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HOMOGENEOUS CARTAN GEOMETRIES

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Abstract. We describe invariant principal and Cartan connections on homogeneous principal bundles and show how to calculate the curvature and the holonomy; in the case of an invariant Cartan connection we give a formula for the infinitesimal automorphisms. The main result of this paper is that the above calculations are purely algorithmic. As an example of an homogeneous parabolic geometry we treat a conformal structure on the product of two spheres.

Introduction

We begin with a discussion of invariant principal connections in section 1 and present a similar treatment of invariant Cartan connections in section 2. For both kinds of connections we give an explicit description of the holonomy Lie algebra which is due to H. C. Wang. Using ideas from [3] we can use the knowledge of the holonomy Lie algebra to calculate the infinitesimal automorphisms of a Cartan geometry in section 2.2.

Nothing in here is really new. Invariant principal connections were already treated by H. C. Wang in [12]. The case of invariant Cartan connections is quite analogous. Infinitesimal automorphisms of Cartan geometries were discussed by A. Čap in [3] and the consequences drawn here for the homogeneous case are quite elementary.

Nevertheless, as a whole, this provides a nice and simple framework for treating homogeneous parabolic geometries: for such structures, among which are conformal structures, almost CR structures of hypersurface type, projective structures, projective contact structures and almost quaternionic structures, one has an equivalence of categories with certain types of Cartan geometries ([11, 13, 5, 6]): this immediately extends all notions discussed for Cartan geometries, like curvature, holonomy and infinitesimal automorphisms, to these structures.

In section 3 we consider as an explicit example the case of a conformal structure on the product of two spheres: We use a well known method to obtain the canonical

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Cartan connection to this parabolic geometry and then calculate the holonomy Lie algebra of this geometry.

1. Invariant principal connections

1.1. Homogeneous P-principal bundles. Let $\mathcal{G} \xrightarrow{\pi} M$ be a P-principal bundle. The right action of P on $\mathcal{G}$ shall be called

$$r\colon \mathcal{G} \times P \to \mathcal{G}\,,$$
$$r^p(u) = r(u,p) = u \cdot p\,.$$

We say that the P-principal bundle $\mathcal{G}$ is homogeneous when there is a Lie group H acting fiber-transitively on $\mathcal{G}$ by principal bundle automorphisms.

We denote the action of an element $h \in H$ on $\mathcal{G}$ by $\lambda_h \in \mathrm{Aut}(\mathcal{G})$ and simply write $\lambda_h(u) = h{\cdot}u$ for $u \in \mathcal{G}$. The diffeomorphism λ_h factorizes to a diffeomorphism $\check{\lambda}_h$ of M and fiber-transitivity of the action of H on $\mathcal{G}$ is equivalent to transitivity of the induced action of H on M.

Let $o \in M$ be an arbitrary point. Then the isotropy group $K = H_o$ of o is a Lie subgroup of H and $M = H/K$. Now the action of K on $\mathcal{G}$ leaves the fiber $\mathcal{G}_o$ of $\mathcal{G}$ over o invariant. Take some $u_0 \in \mathcal{G}_o$; then one sees that there is a unique homomorphism of Lie groups $\Psi : K \to P$ with $k \cdot u_0 = u_0 \cdot \Psi(k)$. Now one checks that in fact $\mathcal{G} = H \times_K P = H \times_\Psi P$, the associated bundle to the principal K-bundle $H \to H/K$ obtained by the action of K on P by Ψ. We denote the equivalence class

$$\tilde{\pi}(h,p) = \{(h \cdot k, \Psi(k) \cdot p), k \in K\} = [h,p]\,.$$

So we described an arbitrary homogeneous P-principal bundle as a quotient of the trivial bundle $H \times P$. We have a K-principal bundle whose base is a P-principal bundle:

$$\begin{tikzcd}
H \times P \arrow[d, "\tilde{\pi}"] & K \arrow[l] \\
H \times_K P \arrow[d, "\pi"] & P \arrow[l] \\
H/K &
\end{tikzcd}$$

We summarize: Every homogeneous P-principal bundle is of the form $\mathcal{G} = H \times_K P \to H/K$ with the canonical left- respectively right- actions of H resp. P on $\mathcal{G}$. We can also say: the data defining an H-homogeneous P-principal bundle is a 4-tuple (H, K, P, Ψ) , where H and P are Lie groups, K is a closed subgroup of H and Ψ is a homomorphism of Lie groups from K to P.

