a metric in the conformal class, by $\bigwedge^{r,s}(w)$, $\bigwedge^{r,s}_0(w)$ the respective tensor products with $\mathcal{D}_w$, and by $\mathcal{E}^{r,s}(w)$, $\mathcal{E}^{r,s}_0(w)$ the sheaves of germs of sections. Each of the bundles $\bigwedge^{r,s}_0(w)$ is an irreducible homogeneous bundle; i.e., it is induced by an irreducible representation of $P$.

We will represent sections of $\bigwedge^{r,s}(w)$ in either of two ways. On $\mathbb{R}^n \hookrightarrow \mathcal{Q}$, we can use $h$ to trivialize the density bundle and can thereby identify a section with a tensor field $u$ on an open subset of $\mathbb{R}^n$ having the symmetries indicated above. Alternately, we can view a section as a homogeneous tensor field $f$ on an open subset of $\mathcal{N}$. Define the dilations $\delta_\lambda\colon \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$ by $\delta_\lambda(x) = \lambda x$ for $\lambda \in \mathbb{R}\setminus\{0\}$. Then for $\mathcal{U} \subset \mathcal{Q}$ open, there is a 1-1 correspondence between $\mathcal{E}^{r,s}_{\mathcal{U}}(w)$ and the set of smooth sections $f$ of $\bigotimes^{r+s} T^*\mathcal{N}$ on $\pi^{-1}(\mathcal{U})$ which have the symmetries above and which satisfy

$$\delta^*_\lambda f = |\lambda|^w f\,, \qquad X \lrcorner f = 0\,. \tag{2.2}$$

Here the condition $X \lrcorner f = 0$ is interpreted to mean that the contraction of $X$ into every index of $f$ vanishes.

We now work on $\mathbb{R}^n$, viewed as a subset of $\mathcal{Q}$, and use its usual coordinates and the flat metric $h$. Define differential operators

$$\begin{aligned}
d_1&\colon \mathcal{E}^{r,s} \to \mathcal{E}^{r+1,s}\\
d_2&\colon \mathcal{E}^{r,s} \to \mathcal{E}\left(\textstyle\bigwedge^r \otimes \bigwedge^{s+1}\right)\\
\delta_1&\colon \mathcal{E}^{r,s} \to \mathcal{E}\left(\textstyle\bigwedge^{r-1} \otimes \bigwedge^{s}\right)\\
\delta_2&\colon \mathcal{E}^{r,s} \to \mathcal{E}^{r,s-1}
\end{aligned} \tag{2.3}$$

by:

$$\begin{aligned}(d_1u)_{i_0i_1\dots i_rj_1\dots j_s} &= \partial_{[i_0}u_{i_1\dots i_r]j_1\dots j_s}\\ (d_2u)_{i_1\dots i_rj_0\dots j_s} &= \partial_{[j_0}u_{|i_1\dots i_r|j_1\dots j_s]}\\ (\delta_1u)_{i_1\dots i_{r-1}j_1\dots j_s} &= -\partial^k u_{i_1\dots i_{r-1}kj_1\dots j_s}\\ (\delta_2u)_{i_1\dots i_rj_1\dots j_{s-1}} &= -\partial^k u_{i_1\dots i_rj_1\dots j_{s-1}k}\,.\end{aligned}$$

Here the $|i_1 \dots i_r|$ indicates indices excluded from the skew-symmetrization. The derivatives are coordinate derivatives on $\mathbb{R}^n$ and the contractions are with respect to $h$. In making this definition, we momentarily ignore the weights and the structure as homogeneous bundles and view these simply as differential operators on tensor fields.

For $n \geq 4$, the deformation complex on $\mathbb{R}^n$ is:

$$\begin{aligned}&0 \to \mathfrak{g} \to \mathcal{E}^1(2) \xrightarrow{D_0} \mathcal{E}_0^{1,1}(2) \xrightarrow{D_1} \mathcal{E}_0^{2,2}(2) \xrightarrow{D_2} \mathcal{E}_0^{3,2}(2)\\ &\qquad\to \cdots \to \mathcal{E}_0^{n-2,2}(2) \xrightarrow{D_{n-2}} \mathcal{E}_0^{n-1,1} \xrightarrow{D_{n-1}} \mathcal{E}^{n-1}(-2) \to 0\,,\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}D_0 &= \operatorname{tf}\operatorname{Sym} d_2\\ D_1 &= \operatorname{tf} d_1d_2\\ D_r &= \operatorname{tf} d_1 \quad r = 2,3,\dots,n-3\\ D_{n-2} &= \delta_2 d_1\\ D_{n-1} &= \delta_2\,.\end{aligned}$$

Here tf denotes the trace-free part with respect to $h$ and Sym denotes symmetrization over the two indices. When $n = 4$, the $\mathcal{E}_0^{2,2}(2)$ on the first line and the $\mathcal{E}_0^{n-2,2}(2)$ on the second line are the same space, so $D_2 = \delta_2 d_1$ maps into $\mathcal{E}_0^{3,1}$ and the space $\mathcal{E}_0^{3,2}(2)$ does not occur. In higher dimensions, the spaces between $\mathcal{E}_0^{2,2}(2)$ and $\mathcal{E}_0^{n-1,1}$ are the $\mathcal{E}_0^{r,2}(2)$ for $3 \leq r \leq n-2$. The $D_r$ are the expressions on $\mathbb{R}^n$ of $G$-equivariant differential operators between the indicated homogeneous vector bundles on $G/P$, or equivalently between the sheaves of their germs of local sections. The space $\mathfrak{g}$ is the locally constant sheaf. The bundle $\bigwedge^1(2)$ is isomorphic to the tangent bundle by raising the index, and in this realization the map $\mathfrak{g} \to \mathcal{E}^1(2)$ is the infinitesimal $G$-action.

The deformation complex is a complex, i.e. the composition of two successive operators vanishes. It can be thought of as analogous to the deRham complex; it has the same length as the deRham complex. The operators $D_1$ and $D_{n-2}$ are second order; all other $D_r$ are first order. The deformation complex was constructed explicitly "by hand" by Gasqui-Goldschmidt in [11] on a general conformally flat manifold. In the homogeneous case it is a particular case of a generalized Bernstein-Gelfand-Gelfand complex (see [17] for the introduction of gBGG complexes in the algebraic setting). In the 3-dimensional case the deformation complex takes a special form:

$$0 \to \mathfrak{g} \to \mathcal{E}^1(2) \xrightarrow{D_0} \mathcal{E}_0^{1,1}(2) \xrightarrow{D_1} \mathcal{E}_0^{2,1} \xrightarrow{D_2} \mathcal{E}^2(-2) \to 0, \tag{2.5}$$

where $D_0$ is as above, $D_2 = \delta_2$, and $D_1 = \operatorname{tf} \delta_2 d_1 d_2$ is third order.

The main fact that we will need about the deformation complex is that it is exact on jets; i.e., if an infinite-order jet of a section of one of the bundles at a point is annihilated by the corresponding operator as a jet, then it is in the image of the previous operator acting on jets at that point. This fact is proved in [11] and is also contained in the theory of the generalized BGG complexes.

This complex is called the deformation complex because its first terms describe the infinitesimal deformation of conformal structures. The first operator $D_0$ corresponds to the conformal Killing operator $\operatorname{tf} \mathcal{L}_V h$, where $\mathcal{L}$ denotes the Lie derivative and $V$ a vector field, which is obtained by linearizing the action of diffeomorphisms on conformal structures. Its kernel $\mathfrak{g}$ consists of the infinitesimal conformal transformations. For $n \geq 4$, the operator $D_1$ is the linearization of the map which takes the Weyl tensor of a metric, and $D_2$ is the linearization of the Bianchi identity satisfied by such a Weyl tensor. For $n = 3$, $D_1$ is the linearization of the Cotton tensor, and $D_2$ the linearization of the "Bianchi identity" satisfied by a Cotton tensor of a metric.

We wish to give an alternate description of the deformation complex for $n$ odd in which the spaces and maps are defined on the ambient space. Other descriptions and curved versions are contained in [5] and [3] (in much greater generality), and in [13]. We begin by introducing the ambient versions of the spaces appearing in the complex.

For $0 \leq s \leq r$, denote by $\widetilde{\bigwedge}^{r,s}$ the vector bundle of tensors of rank $r + s$ on $\mathbb{R}^{n+2}$ having the Young symmetry (2.1) and by $\widetilde{\bigwedge}_0^{r,s}$ the subbundle of those tensors which are trace-free with respect to $\widetilde{h}$. We write $\widetilde{d}_1$, $\widetilde{d}_2$, $\widetilde{\delta}_1$, $\widetilde{\delta}_2$ for the operators on $\mathbb{R}^{n+2}$ analogous to (2.3) and $\widetilde{\Delta} = \widetilde{h}^{IJ}\partial^2_{IJ}$ for the Laplacian with respect to $\widetilde{h}$. $\widetilde{\Delta}$ acts on sections of $\widetilde{\bigwedge}^{r,s}$ componentwise with respect to the standard basis. Recall that $X = x^I \partial_I$ denotes the Euler field on $\mathbb{R}^{n+2}$, whose components are thus given by $X^I = x^I$.

Let $\pi\colon \mathbb{R}^{n+2} \setminus \{0\} \to \mathbb{P}^{n+1}$ be the projection. Let $0 \leq s \leq r$ and $w \in \mathbb{C}$. For $\mathcal{V} \subset \mathbb{P}^{n+1}$ open, define $\widetilde{\mathcal{E}}^{r,s}_{\mathcal{V}}(w)$ to be the space of sections $\widetilde{f}$ of $\widetilde{\bigwedge}^{r,s}$ on $\pi^{-1}(\mathcal{V})$ which satisfy $\delta^*_\lambda \widetilde{f} = |\lambda|^w \widetilde{f}$ for $\lambda \in \mathbb{R} \setminus \{0\}$, and $\widetilde{\mathcal{E}}^{r,s}_{0,\mathcal{V}}(w)$ to be the subspace of trace-free sections. The assignments $\mathcal{V} \to \widetilde{\mathcal{E}}^{r,s}_{\mathcal{V}}(w)$, $\widetilde{\mathcal{E}}^{r,s}_{0,\mathcal{V}}(w)$ define presheaves on $\mathbb{P}^{n+1}$ whose associated sheaves we denote by $\widetilde{\mathcal{E}}^{r,s}(w)$, $\widetilde{\mathcal{E}}^{r,s}_0(w)$, resp. Observe that pullback defines a natural action of $O(\widetilde{h})$ on the total space of these sheaves and $\pm I$ acts by the identity, so that $G = O(\widetilde{h})/\{\pm I\}$ also acts.

Recall that $\mathcal{N}$ is the null cone of $\widetilde{h}$ and that $\pi : \mathcal{N} \to \mathcal{Q}$. If $\mathcal{U} \subset \mathcal{Q}$ is open, define $\mathcal{H}^{r,s}_{\mathcal{U}}(w)$ to be the space of infinite-order jets along $\pi^{-1}(\mathcal{U})$ of sections $\widetilde{f} \in \widetilde{\mathcal{E}}^{r,s}_{0,\mathcal{V}}(w)$ for some $\mathcal{V} \subset \mathbb{P}^{n+1}$ open, $\mathcal{U} \subset \mathcal{V}$, which satisfy the following equations formally to infinite order along $\pi^{-1}(\mathcal{U})$:

$$\widetilde{\Delta}\widetilde{f} = 0\,, \qquad \widetilde{\delta}_1\widetilde{f} = 0\,, \qquad X \lrcorner \widetilde{f} = 0\,. \tag{2.6}$$

Here again $X \lrcorner \widetilde{f} = 0$ is interpreted to mean that the contraction of $X$ into any index of $\widetilde{f}$ vanishes. By the symmetries of $\widetilde{f}$, this is equivalent to $X^{I_1}\widetilde{f}_{I_1\cdots I_r J_1 \cdots J_s} = 0$.

Similarly, $\widetilde{\delta}_1\widetilde{f} = 0$ implies $\widetilde{\delta}_2\widetilde{f} = 0$. The assignment $\mathcal{U} \to \mathcal{H}^{r,s}_{\mathcal{U}}(w)$ defines a presheaf on $\mathcal{Q}$, whose associated sheaf we denote by $\mathcal{H}^{r,s}(w)$. Since the equations (2.6) are invariant under $O(\widetilde{h})$, the conformal group $G$ acts on $\mathcal{H}^{r,s}(w)$ covering the translations of $G$ on $\mathcal{Q} = G/P$. Thus $\mathcal{H}^{r,s}(w)$ is a "homogeneous sheaf" on $G/P$ in the same sense as in the definition of a homogeneous vector bundle.

The ambient realization for $\mathcal{E}^{r,s}_0(w)$ in the unobstructed cases is given by the following theorem.

**Theorem 2.1.** *Suppose $n \geq 3$. Let $0 \leq s \leq r \leq n$ and $w \in \mathbb{C}$.*

- *If $r > s = 0$, assume that $w \neq 2r - n$.*
- *If $s > 0$, assume that $w \neq r + 2s - n - 1$, $2r + s - n$.*

*If $w + n/2 - r - s \notin \mathbb{N}$, then*

$$\mathcal{E}^{r,s}_0(w) \cong \mathcal{H}^{r,s}(w)$$

*$G$-equivariantly as sheaves on $\mathcal{Q}$.*

The disallowed values correspond to the existence of certain particular $G$-invariant differential operators which act on $\mathcal{E}^{r,s}_0(w)$. In particular, if the dual generalized Verma module associated to $\mathcal{E}^{r,s}_0(w)$ is irreducible as a $(\mathfrak{g}, P)$-module, then Theorem 2.1 applies to $\mathcal{E}^{r,s}_0(w)$. It is important to note, as we will see, that not all $G$-invariant differential operators obstruct the isomorphism asserted by Theorem 2.1. Otherwise stated, Theorem 2.1 applies to many homogeneous bundles $\mathcal{E}^{r,s}_0(w)$ for which the associated dual Verma modules are not irreducible as $(\mathfrak{g}, P)$-modules.

The map in one direction in Theorem 2.1 is evident, and exists for all values of $w$, $r$ and $s$. For $\mathcal{U} \subset \mathcal{Q}$ open, view elements of $\mathcal{E}^{r,s}_{0,\mathcal{U}}(w)$ as covariant tensor fields $f$ on $\pi^{-1}(\mathcal{U}) \subset \mathcal{N}$ satisfying (2.2) as described above. Let $\iota\colon \mathcal{N} \to \mathbb{R}^{n+2}$ denote the inclusion. If $\widetilde{f} \in \widetilde{\mathcal{E}}^{r,s}_{0,\mathcal{V}}(w)$ for some $\mathcal{V} \subset \mathbb{P}^{n+1}$ open with $\mathcal{U} \subset \mathcal{V}$, and $\widetilde{f}$ satisfies $(X \lrcorner \widetilde{f})|_{\pi^{-1}(\mathcal{U})} = 0$, it is clear that $f = \iota^*\widetilde{f}$ satisfies (2.2). One checks easily that $f$ also satisfies the trace-free condition with respect to $h$ so that $f$ defines an element of $\mathcal{E}^{r,s}_{0,\mathcal{U}}(w)$. Passing to jets along $\pi^{-1}(\mathcal{U})$ and restricting consideration to $\mathcal{H}^{r,s}(w)$ gives a $G$-equivariant map

$$\iota^*\colon \mathcal{H}^{r,s}(w) \to \mathcal{E}^{r,s}_0(w)\,.$$

The content of Theorem 2.1 is that under the stated restrictions on the parameters, this map is a isomorphism. That is, each element of $\mathcal{E}^{r,s}_{0,\mathcal{U}}(w)$ has a unique extension (ambient lift) as an element of $\mathcal{H}^{r,s}_{\mathcal{U}}(w)$.

There are two main steps in the proof of Theorem 2.1. The first is called the "initial lift", and corresponds to defining on $\pi^{-1}(\mathcal{U})$ the components of $\widetilde{f}$ transverse to $\mathcal{N}$ to obtain a section of $\widetilde{\bigwedge}^{r,s}_0|_{\pi^{-1}(\mathcal{U})}$ homogeneous of degree $w$. The ideas in this step go back to Tracy Thomas for special cases of the symmetries including differential forms; he called the process of defining the transverse components "completeing" the tensor. This step is closely related to what are nowadays called differential splittings, about which there is a substantial literature. This first step leads to the excluded values of the parameters indicated in the bullets above.

The second step involves "harmonic" extension of the completed tensor to higher order off $\mathcal{N}$ in such a way as to make the equations (2.6) as well as the trace-free condition hold to infinite order. The condition $w + n/2 - r - s \notin \mathbb{N}$ arises in this second step. Part of the difficulty of the proof, especially for more complicated symmetries, is making sure that the steps can be carried out consistently so that all the required conditions hold to all orders.

To give an idea how this works, we sketch the details for the scalar case $r = s = 0$ and the case $r = 1$, $s = 0$ of 1-forms. The scalar case is studied in detail in [6]. I am grateful to M. Eastwood for providing the argument below in the case of 1-forms.