1.2. Invariant principal connections on homogeneous P-principal bundles. Any H-invariant P-principal connection $\gamma \in \Omega(H \times_K P, \mathfrak{p})$ can be pulled back to a P-principal connection on the trivial bundle $H \times P$. Now Frobenius reciprocity (see e.g. [10], 22.14) provides a one-to-one correspondence between

K-invariant horizontal forms on $H \times P$ and $\mathfrak{p}$-valued forms on $H \times_K P$. This can be used to show that H-left invariant P-principal connections on $H \times_K P$ correspond exactly to certain linear maps $\alpha : \mathfrak{h} \to \mathfrak{p}$. Precisely: Left-trivialize $H \times P = H \times P \times \mathfrak{h} \times \mathfrak{p}$, then

Theorem 1.1 ([12], Prop. (5.1), Prop. (5.2); [8], Thm. 2.2.6, Thm.4.1.1;)**.** *Every invariant principal connection γ on $\mathcal{G} = H \times_K P$ is obtained by factorizing a $\mathfrak{p}$-valued one form*

$$\hat{\gamma} \in \Omega^1_K(H \times P, \mathfrak{p})^{\mathrm{hor}} ,$$
$$(h, p, X, Y) \mapsto \mathrm{Ad}(p^{-1})\alpha(X) + Y$$

with $h \in H, p \in P, X \in \mathfrak{h}$ and $Y \in \mathfrak{p}$. Here the conditions on $\alpha \in L(\mathfrak{h}, \mathfrak{p})$ such that $\hat{\gamma}$ is indeed K-invariant and horizontal are

i) $\alpha_{|\mathfrak{k}} = \Psi'$

ii) $\alpha(\mathrm{Ad}(k)X) = \mathrm{Ad}(\Psi(k))\alpha(X)$.

I.e.: α is a K-equivariant extension of Ψ' to a linear map from $\mathfrak{h} \to \mathfrak{p}$.

This description of invariant principal connections on $\mathcal{G}$ leads to an easy formula for the curvature. If $\gamma \in \Omega^1(\mathcal{G}, \mathfrak{p})$ is the invariant principal connection corresponding to $\alpha : \mathfrak{h} \to \mathfrak{p}$, the curvature $\bar{\rho} \in \Omega^2(\mathcal{G}, \mathfrak{p})$ of γ is given by

$$\bar{\rho}(\xi, \eta) = -\gamma([\xi_{\mathrm{hor}}, \eta_{\mathrm{hor}}]),$$

where $\xi_{\mathrm{hor}}, \eta_{\mathrm{hor}}$ are the horizontal projections of vector fields $\xi, \eta \in \mathfrak{X}(\mathcal{G})$.

Now $\bar{\rho}$ is P-equivariant and horizontal and thus factorizes to a $H \times_K \mathfrak{p}$-valued 2-form on H/K, i.e., a section of $H \times_K \Lambda^2(\mathfrak{h}/\mathfrak{k}) \otimes \mathfrak{p}$. It is easy to see that H-invariance of γ implies H-invariance of this section, which is therefore determined by a unique K-invariant element ρ of $\Lambda^2(\mathfrak{h}/\mathfrak{k}) \otimes \mathfrak{p}$.

Therefore we will say that the curvature of α is $\rho = \rho_\alpha \in \Lambda^2(\mathfrak{h}/\mathfrak{k}) \otimes \mathfrak{p}$, and one calculates that

$$\rho(X_1, X_2) = [\alpha(X_1), \alpha(X_2)] - \alpha([X_1, X_2])$$

for $X_1, X_2 \in \mathfrak{h}$.

So the curvature of α is its failure to be (an extension of $\Psi' : \mathfrak{k} \to \mathfrak{p}$ to) a homomorphism of Lie algebras $\mathfrak{h} \to \mathfrak{p}$.

Denote the holonomy of the invariant principal connection corresponding to $\alpha : \mathfrak{h} \to \mathfrak{p}$ by $\mathfrak{hol}(\alpha)$. H. C. Wang gave an explicit description of $\mathfrak{hol}(\alpha)$:

Theorem 1.2 ([12], Theorem (B))**.** *Denote by $\hat{R} = \langle \{\rho(X_1, X_2) | X_1, X_2 \in \mathfrak{h}\} \rangle$ the span of the image of ρ in $\mathfrak{p}$. Then $\mathfrak{hol}(\alpha)$ is the $\mathfrak{h}$-module generated by $\hat{R}$. I.e.:*

$$\mathfrak{hol}(\alpha) = \hat{R} + [\alpha(\mathfrak{h}), \hat{R}] + [\alpha(\mathfrak{h}), [\alpha(\mathfrak{h}), \hat{R}]] + \cdots . \tag{1}$$

Note that it follows in particular that $\hat{R} + [\alpha(\mathfrak{h}), \hat{R}] + [\alpha(\mathfrak{h}), [\alpha(\mathfrak{h}), \hat{R}]] + \cdots$ is already a Lie subalgebra of $\mathfrak{p}$. The proof is quite involved. From the Ambrose-Singer theorem ([1]) one knows hat $\hat{R} \subset \mathfrak{hol}(\alpha)$. The essential part is then the construction of a group which can be explicitly described and which contains the holonomy group as a normal subgroup. Then $\mathfrak{hol}(\alpha)$ is a module under this group and is shown to be generated by $\hat{R}$, which can then be reformulated as (1).

2. Homogeneous Cartan Geometries

One has a very similar description of invariant Cartan connections on homogeneous principal bundles. Recall that for Lie groups (G,P) with P a closed subgroup of G a Cartan geometry of type (G,P) is a P-principal bundle $\mathcal{G}$ over a manifold M endowed with a one form $\omega \in \Omega^1(\mathcal{G},\mathfrak{g})$, called the Cartan connection, satisfying

i) ω is P-equivariant: $(r^p)^*(\omega) = \mathrm{Ad}(p^{-1}) \circ \omega$ for all $p \in P$.
ii) ω reproduces fundamental vector fields: $\omega(\frac{d}{dt}_{|t=0} u{\cdot}\exp(tY)) = Y$ for all $Y \in \mathfrak{p}$.
iii) ω is an absolute parallelism: at every $u \in \mathcal{G}$ the map $\omega_u : T_u\mathcal{G} \to \mathfrak{g}$ is an isomorphism.

We will say that $(\mathcal{G},\omega)$ is a Cartan geometry of type (G,P).

A Cartan geometry $(\mathcal{G},\omega)$ is homogeneous if there is a Lie group H acting fiber-transitively on $\mathcal{G}$ by automorphisms of the Cartan geometry. I.e., when we denote the action of H on $\mathcal{G}$ by $\lambda_h : \mathcal{G} \to \mathcal{G}$ we have that for every $h \in H$

i. λ_h is an automorphism of the P-principal bundle $\mathcal{G}$: $\lambda_h(u \cdot p) = \lambda_h(u) \cdot p$
ii. λ_h preserves ω: $\lambda_h^*(\omega) = \omega$.

Now we saw in section 1.1 that $\mathcal{G}$ is of the form $H \times_\Psi P$ for some homomorphism $\Psi : K \to P$ and analogously to the case of principal connections one gets the following description of invariant Cartan connections of type (G,P) on $\mathcal{G} = H \times_K P$:

Theorem 2.1 ([12], Thm. 4; [8], Thm. 4.2.1.)**.** *Invariant Cartan connections on $H \times_K P$ are in 1:1-correspondence with maps $\alpha : \mathfrak{h} \to \mathfrak{g}$ satisfying*

(C.1) $\alpha_{|\mathfrak{k}} = \Psi'$
(C.2) $\alpha(\mathrm{Ad}(k)X) = \mathrm{Ad}(\Psi(k))\alpha(X)$ *for all* $X \in \mathfrak{h}, k \in K$
(C.3) α *induces an isomorphism of* $\mathfrak{h}/\mathfrak{k}$ *with* $\mathfrak{g}/\mathfrak{p}$.

Explicitly: given such an α, the corresponding Cartan connection ω is obtained by factorizing

$$\hat{\omega} \in \Omega^1(H \times_K P, \mathfrak{g}),$$
$$\hat{\omega}((h,p,X,Y)) = \mathrm{Ad}(p^{-1})\alpha(X) + Y .$$

We will say that α is a Cartan connection.

Similarly as in the case of invariant principal connections, the curvature of an invariant Cartan connection $\alpha\colon \mathfrak{h} \to \mathfrak{p}$ is described by an element $\kappa = \kappa_\alpha \in \Lambda^2(\mathfrak{h}/\mathfrak{k}) \otimes \mathfrak{g}$: it is given by

$$\kappa(X_1,X_2) = [\alpha(X_1),\alpha(X_2)] - \alpha([X_1,X_2] .$$

Using the isomorphism induced by α between $\mathfrak{h}/\mathfrak{k}$ and $\mathfrak{g}/\mathfrak{p}$ we can also regard the curvature as

$$\kappa = \kappa(\alpha) \in \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} .$$

2.1. **The holonomy of a Cartan connection.** Consider a Cartan geometry $(\mathcal{G}, \omega)$ of type (G, P). We can extend the structure group of $\mathcal{G}$ from P to G by taking $\mathcal{G}' = \mathcal{G} \times_P G$. Then a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ extends equivariantly to a G-principal connection $\omega' \in \Omega^1(\mathcal{G}', \mathfrak{g})$ on $\mathcal{G}'$. We will say that the holonomy Lie group $\mathrm{Hol}(\omega)$ of the Cartan geometry $(\mathcal{G}, \omega)$ is $\mathrm{Hol}(\omega')$.

When $(\mathcal{G}, \omega)$ is homogeneous, i.e., $\mathcal{G} = H \times_K P$ and ω is induced by an $\alpha \colon \mathfrak{h} \to \mathfrak{g}$ satisfying (C.1)–(C.3) the above construction yields $\mathcal{G}' = H \times_\Psi G$, with Ψ regarded as a map from $K \to P \hookrightarrow G$, and ω' corresponds (again) to $\alpha \colon \mathfrak{h} \to \mathfrak{g}$; Note here that α in particular satisfies (C.1) and (C.2).