In the case $r = s = 0$, Theorem 2.1 asserts that $\mathcal{E}(w) \cong \mathcal{H}(w)$ if $w + n/2 \notin \mathbb{N}$, where $\mathcal{E}(w)$ denotes the sheaf of germs of densities of weight $w$ on $\mathcal{Q}$ and $\mathcal{H}(w)$ denotes the sheaf of jets along $\mathcal{N}$ of homogeneous functions of degree $w$ which satisfy $\widetilde{\Delta}\widetilde{f} = 0$ to infinite order. Set $Q = \widetilde{h}_{IJ}x^I x^J$; then this is the same as showing that given $f$ homogeneous of degree $w$ on $\pi^{-1}(\mathcal{U}) \subset \mathcal{N}$, there exists a unique infinite order jet $\widetilde{f}$ homogeneous of degree $w$ satisfying $\widetilde{\Delta}\widetilde{f} = O(Q^\infty)$ and $\widetilde{f}|_{\pi^{-1}(\mathcal{U})} = f$. The initial lift step is vacuous in this case. For the harmonic extension step, the Taylor expansion of $\widetilde{f}$ is constructed inductively. A key observation is that

$$[\widetilde{\Delta}, Q^k] = 2kQ^{k-1}(2X + n + 2k)\,. \tag{2.7}$$

Suppose that $\widetilde{f}^{(k)}$ has been constructed which satisfies $\widetilde{\Delta}\widetilde{f}^{(k)} = O(Q^{k-1})$. Set

$$\widetilde{f}^{(k+1)} = \widetilde{f}^{(k)} + Q^k\eta \quad \text{for} \quad \eta \in \widetilde{\mathcal{E}}(w - 2k)\,.$$

Then

$$\begin{aligned}
\widetilde{\Delta}\widetilde{f}^{(k+1)} &= \widetilde{\Delta}\widetilde{f}^{(k)} + \widetilde{\Delta}(Q^k\eta) \\
&= \widetilde{\Delta}\widetilde{f}^{(k)} + [\widetilde{\Delta}, Q^k]\eta + O(Q^k) \\
&= \widetilde{\Delta}\widetilde{f}^{(k)} + 2kQ^{k-1}(2X + n + 2k)\eta + O(Q^k) \\
&= \widetilde{\Delta}\widetilde{f}^{(k)} + 2k(n + 2w - 2k)\eta Q^{k-1} + O(Q^k)\,.
\end{aligned}$$

So if $n + 2w \neq 2k$, $\eta$ can be uniquely chosen so that $\widetilde{\Delta}\widetilde{f}^{(k+1)} = O(Q^k)$. Thus if $w + n/2 \notin \mathbb{N}$, then the induction can be carried out to all orders.

If $n/2 + w = m \in \mathbb{N}$, then harmonic extension is obstructed by the conformally invariant operator $\Delta^m = (h^{ij}\partial^2_{ij})^m$ on $\mathbb{R}^n$.

Consider now the case $r = 1$, $s = 0$. Theorem 2.1 asserts that if $w \neq 2 - n$ and $w + n/2 - 1 \notin \mathbb{N}$, then $\mathcal{E}^1(w) \cong \mathcal{H}^1(w)$, where we have written $\mathcal{H}^1(w)$ for $\mathcal{H}^{1,0}(w)$. Recall that $\widetilde{f} \in \mathcal{H}^1(w)$ means that $\widetilde{f}$ is a jet of a section of $\widetilde{\mathcal{E}}^1(w)$ satisfying the equations (2.6) to infinite order. We write $\widetilde{\delta}$ for $\widetilde{\delta}_1$ since $\widetilde{\delta}_2$ vanishes in this case.

Given a 1-form $f$ on $\pi^{-1}(\mathcal{U})$ which is homogeneous of degree $w$ and satisfies $f(X) = 0$, we can choose some $\widetilde{f}$ which is a section in $\widetilde{\mathcal{E}}^1_{\mathcal{V}}(w)$ for some $\mathcal{V} \supset \mathcal{U}$ such that $\iota^*\widetilde{f} = f$. Such an $\widetilde{f}$ is uniquely determined up to addition of $\psi dQ + Q\phi$ with $\psi$ a function and $\phi$ a 1-form, both of homogeneity $w - 2$. We can certainly choose $\widetilde{f}$ to start with so that $\widetilde{f}(X) = O(Q^2)$; in fact we could make $\widetilde{f}(X) = O(Q^\infty)$, but $O(Q^2)$ will suffice. Now try to determine $\psi$, $\phi$ to maintain this condition on vanishing of $\widetilde{f}(X)$:

$$(\widetilde{f}+\psi dQ+Q\phi)(X)=O(Q^2)$$

and also to make

$$\widetilde{\delta}(\widetilde{f}+\psi dQ+Q\phi)=O(Q)\,.$$

The first equation gives $\quad \psi dQ(X)+Q\phi(X)=O(Q^2), \quad$ so

$$2\psi+\phi(X)=O(Q)\,.$$

The second equation gives

$$\widetilde{\delta}\widetilde{f}-2(n+w)\psi-2\phi(X)=O(Q)\,,$$

so combining gives

$$\widetilde{\delta}\widetilde{f}-2(n+w-2)\psi=O(Q)\,.$$

If $n+w\neq 2$, this uniquely determines $\psi \mod Q$. Then $(\widetilde{f}+\psi dQ)|_{\pi^{-1}(\mathcal{U})}$ is the initial lift. Rename $\widetilde{f}+\psi dQ$ to be a new $\widetilde{f}$.

Now all components of $\widetilde{f}|_{\pi^{-1}(\mathcal{U})}$ have been determined. Write $\widetilde{f}=\widetilde{f}_I dx^I$; then each $\widetilde{f}_I$ is a scalar function homogeneous of degree $w-1$. Since $w+n/2-1\notin\mathbb{N}$, by the scalar case we can uniquely extend each $\widetilde{f}_I$ harmonically to infinite order, and this is equivalent to the condition that $\widetilde{f}$ satisfies $\widetilde{\Delta}\widetilde{f}=0$ to infinite order. In particular, we conclude the uniqueness of an extension satisfying (2.6) to infinite order. For existence, we claim that this harmonic extension automatically satisfies $\widetilde{\delta}\widetilde{f}=0$ and $\widetilde{f}(X)=0$ to infinite order. One first checks that the harmonic extension recovers the conditions $\widetilde{\delta}\widetilde{f}=O(Q)$ and $\widetilde{f}(X)=O(Q^2)$ imposed above. Then $\widetilde{\Delta}$ and $\widetilde{\delta}$ commute since they are constant coefficient operators on $\mathbb{R}^{n+2}$, so $\widetilde{\Delta}\widetilde{\delta}\widetilde{f}=0$ to infinite order. But $\widetilde{\delta}\widetilde{f}$ has homogeneity $w-2$ and $w-2+n/2\notin\mathbb{N}$, so uniqueness for the scalar case implies that $\widetilde{\delta}\widetilde{f}=0$ to infinite order. The argument that $\widetilde{f}(X)=O(Q^\infty)$ is similar. One has $\widetilde{\Delta}(\widetilde{f}(X))=O(Q^\infty)$ since $\widetilde{\Delta}\widetilde{f}=O(Q^\infty)$ and $\widetilde{\delta}\widetilde{f}=O(Q^\infty)$. Now $\widetilde{f}(X)$ is homogeneous of degree $w$. Since $w+n/2-1\notin\mathbb{N}$, we can apply the usual statement of uniqueness in the scalar case unless $w+n/2=1$. If $w+n/2=1$, the argument in the scalar case proves uniqueness for densities which are $O(Q^2)$. Thus $\widetilde{f}(X)=O(Q^\infty)$ holds in general.

For general $r$, $s$, the algebra of the initial lift and the consistency verification is more complicated, but the basic idea is the same. When $r>s=0$, the operator $\delta_1$ is conformally invariant for $w=2r-n$ and obstructs the initial lift. If $s>0$, there are two invariant operators obstructing the initial lift, giving rise to the two excluded values of $w$. For $r>s>0$ the invariant operators are $\delta_2$, $\pi^{r-1,s}\delta_1$ for $w=r+2s-n-1$, $2r+s-n$, respectively, where $\pi^{r-1,s}$ is the Young projector onto $\bigwedge^{r-1,s}$. If $r=s>0$, the obstructing invariant operators are $\delta_2$ for $w=3r-n-1$ and $\delta_1\delta_2$, an iterated divergence, for $w=3r-n$.

Theorem 2.1 implies a corresponding isomorphism obtained by taking jets at a point. Define $\mathcal{J}^{r,s}(w)$ to be the space of infinite-order jets at $[e_0]\in\mathcal{Q}$ of sections of $\bigwedge^{r,s}(w)$, and $\mathcal{J}_0^{r,s}(w)$ to be the subspace of jets which are trace-free to infinite order. The $G$-action on the sheaf $\mathcal{E}_0^{r,s}(w)$ induces a $(\mathfrak{g},P)$-module structure on

$\mathcal{J}_0^{r,s}(w)$ dual to a generalized Verma module. Define $\widetilde{\mathcal{J}}^{r,s}(w)$ to be the space of infinite order jets at $e_0 \in \mathbb{R}^{n+2}$ of sections of $\widetilde{\bigwedge}^{r,s}$ which are homogeneous of degree $w$, and by $\widetilde{\mathcal{J}}_0^{r,s}(w)$ the subspace of jets which are trace-free to infinite order. The $G$-action on $\widetilde{\mathcal{E}}^{r,s}(w)$ induces $(\mathfrak{g}, P)$-module structures on $\widetilde{\mathcal{J}}^{r,s}(w)$, $\widetilde{\mathcal{J}}_0^{r,s}(w)$. Define $\widetilde{\mathcal{J}}_{\mathcal{H}}^{r,s}(w) \subset \widetilde{\mathcal{J}}_0^{r,s}(w)$ to be the submodule consisting of those jets for which the equations (2.6) hold to infinite order at $e_0$. It follows from Theorem 2.1 that if $r$, $s$, $w$ satisfy the restrictions of Theorem 2.1, then

$$\mathcal{J}_0^{r,s}(w) \cong \widetilde{\mathcal{J}}_{\mathcal{H}}^{r,s}(w)$$

as $(\mathfrak{g}, P)$-modules. This statement can be regarded as a "jet isomorphism theorem for $\mathcal{J}_0^{r,s}(w)$" providing an ambient description of the dual generalized Verma modules. Upon expressing a jet in $\widetilde{\mathcal{J}}_{\mathcal{H}}^{r,s}(w)$ as the list of tensors which are the successive derivatives of the section, one can realize the $P$-action in the ambient description in terms of reweighted tensor representations analogous to (1.3). See [6] for further discussion.

Observe that $r$, $s$, $w \in \mathbb{Z}$ for all of the spaces $\mathcal{E}_0^{r,s}(w)$ which occur in the deformation complex (2.4), (2.5). Therefore, if $n$ is odd, then the condition $w + n/2 - r - s \notin \mathbb{N}$ in Theorem 2.1 is automatic for these spaces. One verifies easily that for $n$ odd, the bulleted conditions in Theorem 2.1 hold for all spaces which occur in the deformation complex except for the next to last one, $\mathcal{E}_0^{n-1,1}$, for which the second bulleted condition is violated. This corresponds to the fact that the operator in the complex acting on this space is $\delta_2$, which is precisely the operator obstructing the ambient lift on this space. Even though invariant operators act on the other spaces in the deformation complex, namely the operators occuring in the complex itself, the only one obstructing ambient lifts is the one acting on $\mathcal{E}_0^{n-1,1}$. So Theorem 2.1 provides an ambient description of all of the other spaces in the complex. It is not difficult to identify the differential operators on $\mathbb{R}^{n+2}$ which correspond in this realization to the operators in the deformation complex. One thus obtains:

**Theorem 2.2.** *Let $n$ be odd. The deformation complex with last two spaces removed can be realized as:*

$$\begin{array}{ccccccccccccc}
0 & \to & \mathfrak{g} & \to & \mathcal{H}^1(2) & \xrightarrow{\widetilde{D}_0} & \mathcal{H}^{1,1}(2) & \xrightarrow{\widetilde{D}_1} & \mathcal{H}^{2,2}(2) & \xrightarrow{\widetilde{D}_2} & \mathcal{H}^{3,2}(2) & \to \cdots \\
 & & \parallel & & \wr\parallel & & \wr\parallel & & \wr\parallel & & \wr\parallel & \\
0 & \to & \mathfrak{g} & \to & \mathcal{E}^1(2) & \xrightarrow{D_0} & \mathcal{E}_0^{1,1}(2) & \xrightarrow{D_1} & \mathcal{E}_0^{2,2}(2) & \xrightarrow{D_2} & \mathcal{E}_0^{3,2}(2) & \to \cdots
\end{array}$$

$$\begin{array}{cccc}
\cdots \to & \mathcal{H}^{n-3,2}(2) & \xrightarrow{\widetilde{D}_{n-3}} & \mathcal{H}^{n-2,2}(2) \\
 & \wr\parallel & & \wr\parallel \\
\cdots \to & \mathcal{E}_0^{n-3,2}(2) & \xrightarrow{D_{n-3}} & \mathcal{E}_0^{n-2,2}(2)
\end{array}$$

*where the differential operators are:*

$$\begin{aligned}
\widetilde{D}_0 &= \operatorname{Sym} \widetilde{d}_2 \quad (\textit{so } (\widetilde{D}_0 \widetilde{f})_{IJ} = \partial_{(I} \widetilde{f}_{J)}) \\
\widetilde{D}_1 &= \widetilde{d}_2 \widetilde{d}_1 \\
\widetilde{D}_r &= \widetilde{d}_1 \quad r = 2, 3, \ldots, n-3\,.
\end{aligned}$$

*When $n = 3$, the shortened complexes terminate with the spaces $\mathcal{E}_0^{1,1}(2)$, $\mathcal{H}^{1,1}(2)$.*

Observe that the operators $\widetilde{D}_r$ in the lifted complex are simpler than their downstairs counterparts: they do not involve the trace-free part. For example, $\widetilde{D}_0$ is the Killing operator, while $D_0$ is the conformal Killing operator. The conditions defining the $\mathcal{H}^{r,s}(w)$ imply that the images of the $\widetilde{D}_r$ are already contained in trace-free tensors.

As indicated above, the operator $D_{n-1} = \delta_2$ obstructs ambient lifts of the next space $\mathcal{E}_0^{n-1,1}$ in the deformation complex. However, $\operatorname{im} D_{n-2} \subset \ker D_{n-1}$, and a section of $\mathcal{E}_0^{n-1,1}$ which is in $\ker \delta_2$ does have an ambient lift to $\mathcal{H}^{n-1,1}(0)$. This lift is not unique. Nonetheless, by appropriately modifying the lifted space, one can arrange a unique ambient lift. Thus it is possible to extend the above complexes one more term to include an ambient realization of $\ker D_{n-1}$. For this term, the analogue of the restriction operator inverse to the lift effectively involves a differentiation and the operator lifting $D_{n-2}$ has order one less than $D_{n-2}$. When $n > 3$, the operator lifting $D_{n-2} \colon \mathcal{E}_0^{n-2,2}(2) \to \ker D_{n-1}$ is $\widetilde{d}_1 \colon \mathcal{H}^{n-2,2}(2) \to \mathcal{H}^{n-1,2}(2)$. When $n = 3$, the operator lifting $D_1 \colon \mathcal{E}_0^{1,1}(2) \to \ker D_2$ is the second order operator $\widetilde{d}_2 \widetilde{d}_1 \colon \mathcal{H}^{1,1}(2) \to \mathcal{H}^{2,2}(2)$, the same operator which lifts $D_1$ in higher dimensions.

Theorem 2.1 also gives ambient realizations for other gBGG complexes; for example the deRham complex and the complex which resolves the standard representation $\mathbb{R}^{n+2}$ of $\mathfrak{g}$.

Next we indicate how Theorem 1.4 can be proved using the ambient lift of the deformation complex. In the previous section we constructed the map $c : \mathcal{M}/\mathrm{CDiff}_0 \to \widetilde{\mathcal{R}}$ which evaluates the curvature tensors of the ambient metric and outlined why it is $P$-equivariant. So what remains is to show that $c$ is bijective with polynomial inverse. The first step is a linearization argument as in the direct proof mentioned in §1. One truncates all the jet spaces and the map $c$ at finite order to make everything finite-dimensional. Geodesic normal coordinates and the "conformal normal form" mentioned previously provide a slice for the $\mathrm{CDiff}_0$ action, from which it follows that $\mathcal{M}^N/\mathrm{CDiff}_0$ is a smooth manifold, where $\mathcal{M}^N$ indicates the truncation of $\mathcal{M}$ at order $N$. Now either an algebraic induction argument or the inverse function theorem can be used to reduce the conclusion to proving that $dc : T\mathcal{M}/T\mathcal{O} \to T\widetilde{\mathcal{R}}$ is a vector space isomorphism, where $\mathcal{O}$ is the $\mathrm{CDiff}_0$-orbit of the flat metric $h$, $T\mathcal{M}$ and $T\mathcal{O}$ denote the tangent spaces at $h$, and $T\widetilde{\mathcal{R}}$ is the tangent space to $\widetilde{\mathcal{R}}$ at 0.

The second step is to relate the spaces $T\mathcal{M}/T\mathcal{O}$ and $T\widetilde{\mathcal{R}}$ to the spaces appearing in the deformation complex.