Thus we have

Theorem 2.2. *The holonomy Lie algebra of a homogeneous Cartan geometry* $(H \times_K P, \alpha \colon \mathfrak{h} \to \mathfrak{g})$ *is*

$$\mathfrak{hol}(\alpha) = \hat{R} + [\alpha(\mathfrak{h}), \hat{R}] + [\alpha(\mathfrak{h}), [\alpha(\mathfrak{h}), \hat{R}]] + \cdots ,$$

where $\hat{R} = \langle \{\kappa(X_1, X_2) | X_1, X_2 \in \mathfrak{h}\} \rangle$.

2.2. **Infinitesimal Automorphisms.** Let $(\mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P). A vector field $\xi \in \mathfrak{X}(\mathcal{G})$ is an *infinitesimal automorphism* of $(\mathcal{G} \to M, \omega)$ if it satisfies

(1) ξ is P-invariant: $Tr^p \xi(u) = \xi(u \cdot p)\ \forall u \in \mathcal{G}, p \in P$
(2) ξ preserves ω: $\mathcal{L}_\xi \omega = 0$.

We remark that the Lie algebra of the automorphism group of the Cartan geometry $(\mathcal{G}, \omega)$ is formed by the *complete* vector fields on $\mathcal{G}$ which are infinitesimal automorphisms.

Condition (1) is equivalent to $\omega \circ \xi : \mathcal{G} \to \mathfrak{g}$ being P-equivariant. Thus an infinitesimal automorphism ξ of $(\mathcal{G}, \omega)$ gives rise to a section of the adjoint tractor bundle

$$\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g}\,.$$

We want to describe, in terms of geometric data on $\mathcal{A}M$, those sections of $\mathcal{A}M$ which correspond to infinitesimal automorphisms of $(\mathcal{G}, \omega)$. We first show how ω induces a linear connection on $\mathcal{A}M$: Since the action of P on $\mathfrak{g}$ is just the restriction of the adjoint action of G on $\mathfrak{g}$ we have

$$\mathcal{A}M = \mathcal{G} \times_P \mathfrak{g} = (\mathcal{G} \times_P G) \times_G \mathfrak{g} = \mathcal{G}' \times_G \mathfrak{g}\,.$$

Recall from 2.1: $\mathcal{G}' = \mathcal{G} \times_P G$: $\mathcal{G}'$ is the extension of structure group of $\mathcal{G}$ from P to G and $(\mathcal{G}', \omega')$ is a G-principal bundle endowed with a principal connection. Thus the principal connection ω' on $\mathcal{G}'$ induces a linear connection ∇ on $\mathcal{A}M$.

The second ingredient we need to give a condition on a section $s \in \Gamma(\mathcal{A}M)$ to be an automorphism of $(\mathcal{G}, \omega)$ comes from the curvature function

$$\kappa : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}.$$

It is a P-equivariant smooth map, and since $TM = \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$,

$$\kappa \in \Gamma(\Lambda^2(T^*M) \otimes \mathcal{A}M) \subset \Gamma(T^*M \otimes T^*M \otimes \mathcal{A}M)\,.$$

Now note that since $\mathcal{A}M = \mathcal{G} \times_P \mathfrak{g}$ the canonical surjection $\Pi\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{p}$ induces a map

$$\bar{\Pi}\colon \mathcal{A}M \to TM\,.$$

For $s \in \Gamma(\mathcal{A}M))$ we may thus consider

$$i_s\kappa := \kappa(\bar{\Pi}(s), \cdot) \in \Omega^1(M, \mathcal{A}M).$$

The following theorem characterizes infinitesimal automorphisms of a parabolic geometry as parallel sections of a connection on the adjoint tractor bundle $\mathcal{A}M$:

Theorem 2.3 ([3], Prop. 3.2)**.** *A section s of $\mathcal{A}M$ corresponds to an infinitesimal automorphism of $(\mathcal{G}, \omega)$ if and only if*

$$\nabla s + i_s\kappa = 0\,.$$

Proof. Let $\xi \in \mathfrak{X}(\mathcal{G})$ be a P-invariant vector field on $\mathcal{G}$. Then $\omega \circ \xi$ is a P-equivariant map $\mathcal{G} \to \mathfrak{g}$ and the corresponding section $s \in \Gamma(\mathcal{A}M)$ is obtained by factorizing $u \mapsto \big[u, \omega(\xi(u))\big]$. Now $\mathcal{L}_\xi\omega = i_\xi d\omega + d\big(\omega(\xi)\big)$; Take a vector field $\eta \in \mathfrak{X}(\mathcal{G})$ which is π-related to a vector field $\tilde{\eta}$ on M; Then

$$(\mathcal{L}_\xi\omega)(\eta) = d\omega(\xi, \eta) + \eta \cdot \omega(\xi) = \kappa(\xi, \eta) + \eta \cdot \omega(\xi) - \big[\omega(\xi), \omega(\eta)\big]\,.$$