**Lemma 2.3.** $T\mathcal{M}/T\mathcal{O} \cong \mathcal{J}_0^{1,1}(2)/D_0\mathcal{J}^1(2)$.

**Proof.** The definitions give

$$T\mathcal{M} = \left\{ s \in \mathcal{J}^{1,1}(2) : s(0) = 0 \right\},$$
$$T\mathcal{O} = \left\{ \mathcal{L}_V h : V = O\left(\|x\|^2\right) \right\} \oplus \left\{ \Omega^2 h : \Omega = O\left(\|x\|^2\right) \right\}.$$

Recall that $D_0$ is the conformal Killing operator, and corresponds to $V \to \operatorname{tf} \mathcal{L}_V h$ when its argument is viewed as a vector field. Now, as in the construction of geodesic normal coordinates, every 1-jet of an infinitesimal metric is in the range of the Killing operator on jets of vector fields. This shows that all 1-jets in both $T\mathcal{M}$ and $\mathcal{J}_0^{1,1}(2)$ are contained in the respective denominator spaces. For higher order jets, the term $\left\{ \Omega^2 h : \Omega = O\left(\|x\|^2\right) \right\}$ cancels the trace components. □

**Proposition 2.4.** $T\widetilde{\mathcal{R}} \cong \ker \widetilde{d}_1 \subset \widetilde{\mathcal{J}}_{\mathcal{H}}^{2,2}(2)$.

**Proof.** The tangent space $T\widetilde{\mathcal{R}}$ is defined by the same relations (1)–(5) in Definition 1.3, except that the $\widetilde{Q}^{(s)}$ term in (3) is replaced by 0. Thus each $\widetilde{R}_{IJKL,M_1\ldots M_r}$ is symmetric in $M_1 \ldots M_r$. We may identify jets $\widetilde{R}$ at $e_0$ of sections of $\widetilde{\bigwedge}^2 \otimes \widetilde{\bigwedge}^2$ with such lists of tensors by the requirement that

$$\partial^r_{M_1\ldots M_r} \widetilde{R}_{IJKL}(e_0) = \widetilde{R}_{IJKL,M_1\ldots M_r}\,, \qquad r \geq 0\,. \tag{2.8}$$

Clearly conditions (1) and (4) are equivalent to the statement that $\widetilde{R}$ is the jet of a section of $\widetilde{\bigwedge}_0^{2,2}$. Differentiation of the relation $X^L \widetilde{R}_{IJKL} = 0$ and evaluating at $e_0$ shows that condition (5) is equivalent to the statement that $X \lrcorner \widetilde{R} = 0$ to infinite order. Condition (2) holds if and only if $\widetilde{R} \in \ker \widetilde{d}_2$. Since these are all the relations defining $\widetilde{\mathcal{R}}$, it follows that $\ker \widetilde{d}_2|_{\widetilde{\mathcal{J}}_{\mathcal{H}}^{2,2}(2)} \subset T\widetilde{\mathcal{R}}$. Note that $\ker \widetilde{d}_1 = \ker \widetilde{d}_2$ on sections of $\widetilde{\bigwedge}^{2,2}$ by the symmetries of curvature tensors.

Now $\widetilde{\mathcal{J}}_{\mathcal{H}}^{2,2}(2)$ is defined by the conditions considered in the previous paragraph together with the additional requirements that $\widetilde{R}$ be homogeneous of degree 2 as a jet and that $\widetilde{\Delta}\widetilde{R} = 0$ and $\widetilde{\delta}_1 \widetilde{R} = 0$ to infinite order. The homogeneity statement is equivalent to

$$\widetilde{R}_{IJKL,M_1\ldots M_r 0} = (-2-r)\widetilde{R}_{IJKL,M_1\ldots M_r}\,. \tag{2.9}$$

The symmetry of $\widetilde{R}^{(r+1)}$ in the differentiation indices and relation (2) can be used to express the left hand side as a sum of two terms in which the '0' index is before the comma. Then applying (5) and then (2) again establishes (2.9). The relations $\widetilde{\delta}_1 \widetilde{R} = 0$ and $\widetilde{\Delta}\widetilde{R} = 0$ follow similarly using (2) to move contracted derivative indices before the comma and then applying (4). Thus $T\widetilde{\mathcal{R}} = \ker \widetilde{d}_1|_{\widetilde{\mathcal{J}}_{\mathcal{H}}^{2,2}(2)}$ under the identification (2.8). □

Composing with the isomorphisms of Lemma 2.3 and Proposition 2.4, the jet isomorphism theorem reduces to the statement that

$$dc\colon \mathcal{J}_0^{1,1}(2)/D_0\mathcal{J}^1(2) \to \ker \widetilde{d}_1 \subset \widetilde{\mathcal{J}}_{\mathcal{H}}^{2,2}(2)$$

is an isomorphism. Suppose first that $n \geq 5$. According to Theorem 2.1, the lift of the deformation complex on jets contains

$$\begin{array}{ccccccc} \to \ \widetilde{\mathcal{J}}^1_{\mathcal{H}}(2) & \to & \widetilde{\mathcal{J}}^{1,1}_{\mathcal{H}}(2) & \to & \widetilde{\mathcal{J}}^{2,2}_{\mathcal{H}}(2) & \xrightarrow{\widetilde{D}_2} \\ \wr\| & & \wr\| & & \wr\| & \\ \to \ \mathcal{J}^1(2) & \xrightarrow{D_0} & \mathcal{J}^{1,1}_0(2) & \xrightarrow{D_1} & \mathcal{J}^{2,2}_0(2) & \xrightarrow{D_2} \end{array}$$

Since the deformation complex is exact on jets, $D_1$ induces an isomorphism

$$\mathcal{J}^{1,1}_0(2)/D_0\mathcal{J}^1(2) \cong \ker D_2 \cong \ker \widetilde{D}_2 = \ker \widetilde{d}_1 \, .$$

One can show that this map agrees with $dc$, and the result follows.

When $n = 3$, $D_1$ maps into $\ker D_2 \subset \mathcal{J}^{2,1}_0$. But, as discussed after the statement of Theorem 2.2, when $n = 3$ we have a modified lift of $\ker D_2$ to $\widetilde{\mathcal{J}}^{2,2}_{\mathcal{H}}(2)$. The jet isomorphism theorem follows in exactly the same manner.

## 3. Jet isomorphism, even dimensions

When $n$ is even, the construction of the ambient metric is obstructed at order $n/2$. So the map $c$ evaluating the covariant derivatives of curvature of the ambient metric is not defined beyond this order. This is a reflection of a difference in the structure of $\mathcal{M}/\mathrm{CDiff}_0$ as a $P$-space when $n$ is even.

The same phenomenon occurs when constructing the ambient lift for $\mathcal{E}^{r,s}_0(w)$ when $w + n/2 - r - s \in \mathbb{N}$. In this section, an extension of the theory to these cases will be outlined. The main ingredients are the following:

- A weakening of the homogeneity condition on the ambient lift
- The occurrence of logarithm terms in the solutions of the ambient equations
- Existence of an ambiguity (nonuniqueness) in the solutions
- An invariant smooth part for the solutions with log terms
- Jet isomorphism theorem for an enlarged space

We will first illustrate the ideas by discussing the ambient lift with log term and the generalization of Theorem 2.1 for scalars in the obstructed case $w+n/2 \in \mathbb{N}$. Then, by analogy with the discussion in §1, we will formulate the jet isomorphism theorem for conformal structures in even dimensions and will discuss the construction of inhomogeneous ambient metrics containing log terms and the extension of the map $c$ to infinite order in the even dimensional case. This is all joint work with K. Hirachi.

Recall that Theorem 2.1 asserts that if $w + n/2 \notin \mathbb{N}$, then $\mathcal{E}(w) \cong \mathcal{H}(w)$, where $\mathcal{H}(w)$ is the sheaf of harmonic jets along $\mathcal{N}$ homogeneous of degree $w$. But if $w + n/2 = m \in \mathbb{N}$, then harmonic extension is obstructed at order $m$ by the conformally invariant operator $\Delta^m$. The following proposition shows that it is always possible to find a harmonic extension by including a log term in the expansion.

**Proposition 3.1.** *Suppose $w + n/2 = m \in \mathbb{N}$. Let $\mathcal{U} \subset \mathcal{Q}$ be open and let $f \in \mathcal{E}_{\mathcal{U}}(w)$. There exists an infinite order jet along $\pi^{-1}(\mathcal{U})$ of a function $\widetilde{f}$ on $\mathbb{R}^{n+2}$ of the form*

$$\widetilde{f} = \widetilde{s} + \widetilde{l}\, Q^m \log |Q| \tag{3.1}$$

*with $\widetilde{s}$, $\widetilde{l}$ smooth, $\widetilde{s}$ homogeneous of degree $w$, $\widetilde{l}$ homogeneous of degree $w - 2m$, such that $\widetilde{\Delta}\widetilde{f} = 0$ to infinite order and $\widetilde{f}|_{\pi^{-1}(\mathcal{U})} = f$. These conditions uniquely determine $\widetilde{l}$ to infinite order along $\pi^{-1}(\mathcal{U})$ and determine $\widetilde{s}$ modulo $Q^m\mathcal{H}_{\mathcal{U}}(w - 2m)$.*

Note that $\widetilde{s}$ and $\widetilde{l}\, Q^m$ are each homogeneous of degree $w$. Thus $\widetilde{f}$ is almost homogeneous of degree $w$, but is not so because of the appearance of $\log |Q|$. In this sense the homogeneity condition on $\widetilde{f}$ has been weakened. Of course, the appearance of this log term also means that $\widetilde{f}$ is no longer smooth.

A main feature of Proposition 3.1 is that the solution $\widetilde{f}$ is no longer unique. One constructs $\widetilde{f}$ inductively by order as in the proof of the scalar case of Theorem 2.1 sketched in §2. The argument there constructed $\widetilde{f} \mod Q^m$. The inclusion of the $Q^m \log |Q|$ term enables the possibility of finding a harmonic extension at order $Q^m$. The coefficient of $Q^m \log |Q|$ is uniquely determined but not the coefficient of $Q^m$, which can be prescribed arbitrarily on $\mathcal{N}$. The solution is then uniquely determined to all higher orders. The fact that the uniqueness is at best modulo $Q^m\mathcal{H}(w-2m)$ is immediate from (2.7): $[\widetilde{\Delta}, Q^m] = 0$ on functions homogeneous of degree $w - 2m = -n/2 - m$. Note that Theorem 2.1 implies that $\mathcal{H}(-n/2 - m) \cong \mathcal{E}(-n/2-m)$ so that uniqueness modulo $Q^m\mathcal{H}(-n/2-m)$ is the same as saying that the coefficient of $Q^m$ in the expansion of $\widetilde{f}$ is undetermined. This nonuniqueness is called the ambiguity in the solution.

It turns out that $\widetilde{l}$ can be written entirely in terms of $\widetilde{s}$, and the condition that $\widetilde{f}$ be harmonic can be written entirely in terms of $\widetilde{s}$. Thus one can reformulate the extension as a map taking values in a space of jets along $\mathcal{N}$ of smooth homogeneous functions of degree $w$, staying entirely in the smooth category. To see this, straightforward calculation using (2.7) shows that

$$\begin{aligned}\widetilde{\Delta}\widetilde{f} &= \widetilde{\Delta}(\widetilde{s} + \widetilde{l}\, Q^m \log |Q|)\\ &= (\widetilde{\Delta}\widetilde{s} + 4m\widetilde{l}\, Q^{m-1}) + (\widetilde{\Delta}\widetilde{l})\, Q^m \log |Q|\,.\end{aligned}$$

So $\widetilde{\Delta}\widetilde{f} = 0$ to infinite order if and only if $\widetilde{\Delta}\widetilde{l} = 0$ and $\widetilde{\Delta}\widetilde{s} = -4m\widetilde{l}\, Q^{m-1}$ to infinite order. Now iterating (2.7) shows that if $\widetilde{l}$ is homogeneous of degree $-n/2 - m$ and $\widetilde{\Delta}\widetilde{l} = 0$, then

$$c_m\widetilde{\Delta}^{m-1}(Q^{m-1}\widetilde{l}) = \widetilde{l}, \qquad c_m^{-1} = (-4)^{m-1}(m-1)!^2\,. \tag{3.2}$$

Thus applying $c_m\widetilde{\Delta}^{m-1}$ to the second equation gives

$$c_m\widetilde{\Delta}^m\widetilde{s} = -4m\widetilde{l}\,.$$

This gives $\widetilde{l}$ in terms of $\widetilde{s}$. Substituting back, one can write both equations in terms of $\widetilde{s}$:

$$\widetilde{\Delta}\widetilde{s} = c_m Q^{m-1}\widetilde{\Delta}^m\widetilde{s} \quad \text{and} \quad \widetilde{\Delta}^{m+1}\widetilde{s} = 0\,. \tag{3.3}$$

This motivates the following definition.

**Definition 3.2.** Suppose $w+n/2 = m \in \mathbb{N}$. Define $\mathcal{H}_S(w)$ to be the sheaf on $\mathcal{Q}$ of infinite order jets along $\mathcal{N}$ of smooth functions $\widetilde{s}$ on $\mathbb{R}^{n+2}$ which are homogeneous of degree $w$ and which satisfy (3.3) to infinite order along $\mathcal{N}$, with $c_m$ as in (3.2).

The conditions (3.3) are clearly $G$-invariant, so $\mathcal{H}_S(w)$ is a homogeneous sheaf on $\mathcal{Q} = G/P$. Also observe by applying $c_m\widetilde{\Delta}^{m-1}$ that if $c$ is any constant other than $c_m$, then any solution $\widetilde{s}$ to the system obtained by replacing $c_m$ by $c$ in (3.3) which is homogeneous of degree $w$ necessarily satisfies $\widetilde{\Delta}^m\widetilde{s} = 0$, and therefore $\widetilde{\Delta}\widetilde{s} = 0$. The choice $c = c_m$ is the unique choice for which $\mathcal{H}_S(w) \neq \mathcal{H}(w)$.

Now the substitute ambient lift theorem for scalars in the obstructed cases takes the form:

**Theorem 3.3.** *Suppose $w + n/2 = m \in \mathbb{N}$. There is a $G$-equivariant exact sequence of sheaves:*

$$0 \to \mathcal{E}(w-2m) \to \mathcal{H}_S(w) \to \mathcal{E}(w) \to 0\,. \tag{3.4}$$

The map $\mathcal{H}_S(w) \to \mathcal{E}(w)$ is restriction to $\mathcal{N}$. The map $\mathcal{E}(w-2m) \to \mathcal{H}_S(w)$ is harmonic extension followed by multiplication by $Q^m$; we saw in Proposition 3.1 that jets in $Q^m\mathcal{H}(w-2m)$ are already harmonic, so certainly are contained in $\mathcal{H}_S(w)$. These maps are clearly $G$-equivariant. Since the sheaves $\mathcal{E}(w)$ are soft, exactness of the sequence of sheaves is equivalent to exactness of the corresponding sequences of sections on any open set. And this is just the uniqueness statement of Proposition 3.1 reformulated in terms of $\mathcal{H}_S(w)$ as explained above.

By choosing a (necessarily non-$G$-equivariant) splitting of (3.4), one can parametrize $\mathcal{H}_S(w)$ as $\mathcal{E}(w) \times \mathcal{E}(w-2m)$. The space $\mathcal{E}(w)$ corresponds to the initial density and $\mathcal{E}(w-2m)$ to the ambiguity in the lift. The space that has the ambient realization is not the initial space $\mathcal{E}(w)$ in which we were interested, but the enlargement $\mathcal{E}(w) \times \mathcal{E}(w-2m)$ of this space by the ambiguity in the solution. The space $\mathcal{H}_S(w)$ realizing the ambient representation is an enlargement of the space of smooth homogeneous harmonic jets, and consists of the smooth homogeneous jets satisfying the system (3.3) rather than the equation $\widetilde{\Delta}\widetilde{f} = 0$. By taking jets at $e_0$ of the solutions of this system, one obtains the substitute jet isomorphism theorem for scalars in the obstructed cases analogous to the statement $\mathcal{J}(w) \cong \widetilde{\mathcal{J}}_{\mathcal{H}}(w)$ in the unobstructed cases.

It is easy to check that when $n$ is even, all the spaces in the first half of the deformation complex have $w+n/2-r-s \in \mathbb{N}$, so their ambient lifts are obstructed just as in the scalar case discussed above. There is a version of Theorem 3.3 for these spaces which is used in the proof of the jet isomorphism theorem for conformal structures for $n$ even as indicated below.

The jet isomorphism theorem for conformal structures in even dimensions involves similar features as in the obstructed scalar case. This time the ambiguity is a symmetric 2-tensor which is trace-free with respect to the given metric. There is a map from jets of metrics together with jets of the ambiguity to a space of ambient curvature tensors which induces a $P$-equivariant bijection from the quotient by $\mathrm{CDiff}_0$. We formulate the result more precisely.

Define

$$\mathcal{M} \times_{\mathcal{M}} \mathcal{J}_0^{1,1} \equiv \{(g, A) \in \mathcal{M} \times \mathcal{J}^{1,1} : g^{ij} A_{ij} = 0 \text{ to infinite order}\}.$$

Here $\mathcal{J}^{1,1}$ denotes the space of jets of symmetric 2-tensors at $0 \in \mathbb{R}^n$ (ignoring the weight). The space $\mathcal{M} \times_{\mathcal{M}} \mathcal{J}_0^{1,1}$ may be regarded as a fiber bundle over $\mathcal{M}$ by projecting onto the first factor. Set

$$\widetilde{\mathcal{T}} = \prod_{r=0}^{\infty} \bigwedge^{2,2} \mathbb{R}^{n+2*} \otimes \bigotimes^r \mathbb{R}^{n+2*} \otimes \sigma_{-r-2}.$$

Here $\bigwedge^{2,2} \mathbb{R}^{n+2*}$ denotes the finite-dimensional vector space of covariant 4-tensors in $n+2$ dimensions with curvature tensor symmetries. Then $\widetilde{\mathcal{T}}$ has a natural $P$-action. Recall that when $n$ was odd, the space $\widetilde{\mathcal{R}}$ of lists of ambient curvature tensors was a $P$-invariant subset of $\widetilde{\mathcal{T}}$. The conformal jet isomorphism theorem for $n$ even is then the following.