Since $\nabla_{\tilde{\eta}} s$ corresponds to to the P-equivariant map $u \mapsto \eta(u) \cdot \omega(\xi) + \mathrm{ad}_{\omega_u(\eta)}\big(\omega_u(\xi)\big)$ this proves the claim. $\square$

Now

$$\hat{\nabla}_\xi s := \nabla_\xi s + \kappa(\bar{\Pi}(s), \xi) \quad \text{for} \quad \xi \in \Gamma(TM), s \in \Gamma(\mathcal{A}M) \tag{2}$$

is again a linear connection on $\mathcal{A}M$, and thus Theorem 2.3 says that the infinitesimal automorphisms of $(\mathcal{G}, \omega)$ are

$$\mathfrak{inf}(\omega) := \big\{s \in \Gamma(\mathcal{A}M) :\ \hat{\nabla} s = 0\big\}\,.$$

i.e. $\mathfrak{inf}(\omega)$ consists of the parallel sections of $(\mathcal{A}M, \hat{\nabla})$.

This allows us to reformulate the problem of determining $\mathfrak{inf}(\omega)$ in the following way:

Theorem 2.4.

$$\mathfrak{inf}(\omega) = \big\{X \in \mathfrak{g}\colon \mathrm{Ad}(\mathrm{Hol}(\hat{\nabla}))X = \{X\}\big\}\,.$$

In particular, if M is simply connected,

$$\mathfrak{inf}(\omega) = \big\{X \in \mathfrak{g}\colon \mathrm{ad}(\mathfrak{hol}(\hat{\nabla}))X = \{0\}\big\}\,.$$

This follows from the well known fact that parallel sections of vector bundles correspond to holonomy-invariant elements of the modelling vector space.

2.2.1. *Infinitesimal Automorphisms of Homogeneous Cartan Geometries.* We can now apply theorem 2.4 to the case of a Cartan connection $\alpha\colon \mathfrak{h} \to \mathfrak{g}$ on a homogeneous principal bundle $H \times_K P \to H/K$ as discussed in section 2:

$\hat{\nabla}$ as defined in (2) is H-invariant and it is easy to see that it is induced by the G-principal connection

$$\begin{gathered}\hat{\alpha} : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g})\,,\\ \hat{\alpha}(X) = \mathrm{ad}(\alpha(X)) + \kappa(\Pi(\cdot), \alpha(X) + \mathfrak{p})\end{gathered}$$

on $G' = \mathcal{G} \times_P G$.

Thus we have

Theorem 2.5. *Let $\alpha\colon \mathfrak{h} \to \mathfrak{g}$ be a Cartan connection of type (G, P) on a simply connected homogeneous space H/K. Then the Lie algebra of infinitesimal automorphisms of the corresponding homogeneous Cartan geometry consists of all elements of $\mathfrak{g}$ which are stabilized by $\mathfrak{hol}(\hat{\alpha})$, i.e.:*

$$\mathfrak{inf}(\alpha) = \big\{X \in \mathfrak{g} : \mathfrak{hol}(\hat{\alpha})X = \{0\}\big\}\,.$$

Since we can determine $\mathfrak{hol}(\hat{\alpha})$ by using theorem 1.2 it is a purely algorithmic task to calculate the infinitesimal automorphisms of a homogeneous Cartan geometry.

Of course we know that H acts by automorphisms of Cartan geometries on $(\mathcal{G}, \alpha) = (H \times_K P, \alpha)$ from the left. It is clear that the fundamental vector fields on $\mathcal{G}$ for this action are infinitesimal automorphisms and we have seen above that they are thus determined by elements of $\mathfrak{g}$: It is easy to see that these elements are exactly those in the image of $\alpha : \mathfrak{h} \to \mathfrak{g}$. It is not difficult either to verify that indeed $\alpha(\mathfrak{h})$ lies in the kernel of every element of the holonomy Lie algebra of $\hat{\alpha}$; the main observation here is that $\hat{\alpha}(X)\alpha(Y) = \alpha\big([X, Y]\big)$ for $X, Y \in \mathfrak{h}$.

3. The conformal holonomy of the product of two spheres

It is a classical result of Élie Cartan ([7]) that conformal geometries are equivalent to certain parabolic geometries. Thus 2.1 provides a notion of holonomy for conformal structures, which is called *conformal holonomy*. Conformal holonomies induced by bi-invariant metrics on Lie groups have been treated in [9]. Our setting for calculating conformal holonomies works for invariant conformal structures on arbitrary simply connected homogeneous spaces.

Let S^p, S^q be the Euclidean spheres of dimension p, q with $p+q \geq 3$ and let $\overline{g}_1$,$\overline{g}_2$ denote their Riemannian metrics of radius 1. For $s \in \mathbb{R}, s > 0$ and $s' \in \mathbb{R}\backslash\{0\}$ we have the (pseudo-)Riemannian metric $\overline{g}_{(s,s')} = (s\overline{g}_1, s'\overline{g}_2)$ on $M = S^p \times S^q$. When $s' > 0$ $\overline{g}_{(s,s')}$ is positive definite and for $s' < 0$ it has signature (p, q). The conformal class $[\overline{g}_{(s,s')}]$ of this Riemannian metric endows $S^p \times S^q$ with a conformal structure and we are going to calculate its conformal holonomy. To do this, we first switch to a homogeneous or Lie group description of the conformal geometry $(S^p \times S^q, \overline{g}_{(s,s')})$.

Since $S^p = O(p+1)/O(p)$ we have $M = S^p \times S^q = H/K$ with $H = O(p+1) \times O(q+1)$ and $K = O(p) \times O(q)$. We will write elements of $\mathfrak{so}(p+1)$ as

$$v \oplus A = \begin{pmatrix} 0 & -v^t \\ v & A \end{pmatrix}$$

with $v \in \mathbb{R}^p$ and $A \in \mathfrak{so}(p)$. So $\mathfrak{so}(p+1) = \mathbb{R}^p \oplus \mathfrak{so}(p)$ and the Lie bracket is

$$[v_1 \oplus A_1, v_2 \oplus A_2] = \big(A_1 v_2 - A_2 v_1\big) \oplus \big(v_2 v_1^t - v_1 v_2^t + [A_1, A_2]\big). \tag{3}$$

Thus $\mathfrak{k} = \mathfrak{so}(p) \oplus \mathfrak{so}(q)$ and $\mathfrak{h} = (\mathbb{R}^p \oplus \mathbb{R}^q) \oplus \mathfrak{k}$. We will denote $\mathfrak{n} = \mathbb{R}^p \oplus \mathbb{R}^q < \mathfrak{h}$. Let $g_1 = \mathbb{I}_p$, $g_2 = \mathbb{I}_q$ be the standard Euclidean inner products on $\mathbb{R}^p$ and $\mathbb{R}^q$. It will later be useful also to regard g_1,g_2 as (degenerate) bilinear forms on $\mathbb{R}^{p+q}$ by trivial extension.

It is easy to see that the $H = \mathrm{O}(p+1) \times \mathrm{O}(q+1)$-invariant (pseudo-)Riemannian metric $\overline{g}_{(s,s')} = (s\overline{g}_1, s'\overline{g}_2)$ on $M = S^p \times S^q = H/K$ corresponds to the K-invariant (pseudo-) inner product $g_{(s,s')} = sg_1 \oplus s' g_2$ on $\mathbb{R}^p \oplus \mathbb{R}^q$.

3.1. **The prolongation to a canonical Cartan connection.** Now we describe $(H/K = S^p \times S^q, [\overline{g}_{(s,s')}])$ as a (canonical) Cartan geometry of type (G, P): here $G = \mathrm{PO}(p+q+1, 1)$ for $s' > 0$ and $G = \mathrm{PO}(p+1, q+1)$ for $s' < 0$. Let $g := g_1 + \mathrm{sgn}(s') g_2$. This is the standard inner product of the same signature as $g_{(s,s')}$. The Lie algebra of G is graded

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \mathbb{R}^{p+q} \oplus \mathfrak{co}(\mathbb{R}^{p+q}, g) \oplus (\mathbb{R}^{p+q})^*$$

and P is the stabilizer of the induced filtration of $\mathfrak{g}$. The adjoint action identifies $\mathfrak{g}_0$ with $\mathfrak{co}(\mathfrak{g}_{-1}, g)$.

There is a unique (up to isomorphisms of $\mathfrak{g}$) Cartan connection

$$\alpha \colon \mathfrak{h} \to \mathfrak{g}$$

which induces an isomorphism of the Euclidean spaces $(\mathfrak{n}, g_{(s,s')})$ and $\mathfrak{g}/\mathfrak{p} = \mathfrak{g}_{-1} = (\mathbb{R}^{p+q}, g)$ and satisfies the following two normalization conditions:

(Conf.1) $\mathrm{Im}\kappa \subset \mathfrak{p}$: its curvature has vanishing $\mathfrak{g}_-$-part
(Conf.2) The Ricci-type contraction of the $\mathfrak{g}_0$-component of κ vanishes.