**Theorem 3.4.** *Let $n \geq 4$ be even. There is a $P$-equivariant polynomial injection $c : (\mathcal{M} \times_{\mathcal{M}} \mathcal{J}_0^{1,1})/\mathrm{CDiff}_0 \to \widetilde{\mathcal{T}}$, whose image $\widetilde{\mathcal{R}}$ is a submanifold of $\widetilde{\mathcal{T}}$ whose tangent space $T\widetilde{\mathcal{R}}$ at $0$ is the space of jets $\widetilde{R} \in \widetilde{\mathcal{J}}^{2,2}(2)$ which are solutions to the following equations to infinite order at $e_0$:*

1. $\widetilde{R}_{IJ[KL,M]} = 0$
2. $X \lrcorner \widetilde{R} = 0$
3. $\widetilde{\mathrm{tr}}\, \widetilde{R} = c_{n/2} Q^{n/2-1} \widetilde{\Delta}^{n/2-1} \widetilde{\mathrm{tr}}\, \widetilde{R}$
4. $\widetilde{\Delta}^{n/2} \widetilde{\mathrm{tr}}\, \widetilde{R} = 0.$

*Also, $c^{-1} : \widetilde{\mathcal{R}} \to (\mathcal{M} \times_{\mathcal{M}} \mathcal{J}_0^{1,1})/\mathrm{CDiff}_0$ is polynomial. Here $(\widetilde{\mathrm{tr}}\, \widetilde{R})_{IJ} = \widetilde{h}^{KL} \widetilde{R}_{IKJL}$ corresponds to the Ricci tensor, and $\widetilde{\mathcal{J}}^{2,2}(2)$ is identified with a $P$-submodule of $\widetilde{\mathcal{T}}$ via* (2.8). *The constant $c_{n/2}$ is given in* (3.2).

The formulation of Theorem 3.4 requires some explanation. First, the statement that $\widetilde{\mathcal{R}}$ is a submanifold of $\widetilde{\mathcal{T}}$ is to be interpreted in terms of finite-order truncations of these spaces; the full spaces are projective limits of their truncations. The truncations are finite dimensional so these notions make sense in this context. Next, we have not yet defined the CDiff-action on $\mathcal{M} \times_{\mathcal{M}} \mathcal{J}_0^{1,1}$ which gives rise to the quotient by $\mathrm{CDiff}_0$ and the $P$-action on the quotient. There is a natural action of CDiff, but as in Theorem 3.3, it is not a product action. This action has the property that the projection $\mathcal{M} \times_{\mathcal{M}} \mathcal{J}_0^{1,1} \to \mathcal{M}$ is CDiff-equivariant, where the action on $\mathcal{M}$ is that defined in §1. The CDiff-action on $\mathcal{M} \times_{\mathcal{M}} \mathcal{J}_0^{1,1}$ will be defined below.

When $n$ was odd, the nonlinear space $\widetilde{\mathcal{R}}$ was identified explicitly; see Definition 1.3. For $n$ even, Theorem 3.4 asserts instead that $c$ is a bijection onto a submanifold $\widetilde{\mathcal{R}}$ of $\widetilde{\mathcal{T}}$ and identifies explicitly the tangent space $T\widetilde{\mathcal{R}}$. This suffices for all the applications; the explicit form of the nonlinear terms is not needed. One can be somewhat more explicit about the equations defining $\widetilde{\mathcal{R}}$. The equations (1)–(3) and (5) in Definition 1.3 hold also for $\widetilde{\mathcal{R}}$ in even dimensions. Equation (4) in Definition 1.3 is replaced by nonlinear versions of (3) and (4) above.

As shown by the proof of Proposition 2.4, for $n$ odd $T\widetilde{\mathcal{R}}$ is defined by exactly the same relations (1)–(4) above, except that (3) and (4) are replaced by the single equation $\widetilde{\operatorname{tr}}\,\widetilde{R} = 0$. The relations (3) and (4) in Theorem 3.4 are completely analogous to the equations (3.3) for the obstructed scalar problem; the Ricci tensor $\widetilde{\operatorname{tr}}\,\widetilde{R}$ plays the role of $\widetilde{\Delta}\widetilde{s}$.

In the rest of this section we will describe the extension of the ambient metric construction to all orders in even dimensions and the construction of the map $c$. Recall that in odd dimensions an ambient metric is a smooth metric defined by the conditions (1)–(3) of Definition 1.5. In even dimensions there is a formal obstruction at order $n/2$ to the existence of such a metric, analogous to the obstruction to finding a smooth harmonic extension of a density in the scalar problem. It is natural to try to continue the expansion by including log terms. In the scalar problem, $Q$ was a $G$-invariant defining function for $\mathcal{N}$ and terms involving $Q^m \log|Q|$ contained a built-in $G$-invariance. But there is no canonical analogue of $Q$ for the nonlinear problem, so it is not clear what the argument of the logarithm should be to obtain an invariant construction. Another distinction is that the nonlinearity of the Ricci curvature operator will force the inclusion of powers of the logarithm as well.

These considerations motivate the following definition. Let $r$ denote an arbitrary smooth defining function for $\mathcal{G} \subset \widetilde{\mathcal{G}}$ homogeneous of degree 2.

**Definition 3.5.** Let $\mathcal{A}_{\log}$ denote the space of formal asymptotic expansions of metrics of signature $(p+1, q+1)$ on $\widetilde{\mathcal{G}}$ of the form

$$\widetilde{g} \sim \widetilde{g}^{(0)} + \sum_{N\geq 1} \widetilde{g}^{(N)} r (r^{n/2-1} \log|r|)^N, \tag{3.5}$$

where $\widetilde{g}^{(N)}$, $N \geq 0$, are smooth symmetric 2-tensor fields on $\widetilde{\mathcal{G}}$ satisfying $\delta_s^* \widetilde{g}^{(0)} = s^2 \widetilde{g}^{(0)}$ and $\delta_s^* \widetilde{g}^{(N)} = s^{(2-n)N} \widetilde{g}^{(N)}$ for $N \geq 1$, and such that $\iota^* \widetilde{g} = \mathbf{g_0}$.

It is easy to see that the space $\mathcal{A}_{\log}$ is independent of the choice of $r$; upon changing $r$, one obtains an expansion of the same form but with different coefficients. Also, the space $\mathcal{A}_{\log}$ is invariant under pullback by smooth homogeneous diffeomorphisms $\Phi$ of $\widetilde{\mathcal{G}}$ satisfying $\Phi|_{\mathcal{G}} = I$.

Recall that ambient metrics in odd dimensions automatically had an additional geometric property. We call metrics having this property straight:

**Definition 3.6.** A metric $\widetilde{g} \in \mathcal{A}_{\log}$ is *straight* if for each $p \in \widetilde{\mathcal{G}}$, the dilation orbit $s \to \delta_s p$ is a geodesic for $\widetilde{g}$. (Since $\widetilde{g}$ is only defined as an asymptotic expansion, this means that the geodesic equations hold to infinite order along $\mathcal{G}$.)

The ambient metrics involving log terms in even dimensions are then defined as follows. We call these inhomogeneous ambient metrics because the occurence of the log terms means that the metrics are no longer homogeneous.

**Definition 3.7.** Let $n \geq 4$ be even. An inhomogeneous ambient metric for $(M,[g])$ is a straight metric $\widetilde{g} \in \mathcal{A}_{\log}$ satisfying $\mathrm{Ric}(\widetilde{g}) = 0$ formally to infinite order.

The straightness condition is crucial in the inhomogeneous case because of the following proposition.

**Proposition 3.8.** *Let $\widetilde{g} \in \mathcal{A}_{\log}$ be straight. Then $\widetilde{g}(T,T)$ is a smooth defining function for $\mathcal{G}$ homogeneous of degree 2.*

Recall that in the flat case, the vector field $X$ on $\mathbb{R}^{n+2}$ plays the role of $T$ and satisfies $\widetilde{h}(X,X) = Q$. Thus $\widetilde{g}(T,T)$ is a generalization of $Q$. For general $\widetilde{g} \in \mathcal{A}_{\log}$, $\widetilde{g}(T,T)$ has an asymptotic expansion involving $\log|r|$, but Proposition 3.8 asserts that if $\widetilde{g}$ is straight, then $\widetilde{g}(T,T)$ is actually smooth. The proof of Proposition 3.8 is a straightforward analysis of the geodesic equations for the dilation orbits.