We will now obtain maps $\Psi \colon K \to P$ and $\alpha_0 \colon \mathfrak{h} \to \mathfrak{g}$ such that the conformal structure $\overline{g}_{(s,s')}$ is the underlying structure of the homogeneous Cartan geometry corresponding to (Ψ, α_0). First note that we have a canonical embedding Ψ of $K = O(p) \times O(q)$ into $O(g) < CO(g) = G_0 < P$; being less formal, one could say that $\Psi = \mathrm{Ad}_{|K}$. Now $\Psi' \colon \mathfrak{k} \to \mathfrak{h}$ is extended to a map $\alpha_0 \colon \mathfrak{h} \to \mathfrak{g}$ by the obvious isometry of $(\mathfrak{n}, g_{(s,s')}) = (\mathbb{R}^p \oplus \mathbb{R}^q, sg_1 \oplus |s'| g_2)$ with $(\mathfrak{g}_{-1}, g) = (\mathbb{R}^{p+q}, g_1 \oplus \mathrm{sgn}(s') g_2)$:

$$\begin{gathered} \alpha_{0|\mathfrak{n}} \colon \mathfrak{n} \to \mathbb{R}^{p+q} < \mathfrak{g}\,, \\ \big(v \oplus 0\big) \oplus \big(w \oplus 0\big) \mapsto \Big(\frac{1}{\sqrt{s}} v \oplus \frac{1}{\sqrt{|s'|}} w\Big) \oplus 0 \oplus 0\,. \end{gathered}$$

It is clear that $\alpha_0 : \mathfrak{h} \to \mathfrak{g}$ is K-equivariant; also, α_0 satisfies (C.1) and (C.3) by construction and thus it is a Cartan connection of type (G, P).

Since in $\mathfrak{so}(p+1)=\mathbb{R}^p\oplus\mathfrak{so}(p)$ the $\mathbb{R}^p$-part brackets into $\mathfrak{so}(p)$ by (3) we also have $[\mathfrak{n},\mathfrak{n}]\subset\mathfrak{k}$; thus

$$\kappa(v_1\oplus w_1, v_2\oplus w_2)\subset\mathfrak{p}\,,$$

which is the first normalizing condition (Conf.1) on the Cartan connection to this conformal structure. So it remains to find a map $A\colon\mathfrak{g}_{-1}\to\mathfrak{g}_1$ such that $\alpha=\alpha_0+A\circ\alpha_0$ also satisfies (Conf.2).

This problem is solved by the rho-tensor $A_{ij}\in L(\mathbb{R}^{p+q},(\mathbb{R}^{p+q})^*)$:

$$A_{ij}=-\frac{1}{p+q-2}\Big(R_{ij}-\frac{\tilde R}{2(p+q-1)}g_{ij}\Big)\,.$$

Here $R=\kappa_0$, the $\mathfrak{g}_0$-component of the curvature of α_0, $R_{ij}=R_{ai}{}^a{}_j$ is the Ricci curvature and $\tilde R=g^{ij}R_{ij}$ is the scalar curvature. It is a straightforward calculation that the Ricci curvature is

$$R_{ij}=s(p-1)g_{1ij}+s'(q-1)\mathrm{sgn}(s')g_{2ij}$$

and the scalar curvature is

$$\tilde R=sp(p-1)+s'q(q-1)\,.$$

Thus the rho-tensor is given by

$$A_{ij}=-\frac{1}{2\delta}\big((2s\Delta+m(s,s'))g_{1ij}+(2s'\Delta-m(s,s'))\mathrm{sgn}(s')g_{2ij}\big), \tag{4}$$

where

$$\begin{aligned}\delta&=(p+q-1)(p+q-2)\,;\\ \Delta&=(p-1)(q-1)\,;\\ m(s,s')&=sp(p-1)-s'q(q-1)\,.\end{aligned}$$

3.2. **Calculation of the curvature and the holonomy.** Now we can calculate the curvature κ of the normal Cartan connection $\alpha=\alpha_0+A\circ\alpha_0$. Since, as we have already observed,$[\mathfrak{n},\mathfrak{n}]\subset\mathfrak{k}$, and since the projection of α to $\mathfrak{g}_0$ vanishes, it is easy to see that the $\mathfrak{g}_1$-component of κ vanishes. Thus $\kappa=\kappa_{\mathfrak{g}_0}$ and for $X_1,X_2\in\mathbb{R}^{p+q}$

$$\kappa_{\mathfrak{g}_0}(X_1,X_2)=R(X_1,X_2)+[X_1,A(X_2)]-[X_2,A(X_1)].$$

One obtains that $\kappa_{ij}{}^r{}_s$ is the skew-symmetrization of $\tilde\kappa_{ij}{}^r{}_s$ in the variables i,j, where

$$\begin{aligned}\tilde\kappa_{ij}{}^r{}_s=\big(s-\frac{1}{\delta}(m(s,s')+2s\Delta)\big)\delta_{1}{}^r_i g_{1js}+\big(s'+\frac{1}{\delta}(m(s,s')-2s'\Delta)\big)\delta_{2}{}^r_i\mathrm{sgn}(s')g_{2js}\\ -\frac{2\Delta}{\delta}(s+s')\big(\delta_{1}{}^r_i g_{2js}-\delta_{2}{}^r_i\mathrm{sgn}(s')g_{1js}\big).\end{aligned} \tag{5}$$

Theorem 3.1 (Special cases)**.**

i) *If either p or q is 1 or if $s'=-s$, $(S^p\times S^q,g_{(s,s')})$ is conformally flat.*

ii) *If $p,q\geq 2$ and $s'=\frac{p-1}{q-1}s$ then $(S^p\times S^q,g_{(s,s')})$ is Einstein and*

$$\mathfrak{hol}(S^p\times S^q,g_{(s,s')})=\mathfrak{so}(p+q+1)\,.$$

Proof. Of course the case $s' = -s$ just reflects the fact that then $S^p \times S^q$ is simply a twofold covering of the homogeneous model of conformal Cartan geometries of signature (p, q). But both conformally flat cases can be seen immediately by choosing an orthogonal basis for $\mathbb{R} \oplus \mathbb{R}^{p+q} \oplus \mathbb{R}$ and using the embeddings

$$\begin{pmatrix} 0 & -v^t \\ v & A_1 \end{pmatrix} \oplus \begin{pmatrix} 0 & -w^t \\ w & A_2 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -v^t & 0 & 0 \\ v & A_1 & 0 & 0 \\ 0 & 0 & A_2 & w \\ 0 & 0 & -w^t & 0 \end{pmatrix}$$

for $s = 1, s' = -1$ resp.

$$\begin{pmatrix} 0 & -v^t \\ v & A_1 \end{pmatrix} \oplus \begin{pmatrix} 0 & -w \\ w & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -v^t & 0 & 0 \\ v & A_1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{|s'|}} w \\ 0 & 0 & \operatorname{sgn}(s') \frac{1}{\sqrt{|s'|}} w & 0 \end{pmatrix}$$

for $q = 1, s = 1$ and s' of arbitrary signature.
To see the second claim we first notice that $\mathfrak{hol}(\alpha)$ contains $\mathfrak{so}(\mathbb{R}^n, g)$: Since

$$\frac{2\Delta}{\delta}(s + s') \neq 0$$

the image of κ contains all matrices of the form

$$\begin{pmatrix} 0 & B \\ -\operatorname{sgn}(s') B^t & 0 \end{pmatrix} \in \mathfrak{so}(\mathbb{R}^n, g) \,.$$

But since

$$\alpha\left(\begin{pmatrix} 0 & 0 \\ 0 & A_1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & A_2 \end{pmatrix}\right) = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \in \mathfrak{so}(\mathbb{R}^n, g)$$

and $\mathfrak{hol}(\alpha)$ is the $(\mathfrak{h}, \alpha)$-module generated by the image of κ, simplicity of $\mathfrak{so}(\mathbb{R}^n, g)$ shows that indeed $\mathfrak{so}(\mathbb{R}^n, g) \subset \mathfrak{hol}(\alpha)$.
Now the condition

$$s' = \frac{p-1}{q-1} s$$

means exactly that the Ricci curvature R_{ij} is a multiple of g, and thus $(S^p \times S^q, g_{(s,s')})$ is Einstein. Then by (4) also A is a multiple of g, and, explicitly: $A = rg$ with

$$r = -\frac{p-1}{2(p+q-1)} \,.$$

Thus

$$\alpha((v \oplus A_1) \oplus (w \oplus A_2) = (v \oplus w) \oplus (A_1 \oplus A_2) \oplus r(v \oplus w)^t \,,$$

and we see that the smallest Lie subalgebra of $\mathfrak{g}$ containing $\mathrm{Im}\kappa$ and being invariant under $(\mathfrak{h}, \alpha)$ consists of matrices of the form

$$\begin{pmatrix} 0 & rX^t & 0 \\ X & A & -rX \\ 0 & -X^t & 0 \end{pmatrix},$$

which proves the claim since $r < 0$. □

When both p and q are at least two a generic ratio of radii (s, s') of arbitrary signature yields full holonomy, which is straightforward to check. A different argument for this case can be found in [2].

Theorem 3.2 (The generic case)**.** *If $p, q \geq 2$, then for $s' \notin \{-s, \frac{p-1}{q-1}s\}$,*

$$\mathfrak{hol}(S^p \times S^q, g_{(s,s')}) = \mathfrak{g} = \mathfrak{so}(p+q+1, 1) \text{ for } s' > 0 \quad \text{resp.}$$
$$\mathfrak{hol}(S^p \times S^q, g_{(s,s')}) = \mathfrak{g} = \mathfrak{so}(p+1, q+1) \text{ for } s' < 0\,.$$

Remark 3.3. The treatment of the holonomy of homogeneous parabolic geometries other than conformal structures is closely parallel, the only additional problem which appears is that there are no longer general formulas for the prolongation of the given geometric data to the corresponding Cartan geometries. To see how this problem boils down to basic representation theory see e.g. [4] or [8] for explicit examples of prolongations in the realm of CR-structures.

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