Let $\widetilde{g} \in \mathcal{A}_{\log}$ be straight. Then $\widetilde{g}(T,T)$ is a canonically determined smooth defining function for $\mathcal{G}$ homogeneous of degree 2. We may therefore take $r = \widetilde{g}(T,T)$ in (3.5). The term $\widetilde{g}^{(0)}$ appearing in the resulting expansion is then a smooth metric uniquely determined by $\widetilde{g}$ independently of any choices. We call this metric $\widetilde{g}^{(0)}$ the *smooth part* of $\widetilde{g}$. Observe that $\widetilde{g}^{(0)}$ is homogeneous of degree 2. One checks that $\widetilde{g}^{(0)}$ is also straight. If $\Phi$ is a smooth homogeneous diffeomorphism satisfying $\Phi|_{\mathcal{G}} = I$ and $\widetilde{g} \in \mathcal{A}_{\log}$ is straight, then $(\Phi^*\widetilde{g})^{(0)} = \Phi^*(\widetilde{g}^{(0)})$.

We extend Definition 1.7 to the inhomogeneous case: a straight metric $\widetilde{g} \in \mathcal{A}_{\log}$ is said to be in normal form relative to a metric $g$ in the conformal class if its smooth part $\widetilde{g}^{(0)}$ is in normal form relative to $g$. If $\widetilde{g} \in \mathcal{A}_{\log}$ is straight, then there is a smooth homogeneous diffeomorphism $\Phi$ uniquely determined to infinite order at $\rho = 0$ such that $\Phi|_{\mathcal{G}} = I$ and such that $\Phi^*\widetilde{g}$ is in normal form relative to $g$.

The main theorem concerning the existence and uniqueness of inhomogeneous ambient metrics is the following.

**Theorem 3.9.** *Let $n \geq 4$ be even. Up to pullback by a smooth homogeneous diffeomorphism which restricts to the identity on $\mathcal{G}$, the inhomogeneous ambient metrics for $(M,[g])$ are parametrized by the choice of an arbitrary trace-free symmetric 2-tensor field* (*the ambiguity tensor*) *on $M$.*

We describe more concretely the parametrization of inhomogeneous ambient metrics in terms of the ambiguity tensor. Choose a representative metric $g$ in the conformal class; we normalize the diffeomorphism invariance by requiring that $\widetilde{g}$ be in normal form relative to $g$. Let $\widetilde{g}^{(0)}$ be the smooth part of $\widetilde{g}$ and let $\widetilde{\mathcal{G}} \cong \mathbb{R}_+ \times M \times \mathbb{R}$ be the decomposition induced by the choice of $g$. Consider the component of $\widetilde{g}^{(0)}$ obtained by restricting to vectors tangent to $M$ in the decomposition $\mathbb{R}_+ \times M \times \mathbb{R}$; by homogeneity this may be written $\widetilde{g}^{(0)}_{ij} = t^2 g^{(0)}_{ij}(x,\rho)$, where $g^{(0)}_{ij}(x,\rho)$ is a smooth 1-parameter family of metrics on $M$ with $g^{(0)}_{ij}(x,0)$ equal to the chosen metric $g_{ij}(x)$. The ambiguity tensor of $\widetilde{g}$ relative to $g$ is:

$$A_{ij} = \mathrm{tf}\left((\partial_\rho)^{n/2} g^{(0)}_{ij}|_{\rho=0}\right).$$

Theorem 3.9 asserts that for each representative metric $g$ and each choice of trace-free symmetric 2-tensor $A$, there is a unique inhomogeneous ambient metric $\widetilde{g}$ in normal form relative to $g$ with ambiguity tensor $A$.

The choice of $g$ and $A$ uniquely determine $\widetilde{g}$ in normal form, and therefore also the smooth part $\widetilde{g}^{(0)}$. As in the scalar case, the inhomogeneous ambient metric serves as an intermediate tool used to determine the smooth homogeneous metric $\widetilde{g}^{(0)}$. However, because of the nonlinearity we do not have a simple way of writing directly the system of equations defining $\widetilde{g}^{(0)}$.

The map $c$ is now defined exactly as in the odd-dimensional case, using the smooth part $\widetilde{g}^{(0)}$ in place of $\widetilde{g}$. Given $(g, A) \in \mathcal{M} \times_{\mathcal{M}} \mathcal{J}_0^{1,1}$, choose tensors also denoted $g$ and $A$ in a neighborhood of $0 \in \mathbb{R}^n$ with the prescribed Taylor expansions and such that $g^{ij}A_{ij} = 0$ in the whole neighborhood. According to Theorem 3.9, there is a unique inhomogeneous ambient metric $\widetilde{g}$ in normal form relative to $g$ with ambiguity tensor $A$. Define the tensors $\widetilde{R}^{(r)}$ to be the iterated covariant derivatives of the curvature tensor of $\widetilde{g}^{(0)}$ evaluated at $t = 1$, $x = 0$, $\rho = 0$. This gives a map $\mathcal{M} \times_{\mathcal{M}} \mathcal{J}_0^{1,1} \to \widetilde{\mathcal{T}}$. Because these $\widetilde{R}^{(r)}$ are the curvature tensors of a smooth, homogeneous, straight metric, the relations (1)–(3) and (5) in Definition 1.3 hold for these tensors. Since $\widetilde{g}^{(0)}$ is not Ricci-flat, equation (4) in Definition 1.3 does not hold. But a study of the linearized problem shows that relations (3) and (4) in Theorem 3.4 hold for the linearized tensors.

The CDiff action on $\mathcal{M} \times_{\mathcal{M}} \mathcal{J}_0^{1,1}$ is defined as follows. First consider metrics $g$ and ambiguity tensors $A$ defined on a manifold $M$; the CDiff action will be obtained by passing to jets at the origin in $\mathbb{R}^n$. Let $\widetilde{g}$ be the inhomogeneous ambient metric in normal form relative to $g$ with ambiguity tensor $A$. If $0 < \Omega \in C^\infty(M)$, set $\widehat{g} = \Omega^2 g$. Now there is a smooth homogeneous diffeomorphism $\Phi$ satisfying $\Phi|_{\mathcal{G}} = I$, uniquely determined to infinite order at $\rho = 0$, such that $\Phi^*\widetilde{g}$ is in normal form relative to $\widehat{g}$. Since $\Phi^*\widetilde{g}$ is also an inhomogeneous ambient metric, it uniquely determines an ambiguity tensor $\widehat{A}$ with the property that $\Phi^*\widetilde{g}$ is the inhomogeneous ambient metric in normal form relative to $\widehat{g}$ with ambiguity tensor $\widehat{A}$. The correspondence $(g, A, \Omega) \to \widehat{A}$ gives a well-defined transformation law for the ambiguity tensor under conformal change, described more explicitly in [14]. The jet of $\widehat{A}$ at a point depends only on the jets of $(g, A, \Omega)$ at that point. Now for $(\varphi, \Omega) \in \mathrm{CDiff}$ and $(g, A) \in \mathcal{M} \times_{\mathcal{M}} \mathcal{J}_0^{1,1}$, the CDiff action is defined by

$$(\varphi, \Omega) \cdot (g, A) = \big((\varphi^{-1})^*\widehat{g}, (\varphi^{-1})^*\widehat{A}\big)\,.$$

One can identify the Jacobian along $\mathcal{G}$ of the diffeomorphism $\Phi$ above to derive the transformation laws for the tensors $\widetilde{R}^{(r)}$. From this it follows that the map $\mathcal{M} \times_{\mathcal{M}} \mathcal{J}_0^{1,1} \to \widetilde{\mathcal{T}}$ passes to a map $c\colon (\mathcal{M} \times_{\mathcal{M}} \mathcal{J}_0^{1,1})/\mathrm{CDiff}_0 \to \widetilde{\mathcal{T}}$ which is $P$-equivariant, as claimed in Theorem 3.4.

The completion of the proof of Theorem 3.4 requires showing that

$$dc\colon T\big((\mathcal{M} \times_{\mathcal{M}} \mathcal{J}_0^{1,1})/\mathrm{CDiff}_0\big) \to T\widetilde{\mathcal{R}}$$

is a vector space isomorphism. This uses the same idea as for $n$ odd: lift the deformation complex. However, the algebra is substantially more complicated, as there is an ambiguity for the lift of each term in the first half of the complex.

It is possible to extend the parabolic invariant theory of [1] to characterize scalar $P$-invariants of $T\widetilde{\mathcal{R}}$ for $n$ even. Theorem 3.4 then enables one to transfer the results to characterize scalar invariants of conformal structures in even dimensions similarly to the arguments of [10] in odd dimensions. These results are described in [14]; details will be forthcoming.

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